BESSEL FUNCTIONS

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Differential Equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

Solution

$$y(x) = A_1 J_n(x) + A_2 Y_n(x) \quad \text{for all n}$$

$$y(x) = A_1 J_n(x) + A_2 J_{-n}(x) \quad \text{for all } n \neq 0, 1, 2, \ldots$$

Bessel Function of the First Kind

Zero Order

$$J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \ldots$$

First Order

$$J_1(x) = \frac{x}{2}\left[1 - \frac{(x/2)^2}{2(1!)^2} + \frac{(x/2)^4}{3(2!)^2} - \frac{(x/2)^6}{4(3!)^2} + \ldots\right] = -\frac{d}{dx}[J_0(x)]$$

Order n

$n = 0, 1, 2, \ldots$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,\Gamma(k+1+n)}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\,\Gamma(k+1-n)} = (-1)^n J_n(x)$$

<u>Modified Bessel Function of the First Kind</u>

Zero Order

$$I_0(x) = 1 + \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} + \frac{(x/2)^6}{(3!)^2} + \ldots$$

First Order

$$I_1(x) = \frac{x}{2}\left[1 + \frac{(x/2)^2}{2(1!)^2} + \frac{(x/2)^4}{3(2!)^2} + \frac{(x/2)^6}{4(3!)^2} + \ldots\right] = \frac{d}{dx}\left[I_0(x)\right]$$

Higher Order Modified Bessel Function

$$I(x) = (j)^{-n} J_n(jx)$$

Bessel Function of the Second Kind (Neumann Functions)

Zero Order

$$Y_o(x) = \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + c\right)J_o(x) + \frac{2}{1}J_2(x) - \frac{2}{2}J_4(x) + \frac{2}{3}J_6(x) - \dots\right]$$

where $c = 0.577\ 215\ 665$

First Order

$$Y_1(x) = \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + c\right)J_1(x) - \left(\frac{1}{x}\right) - \frac{1}{2}J_1(x) + \frac{9}{4}J_3(x) - \dots\right] = -\frac{d}{dx}\left[Y_o(x)\right]$$

Modified Bessel Functions of the Second Kind

Zero Order

$$K_o(x) = -\left[\left(\ln\frac{x}{2} + c\right) I_o(x) - \frac{2}{1}I_2(x) - \frac{2}{2}I_4(x) - \frac{2}{3}J_6(x) - \dots\right]$$

Half-Odd Integers

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \qquad Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos(x)$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin(x)}{x} - \cos(x)\right] \qquad Y_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos(x)}{x} + \sin(x)\right]$$

Recurrence Relations

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$I_{n+1}(x) = -\frac{2n}{x} I_n(x) + I_{n-1}(x)$$

APPENDIX A

Bessel Function of the First Kind of Order n

First Derivative

$$
\begin{aligned}
\frac{d}{dx} J_n(x) &= J_{n-1}(x) - \frac{n}{x} J_n(x) \\
&= -J_{n+1}(x) + \frac{n}{x} J_n(x) \\
&= \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]
\end{aligned}
$$

Second Derivative

$$
\begin{aligned}
\frac{d^2}{dx^2} J_n(x) &= \frac{1}{2}\frac{d}{dx}\left[J_{n-1}(x) - J_{n+1}(x)\right] \\
&= \frac{1}{2}\left\{\frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right] - \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]\right\} \\
&= \frac{1}{4}\left\{ J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right\}
\end{aligned}
$$

Third Derivative

$$\frac{d^3}{dx^3}J_n(x) = \frac{1}{4}\frac{d}{dx}\left\{ J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)\right\}$$

$$= \frac{1}{4}\left\{ \frac{d}{dx}J_{n-2}(x) - 2\frac{d}{dx}J_n(x) + \frac{d}{dx}J_{n+2}(x)\right\}$$

$$= \frac{1}{4}\left\{ \frac{1}{2}\left[J_{n-3}(x) - J_{n-1}(x)\right] - \left[J_{n-1}(x) - J_{n+1}(x)\right] + \frac{1}{2}\left[J_{n+1}(x) - J_{n+3}(x)\right]\right\}$$

$$= \frac{1}{8}\left\{\left[J_{n-3}(x) - J_{n-1}(x)\right] - 2\left[J_{n-1}(x) - J_{n+1}(x)\right] + \left[J_{n+1}(x) - J_{n+3}(x)\right]\right\}$$

$$= \frac{1}{8}\left\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\right\}$$

Fourth Derivative

$$\frac{d^4}{dx^4}J_n(x) \quad = \frac{1}{8}\frac{d}{dx}\{J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)\}$$

$$= \frac{1}{8}\left\{\frac{d}{dx}J_{n-3}(x) - 3\frac{d}{dx}J_{n-1}(x) + 3\frac{d}{dx}J_{n+1}(x) - \frac{d}{dx}J_{n+3}(x)\right\}$$

$$= \frac{1}{8}\left\{\frac{d}{dx}J_{n-3}(x) - 3\frac{d}{dx}J_{n-1}(x) + 3\frac{d}{dx}J_{n+1}(x) - \frac{d}{dx}J_{n+3}(x)\right\}$$

$$= \frac{1}{16}\left[J_{n-4}(x) - J_{n-2}(x)\right]$$

$$- \frac{3}{16}\left[J_{n-2}(x) - J_n(x)\right]$$

$$+ \frac{3}{16}\left[J_n(x) - J_{n+2}(x)\right]$$

$$- \frac{1}{16}\left[J_{n+2}(x) - J_{n+4}(x)\right]$$

$$= \frac{1}{16}\left[J_{n-4}(x) - 4J_{n-2}(x) + 6J_n(x) - 4J_{n+2}(x) + J_{n+4}(x)\right]$$

APPENDIX B

Modified Bessel Function of the First Kind of Order n

First Derivative

$$\frac{d}{dx} I_n(x) = I_{n-1}(x) - \frac{n}{x} I_n(x)$$

$$= I_{n+1}(x) + \frac{n}{x} I_n(x)$$

$$= \frac{1}{2}\left[I_{n-1}(x) + I_{n+1}(x)\right]$$

Second Derivative

$$\frac{d^2}{dx^2} I_n(x) = \frac{1}{2}\frac{d}{dx}\left[I_{n-1}(x) + I_{n+1}(x)\right]$$

$$= \frac{1}{2}\left[\frac{d}{dx} I_{n-1}(x) + \frac{d}{dx} I_{n+1}(x)\right]$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[I_{n-2}(x) + I_n(x)\right] + \frac{1}{2}\left[I_0(x) + I_{n+2}(x)\right]\right\}$$

$$= \frac{1}{4}\left\{I_{n-2}(x) + 2I_n(x) + I_{n+2}(x)\right\}$$

Third Derivative

$$\frac{d^3}{dx^3} I_n(x) = \frac{1}{4}\frac{d}{dx}\left\{I_{n-2}(x) + 2I_n(x) + I_{n+2}(x)\right\}$$

$$+\frac{1}{4}\left\{\frac{d}{dx} I_{n-2}(x) + 2\frac{d}{dx} I_n(x) + \frac{d}{dx} I_{n+2}(x)\right\}$$

$$+\frac{1}{4}\left\{\frac{1}{2}\left[I_{n-3}(x) + I_{n-1}(x)\right] + \left[I_{n-1}(x) + I_{n+1}(x)\right] + \frac{1}{2}\left[I_{n+1}(x) + I_{n+3}(x)\right]\right\}$$

$$+\frac{1}{8}\left\{\left[I_{n-3}(x) + I_{n-1}(x)\right] + 2\left[I_{n-1}(x) + I_{n+1}(x)\right] + \left[I_{n+1}(x) + I_{n+3}(x)\right]\right\}$$

$$+\frac{1}{8}\left\{I_{n-3}(x) + 3I_{n-1}(x) + 3I_{n+1}(x) + I_{n+3}(x)\right\}$$

Fourth Derivative

$$\frac{d^4}{dx^4} I_n(x) \quad = \frac{1}{8}\frac{d}{dx}\{I_{n-3}(x) + 3I_{n-1}(x) + 3I_{n+1}(x) + I_{n+3}(x)\}$$

$$= \frac{1}{8}\left\{\frac{d}{dx} I_{n-3}(x) + 3\frac{d}{dx} I_{n-1}(x) + 3\frac{d}{dx} I_{n+1}(x) + \frac{d}{dx} I_{n+3}(x)\right\}$$

$$= \frac{1}{8}\left\{\frac{d}{dx} I_{n-3}(x) + 3\frac{d}{dx} I_{n-1}(x) + 3\frac{d}{dx} I_{n+1}(x) + \frac{d}{dx} I_{n+3}(x)\right\}$$

$$= \frac{1}{16}[I_{n-4}(x) + I_{n-2}(x)]$$

$$+ \frac{3}{16}[I_{n-2}(x) + I_n(x)]$$

$$+ \frac{3}{16}[I_n(x) + I_{n+2}(x)]$$

$$+ \frac{1}{16}[I_{n+2}(x) + I_{n+4}(x)]$$

$$= \frac{1}{16}[I_{n-4}(x) + 4I_{n-2}(x) + 6I_n(x) + 4I_{n+2}(x) + I_{n+4}(x)]$$

APPENDIX C

Bessel Function of the First Kind of Order Zero, Derivatives

First Derivative

$$\frac{d}{dx} J_0(x) \quad = -J_1(x)$$

Second Derivative

$$\frac{d^2}{dx^2} J_0(x) \quad = \frac{1}{4}\left\{ J_{-2}(x) - 2J_0(x) + J_2(x) \right\}$$

Note that

$$J_{n+1}(x) = -J_{n-1}(x) \quad \text{for } \; n=0$$

Thus

$$\frac{d^2}{dx^2} J_0(x) \quad = \frac{1}{4}\left\{ -J_0(x) - 2J_0(x) - J_0(x) \right\}$$

$$\frac{d^2}{dx^2} J_0(x) \quad = -J_0(x)$$

Third Derivative

$$\frac{d^3}{dx^3}J_0(x) = \frac{1}{8}\{J_3(x) - 3J_{-1}(x) + 3J_1(x) - J_3(x)\}$$

$$= \frac{1}{4}\{3J_1(x) - J_3(x)\}$$

$$= \frac{1}{4}\{3J_1(x) + J_1(x)\}$$

$$= J_1(x)$$

Fourth Derivative

$$\frac{d^4}{dx^4}J_0(x) = \frac{1}{16}\left[J_{-4}(x) - 4J_{-2}(x) + 6J_0(x) - 4J_2(x) + J_4(x)\right]$$

$$\frac{d^4}{dx^4}J_0(x) = \frac{1}{16}\left[-J_{-2}(x) - 4J_{-2}(x) + 6J_0(x) - 4J_2(x) - J_2(x)\right]$$

$$\frac{d^4}{dx^4}J_0(x) = \frac{1}{16}\left[-5J_{-2}(x) + 6J_0(x) - 5J_2(x)\right]$$

$$\frac{d^4}{dx^4}J_0(x) = \frac{1}{16}\left[5J_0(x) + 6J_0(x) + 5J_0(x)\right]$$

$$\frac{d^4}{dx^4}J_0(x) = J_0(x)$$

APPENDIX D

Modified Bessel Function of the First Kind of Order Zero, Derivatives

First Derivative

$$\frac{d}{dx} I_n(x) \quad = \frac{1}{2}\left[I_{-1}(x) + I_1(x)\right]$$

$$= I_1(x)$$

Second Derivative

$$\frac{d^2}{dx^2} I_0(x) \quad = \frac{1}{4}\left\{ I_{-2}(x) + 2I_0(x) + I_2(x)\right\}$$

Note that

$$I_{n+1}(x) = I_{n-1}(x) \quad \text{for } n=0$$

Thus

$$\frac{d^2}{dx^2} I_0(x) \quad = \frac{1}{4}\left\{ I_0(x) + 2I_0(x) + I_0(x)\right\}$$

$$\frac{d^2}{dx^2} I_0(x) \quad = I_0(x)$$

Third Derivative

$$\frac{d^3}{dx^3} I_0(x) = \frac{1}{8}\left\{I_3(x) + 3I_{-1}(x) + 3I_1(x) + I_3(x)\right\}$$

$$= \frac{1}{4}\left\{3I_1(x) + I_3(x)\right\}$$

$$= \frac{1}{4}\left\{3I_1(x) + I_1(x)\right\}$$

$$= I_1(x)$$

Fourth Derivative

$$\frac{d^4}{dx^4} I_0(x) = \frac{1}{16}\left[I_{-4}(x) + 4I_{-2}(x) + 6I_0(x) + 4I_2(x) + I_4(x)\right]$$

$$\frac{d^4}{dx^4} I_0(x) = \frac{1}{16}\left[I_{-2}(x) + 4I_{-2}(x) + 6I_0(x) + 4I_2(x) + I_2(x)\right]$$

$$\frac{d^4}{dx^4} I_0(x) = \frac{1}{16}\left[5I_{-2}(x) + 6I_0(x) + 5I_2(x)\right]$$

$$\frac{d^4}{dx^4} I_0(x) = \frac{1}{16}\left[5I_0(x) + 6I_0(x) + 5I_0(x)\right]$$

$$\frac{d^4}{dx^4} I_0(x) = I_0(x)$$