SHOCK AND VIBRATION RESPONSE SPECTRA COURSE Unit 19
Force Shock: Classical Pulse

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Introduction

Consider a structure subjected to a force shock pulse.

For example, an object might be purposely struck with an impulse hammer. The force impulse usually takes the form of a half-sine pulse. The object's resulting displacement is typically a decaying sinusoidal pulse. The natural frequency and damping ratio of the object can thus be determined. This is one form of modal testing.

As another example, a certain rocket vehicle must withstand the force shock from a motor with a short burn time. The thrust versus time curve might have either a half-sine or rectangular shape.

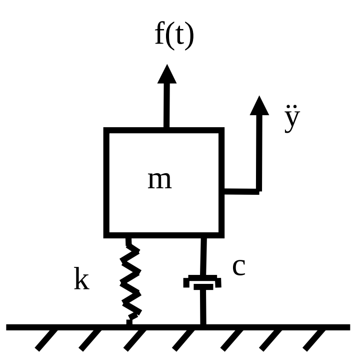
The same vehicle may also be required to withstand the force shock pulses from a attitude control system, which uses bursts of nitrogen gas to reorient the vehicle during coast periods.

The purpose of this Unit is to consider the case where the force input is in the form of a classical pulse, such as a half-sine or rectangular pulse. The force input is applied analytically to a single-degree-of-freedom system.

The maximum displacement response of a system with a variable natural frequency is plotted as a force shock response spectrum. The shock response spectrum is useful for evaluating the damage potential of the shock pulse.

Derivation of Equations

Consider a single-degree-of-freedom system.



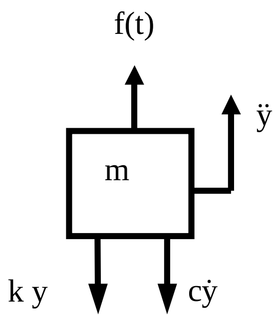
where

m is the mass,

c is the viscous damping coefficient,
k is the stiffness,
y is the absolute displacement of the mass,
f(t) is the applied force.

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n^{\,2} \tag{6}$$

where

ω_n is the natural frequency in (radians/sec),
ξ is the damping ratio.

By substitution,

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{1}{m} f(t) \tag{7}$$

Now assume a sinusoidal force function.

$$f(t) = f_o \sin(\omega t) \tag{8}$$

The governing equation becomes.

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 x = \frac{1}{m} f_o \sin(\omega t) \tag{9}$$

The right-hand-side can be rewritten as

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{\omega_n^2}{k} f_o \sin(\omega t) \tag{10}$$

Equation (10) can be solved via Laplace transforms. Details are given in Reference 1. The resulting displacement is

$$
\begin{aligned}
y(t) = {} & y(0)\, e^{-\xi\omega_n t}\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\
& + y'(0)\left[\frac{1}{\omega_d}\right] e^{-\xi\omega_n t}\sin(\omega_d t) \\
& + \left\{\frac{\omega_n{}^2 f_o / k}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]}\right\}\left\{-2\xi\omega_n\omega\cos(\omega t) - \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right\} \\
& + \left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{2\xi\omega_n\omega_d\cos(\omega_d t)\right\} \\
& + \left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\left[\omega^2 + \omega_n^2\left[-1 + 2\xi^2\right]\right]\sin(\omega_d t)\right\}
\end{aligned}
\tag{11}
$$

Equation (11) gives the response for a steady-state sine input. It is also valid for the first half-cycle of a half-sine input. Thus, modify equation (11) for the half-sine input case. Also, assume zero initial displacement and zero initial velocity.

$$
\begin{aligned}
y(t) = \\
&+\left\{\frac{\omega_n^{2} f_o / k}{\left[\left(\omega^{2}-\omega_n^{2}\right)^{2}+\left(2\xi\,\omega\omega_n\right)^{2}\right]}\right\}\left\{-2\xi\omega_n\omega\cos(\omega t)-\left(\omega^{2}-\omega_n^{2}\right)\sin(\omega t)\right\} \\
&+\left\{\frac{\omega\,\omega_n^{2} f_o / k}{\omega_d\left[\left(\omega^{2}-\omega_n^{2}\right)^{2}+\left(2\xi\,\omega\omega_n\right)^{2}\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{2\xi\omega_n\omega_d\cos(\omega_d t)\right\} \\
&+\left\{\frac{\omega\,\omega_n^{2} f_o / k}{\omega_d\left[\left(\omega^{2}-\omega_n^{2}\right)^{2}+\left(2\xi\,\omega\omega_n\right)^{2}\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\left[\omega^{2}+\omega_n^{2}\left[-1+2\xi^{2}\right]\right]\sin(\omega_d t)\right\},
\end{aligned}
$$

for $\quad 0 \le t \le T$

(12)

Note that T is the half-sine duration and that

$$\omega = \frac{\pi}{T} \tag{13}$$

The velocity is

$$\dot{y}(t) =$$

$$+\left\{\frac{\omega_n{}^2 f_o / k}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\omega_n)^2\right]}\right\}\left\{-2\xi\omega_n\omega\cos(\omega t)-\left(\omega^2-\omega_n^2\right)\sin(\omega t)\right\}$$

$$+\left\{-\omega_d\right\}\left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{2\xi\omega_n\omega_d\sin(\omega_d t)\right\}$$

$$+\left\{\omega_d\right\}\left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\left[\omega^2+\omega_n^2\left[-1+2\xi^2\right]\right]\cos(\omega_d t)\right\}$$

$$+\left\{-\xi\omega_n\right\}\left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{2\xi\omega_n\omega_d\cos(\omega_d t)\right\}$$

$$+\left\{-\xi\omega_n\right\}\left\{\frac{\omega\,\omega_n{}^2 f_o / k}{\omega_d\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\,\omega\omega_n)^2\right]}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\left[\omega^2+\omega_n^2\left[-1+2\xi^2\right]\right]\sin(\omega_d t)\right\},$$

for $\quad 0 \le t \le T$

$$(14)$$

The acceleration can be found by taking the derivative of the velocity equation. A more expedient method for computational purposes, however, is to simply take

$$\ddot{y} = \frac{\omega_n^2}{k} f_o \sin(\omega t) - 2\xi\omega_n\dot{y} - \omega_n^2 y\,, \quad \text{for } 0 \le t \le T \tag{15}$$

For t > T, the free vibration equation may be used to determine the velocity.

$$y(t) = \exp\left[-\xi\omega_n(t-T)\right]\left\{\left[\, y(T)\,\right]\cos\left[\omega_d(t-T)\right] + \left[\frac{\dot{y}(T) + (\xi\omega_n)y(T)}{\omega_d}\right]\sin\left[\omega_d(t-T)\right]\right\},$$

$$t > T \tag{16}$$

Equation (16) is taken from Reference 2.

The velocity is found by taking the derivative.

$$\dot{y}(t) = \exp\left[-\xi\omega_n(t-T)\right]\left\{\dot{y}(T)\cos\left[\omega_d(t-T)\right] - \frac{\omega_n}{\omega_d}\left\{\xi\,\dot{y}(T) + \omega_n y(T)\right\}\sin\left[\omega_d(t-T)\right]\right\},$$

$$t > T \tag{17}$$

The acceleration can be found by taking the derivative of the velocity equation. A more expedient method for computational purposes, however, is to simply take

$$\ddot{y} = -2\xi\omega_n\dot{y} - \omega_n^2 y\,, \quad \text{for} \quad t > T \tag{18}$$

Finally, the nondimensional acceleration can be found by multiplying the acceleration by a factor of $\left[m/f_o\right]$.

Equations (13) through (18) thus provide a method for determining the response of single-degree-of-freedom system to a half-sine force input. These equations can be readily implemented in a computer program.

Example

Consider the example in Table 1. The input force pulse is shown in Figure 1. The calculations were made using equations (13) through (18).

Table 1. Force Shock Response Spectrum, Q=10, m=1 kg, Force Input = 0.010 sec, 1 N, Half-sine Pulse			
Natural Frequency (Hz)	Peak Positive Acceleration (m/sec^2)	Peak Negative Acceleration (m/sec^2)	Figure
10.00	0.96	-0.37	2
20.00	0.89	-0.72	-
30.00	0.88	-1.03	-
40.00	1.09	-1.28	-
50.00	1.25	-1.45	-
60.00	1.34	-1.43	-
70.00	1.36	-1.25	3
80.00	1.32	-1.13	-
90.00	1.22	-1.04	-
100.00	1.06	-0.91	-
110.00	0.87	-0.75	-
120.00	0.66	-0.60	4
130.00	0.44	-0.53	-
140.00	0.31	-0.47	-
150.00	0.29	-0.42	
160.00	0.27	-0.38	
170.00	0.28	-0.35	
180.00	0.35	-0.32	-
190.00	0.38	-0.33	-
200.00	0.38	-0.32	-

Note that only the peak positive and negative values are retained for each time history response. The peak values are found via a simple search method rather than a calculus method.

Furthermore, note that the peak response can occur either during or after the half-sine pulse.

The overall shock response spectrum is shown in Figure 5. It is constructed by plotting the peak positive and negative acceleration amplitudes versus natural frequency in (Hz).

The are other types of shock response spectra which could be plotted. For example, the absolute value acceleration response spectrum could be plotted, instead of the individual positive and negative spectra.

Furthermore, the peak displacement or peak velocity could be plotted versus the natural frequency.

In addition, this process could be repeated for other classical pulses, such as trapezoidal, sawtooth, and rectangular pulses.

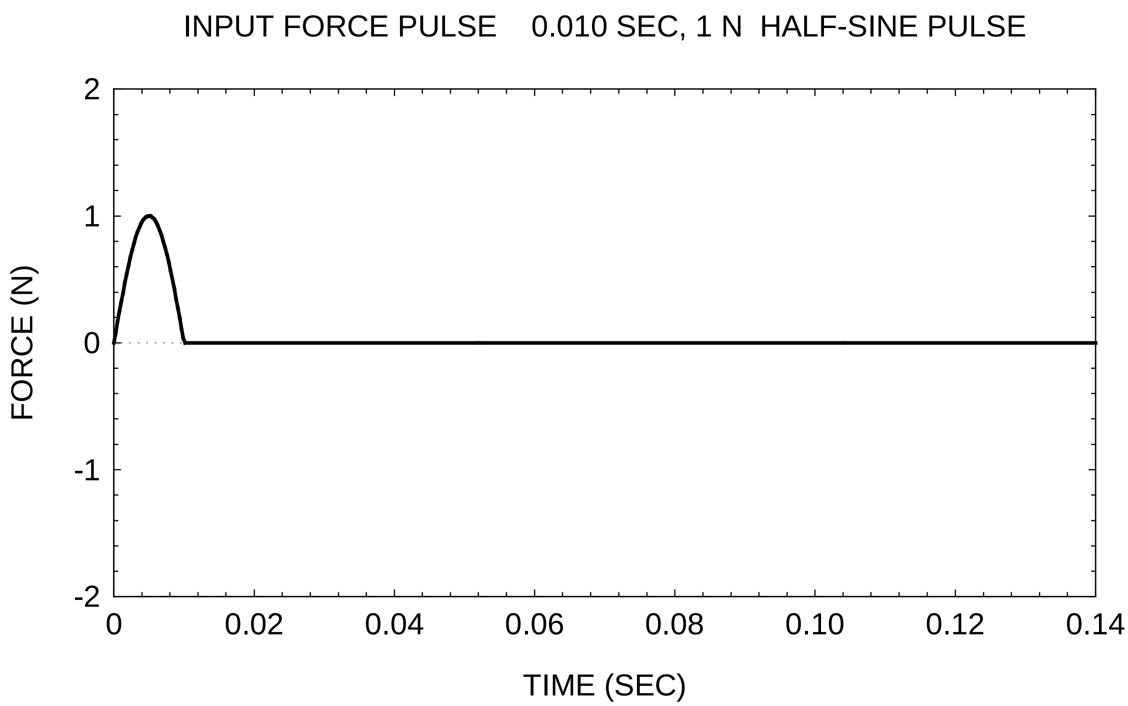


Figure 1.

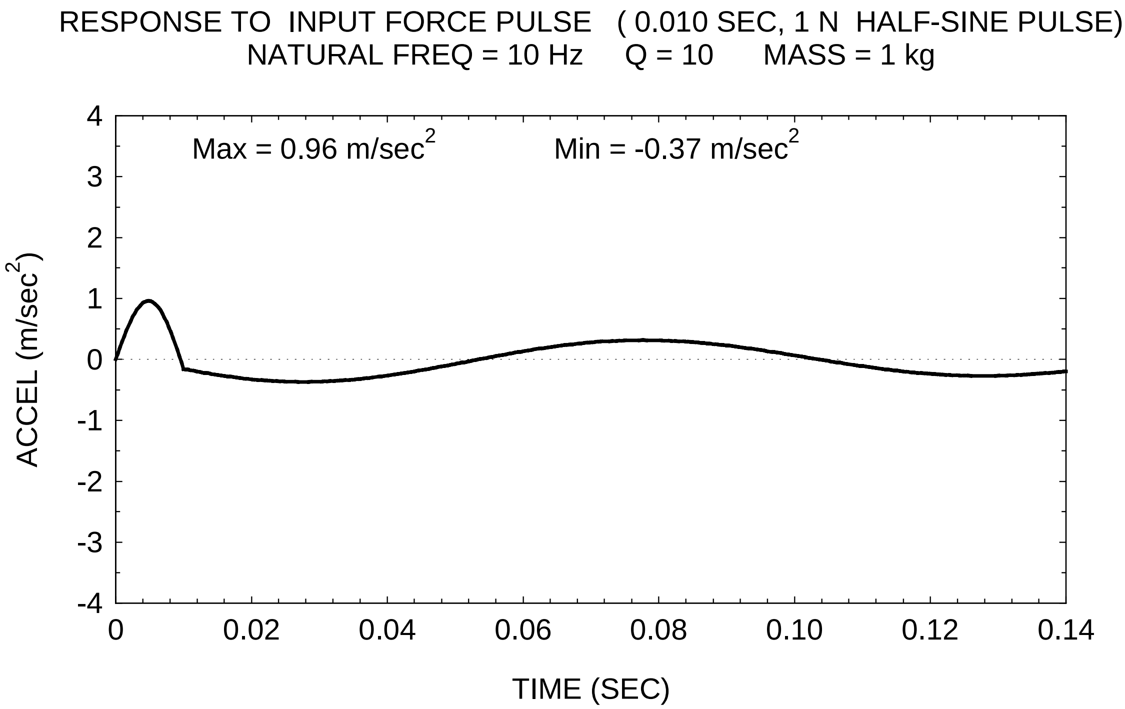


Figure 2.

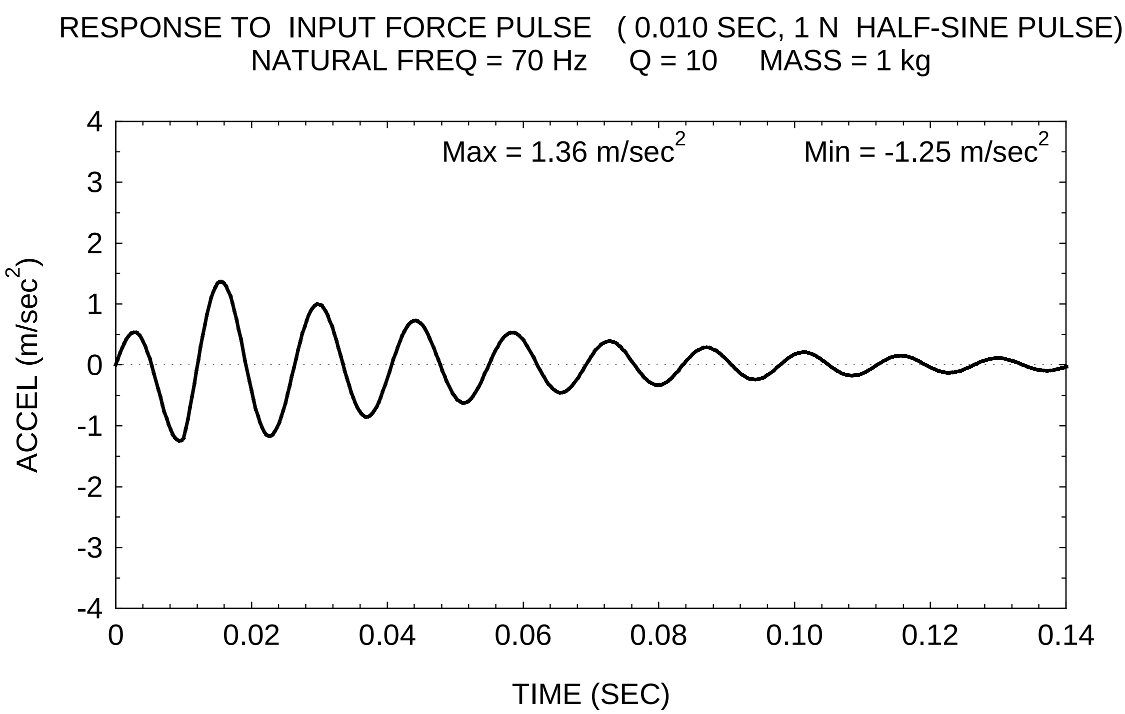


Figure 3.

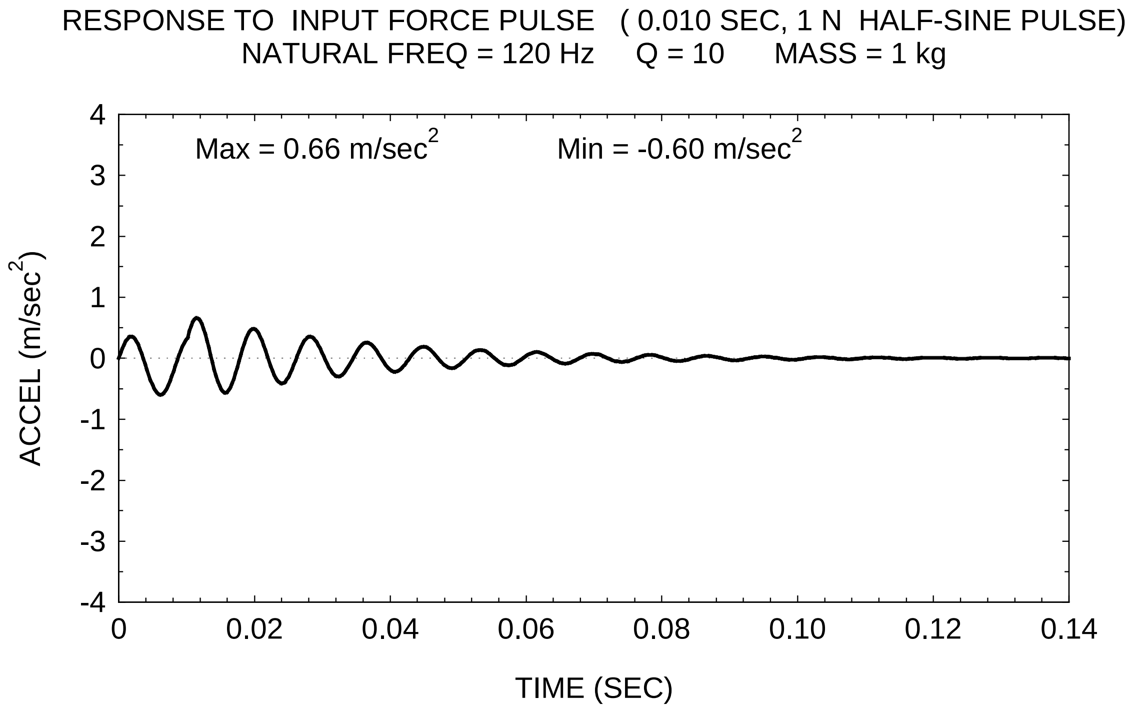

RESPONSE TO INPUT FORCE PULSE (0.010 SEC, 1 N HALF-SINE PULSE)
NATURAL FREQ = 120 Hz Q = 10 MASS = 1 kg
Max = 0.66 m/sec^2
Min = -0.60 m/sec^2
ACCEL (m/sec^2)
4
3
2
1
0
-1
-2
-3
-4
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
TIME (SEC)

Figure 4.

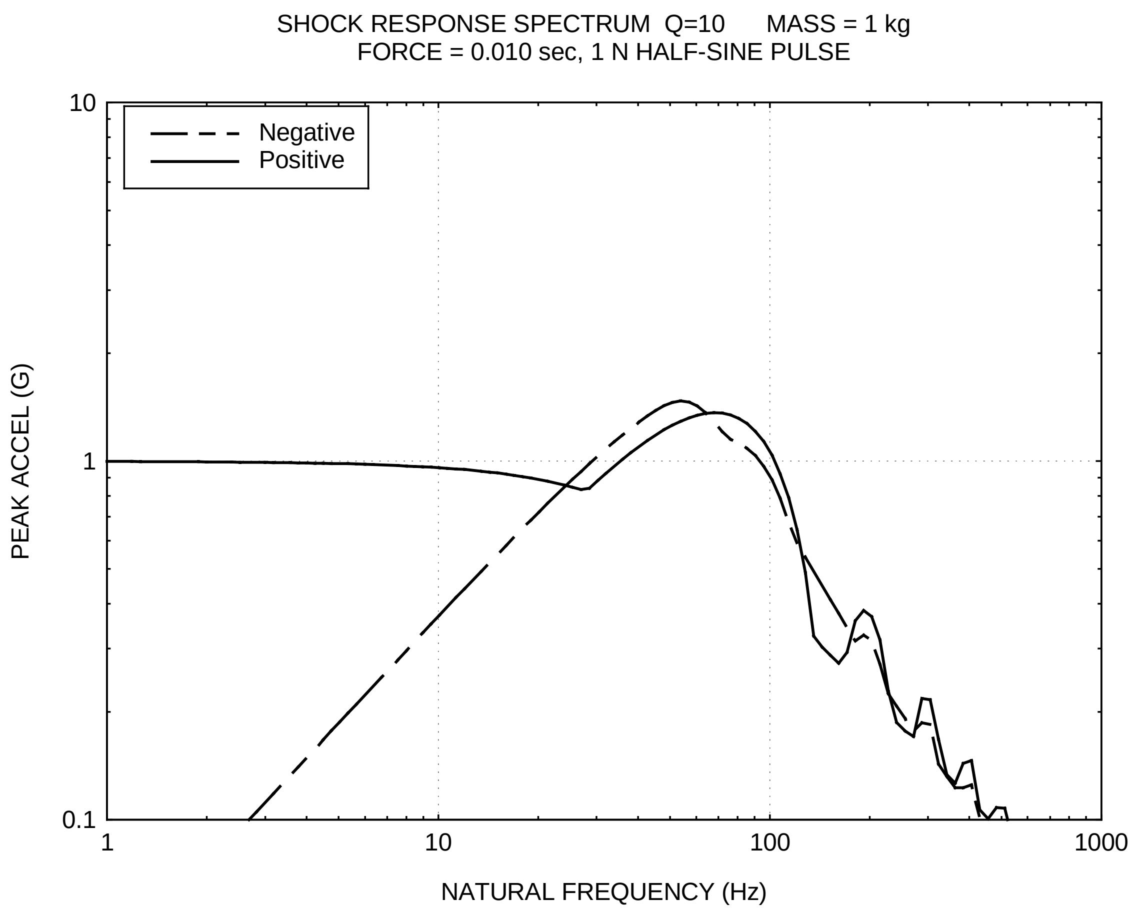


Figure 5.

Note that the spring stiffness approaches zero as the natural frequency decreases to zero. Rigid-body dynamic effects thus occur at very low natural frequencies. The problem simplifies to F=ma at these low frequencies.

The peak response, however, occurs at an intermediate frequency.

<u>References</u>

1. T. Irvine, Total Response of a Single-degree-of-freedom System to a Harmonic Forcing Function, Vibrationdata.com Publications, 1999.
2. T. Irvine, Free Vibration of a Single-degree-of-freedom System, Vibrationdata.com Publications, 1999.

Homework

1. Repeat the example for a half-sine force input of 0.008 seconds, 1 N. Assume a damping value of Q = 10 and a mass of 1 kg. Use program fhsine.exe for both the time history response and shock response spectra calculations.

2. What is the natural frequency which has the highest absolute value response for problem 1?