SHOCK AND VIBRATION RESPONSE SPECTRA COURSE
Unit 2A. Sine Vibration Characteristics

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Introduction

Consider a single-degree-of-freedom system undergoing excitation from an applied force or base excitation. Either the input forcing function or the output response may take the form of a sinusoidal oscillation. This form of oscillation is very important in vibration analysis. Even complex, random vibration can be represented by a series of sine functions.

Statistical Parameters

There are certain descriptive statistical parameters used to characterize a signal. These parameters can be used for any type of signal whatsoever, although the focus of this unit is sinusoidal signals.

The first parameter is the average or mean value. Consider a continuous function a(t) with duration T. The mean value $\overline{X}$ is

$$\overline{X} = \lim_{T \to \infty} \frac{1}{T} \int_0^T a(t)\, dt \tag{1}$$

A practical signal would not have an infinite duration, however. The formula can be modified for finite signals.

Consider a sine function with amplitude A which has a duration of one cycle.

$$\overline{X} = \frac{1}{T} \int_0^T A \sin\left[\frac{2\pi t}{T}\right] dt \tag{2}$$

$$\overline{X} = -\frac{1}{T} A \left[\frac{T}{2\pi}\right] \cos\left[\frac{2\pi t}{T}\right] \Bigg|_0^T \tag{3}$$

$$\overline{X} = -\frac{1}{T} A \left[\frac{T}{2\pi}\right] \left\{\cos(2\pi) - 0\right\} \tag{4}$$

$$\overline{X} = 0 \tag{5}$$

The mean value of a single cycle of a sine wave is zero. Likewise, the mean value of n cycles is zero, where n is an integer.

Now assume that the number of cycles is not an integer. An example is 78.683 cycles.

The mean value equation becomes

$$\overline{X} = \frac{1}{cT}\int_0^{cT} A\sin\left[\frac{2\pi t}{T}\right]dt \tag{6}$$

where c is the number of cycles.

The mean value approaches zero as c becomes large. Thus, the mean value of pure sine oscillation is considered as zero regardless of the number of cycles.

Usually, the mean value of a signal is of little use in vibration analysis. Here are two exceptions:

1. A segment of measured accelerometer data has a spurious baseline offset due to instrumentation problems. This "DC offset" must be removed to accurately portray the data.
2. The test item is undergoing "rigid-body" acceleration. The accelerometer is capable of measuring this acceleration, down to a frequency of zero Hz. Furthermore, measurement of rigid-body acceleration is a requirement.

Rigid-body acceleration must be measured in a guided rocket vehicle. This is done via the servo accelerometers in the inertial navigation system. The flight computer then "double integrates" the acceleration in order to determine the displacement.[1]

Most accelerometer measurements, however, are taken on items such as machines and other structures which experience zero net displacement. Hence, the expected mean acceleration value is zero. Some data acquisition systems thus have "AC coupling" or high-pass filtering to remove any spurious DC component. Otherwise, the removal of the DC component must be performed on the digital data in post-processing.

Further descriptive statistics are needed. The next candidate is the mean square value $\overline{X^2}$.

$$\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^{T} [a(t)]^2\, dt \tag{7}$$

[1] Rotational data from gyroscopes is also used in the displacement calculation.

The mean square value for one cycle of a sine wave is

$$\overline{X^2} = \frac{1}{T}\int_0^T \left\{ A \sin\left[\frac{2\pi t}{T}\right]\right\}^2 dt \tag{8}$$

Apply a trig identity.

$$\overline{X^2} = \frac{A^2}{2T}\int_0^T \left\{ 1 - \cos\left[\frac{4\pi t}{T}\right]\right\} dt \tag{9}$$

$$\overline{X^2} = \frac{A^2}{2T}\left\{ T - \left[\frac{T}{4\pi}\right]\sin\left[\frac{4\pi t}{T}\right]\right\}\Bigg|_0^T \tag{10}$$

$$\overline{X^2} = \frac{A^2}{2} \tag{11}$$

The root-mean-square (RMS) value X_{RMS} is simply the square root of the mean value. Thus, the RMS value for one cycle of a sine function is

$$X_{RMS} = \frac{1}{\sqrt{2}} A \tag{12}$$

$$X_{RMS} \approx 0.707 \text{ A} \tag{13}$$

Note that equations (11) through (13) are also true for a sine function with numerous cycles.

Warning: equations (11) through (13) apply to pure sine vibration. Random vibration has a completely different relationship as discussed in a later unit.

Standard Deviation

Another important parameter is the standard deviation σ. It is a measure of the dispersion about the mean.

The standard deviation is related to the mean and mean-square values as follows:

$$\sigma^2 = \overline{X^2} - \left[\overline{X}\right]^2 \tag{14}$$

Note that σ^2 is called the variance.

$$\sigma = \sqrt{\overline{X^2} - \left[\overline{X}\right]^2} \tag{15}$$

Thus, the standard deviation is equal to the RMS value if the mean is zero.

Discrete Forms

Vibration data is often represented in digital format. Consider a series of amplitude points a_i where there is a total of n points.

The mean value is

$$\overline{X} = \frac{1}{n} \sum_{i=0}^{n} x_i \tag{16}$$

The mean-square value is

$$\overline{X^2} = \frac{1}{n} \sum_{i=0}^{n} \left(x_i^2 \right) \tag{17}$$

The standard deviation equation is still

$$\sigma = \sqrt{\overline{X^2} - \left[\overline{X}\right]^2} \tag{18}$$

An equivalent formula for the standard deviation is

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=0}^{n} (x_i - \overline{X})^2} \tag{19}$$

Crest Factor

The relationship of the peak amplitude to the standard deviation value is another parameter of interest. This ratio is called the crest factor. The mean value should be removed form the peak amplitude to make this calculation.

The crest factor of a pure sine wave is $\sqrt{2}$. The standard deviation of a sine wave is thus relatively close to its peak value.

Random vibration does not have a well-defined crest factor. A value of 3 is often assumed, however.

Kurtosis

Kurtosis is a measurement of the “peakedness" of the data. A sine wave, for example, has a kurtosis value of 1.5. A pure Gaussian random signal has a kurtosis value of 3.0. A higher kurtosis value may indicate the presence of higher sigma peaks on the time history than would be expected from a Gaussian distribution. These peaks could, for example, represent a transient event. On the other hand, they may simply represent random vibration which has a probability density function which departs from the Gaussian ideal.

The kurtosis equation is:

$$\text{kurtosis} = \frac{\sum (x - \overline{X})^4}{n\sigma^4} \tag{20}$$

Note that the kurtosis value appears dimensionless.

Histogram

Consider the sine function in Figure 1. The sine function is a digital signal with 4000 points. The amplitude is 10 units. Now divide the amplitude into bands, each representing a range of 1. Count how many points fall into each band. The result is a histogram, as shown in Figure 2.

Note that the histogram has a "bathtub" shape. The amplitude tends to remain either at the positive or negative peak. Specifically, the probability is 28% that the signal amplitude has an absolute value greater than 9 units.

A related function is the probability density function. This is obtained by dividing each histogram amplitude by the number of total points.

Amplitude Specification

There are several methods for specifying the amplitude of a sine oscillation. Assume that the oscillation begins at zero and reaches a maximum value of A.

Table 1.
Amplitude Parameter for Sine Vibration

Amplitude Parameter	Formula
Zero-to-Peak	A
Peak-to-Peak	2A
RMS	0.707 A

Care should be taken to make sure the proper format is understood. For example, household electricity in the U.S. is 110 V AC. This is actually the RMS value.

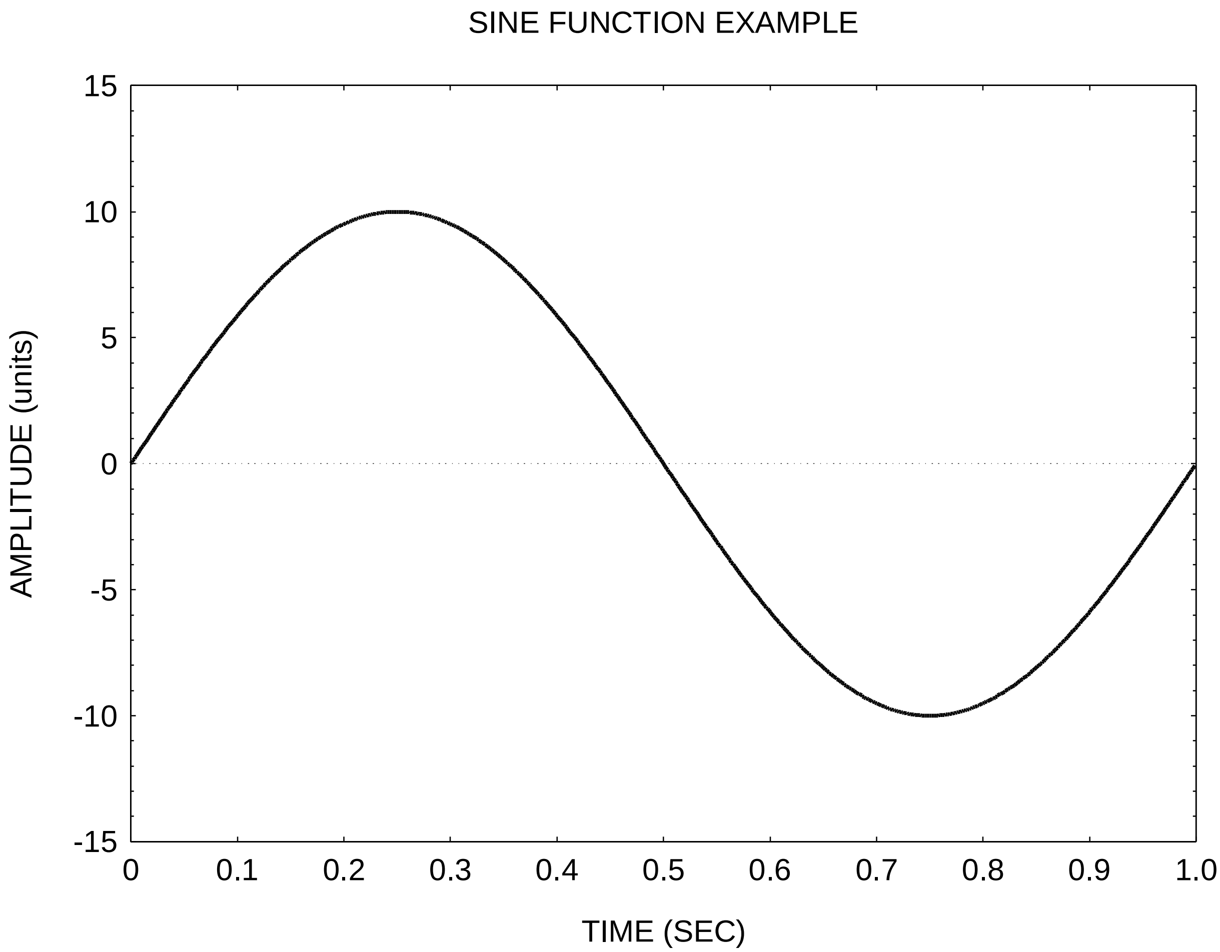
SINE FUNCTION EXAMPLE
15
10
5
0
-5
-10
-15
AMPLITUDE (units)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
TIME (SEC)

Figure 1.

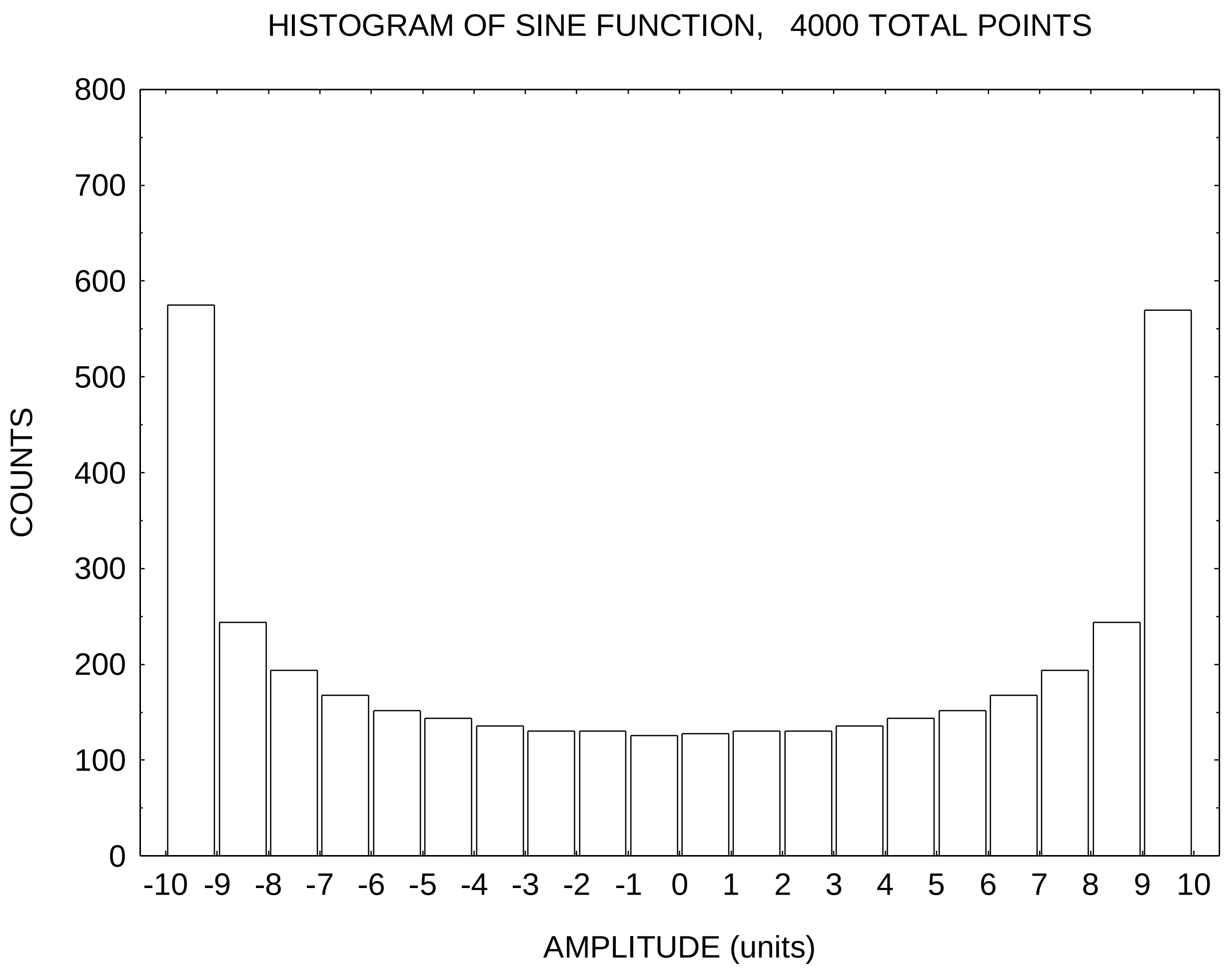
HISTOGRAM OF SINE FUNCTION,   4000 TOTAL POINTS
COUNTS
800
700
600
500
400
300
200
100
0
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
AMPLITUDE (units)

Figure 2.

Homework

1. Calculate the RMS value for a constant amplitude signal x(t) = A.

2. Calculate the standard deviation for the signal in problem 1.

3. A signal consists of two components:

$$x(t) = A + B\sin\left[\frac{2\pi t}{T}\right]$$

what is the standard deviation?

4. Plot the data file sine.txt. This is measured data taken on a floor adjacent to a wafer polishing machine in a semiconductor facility. It is a velocity time history with amplitude in units of (in/sec). It is not a pure sine signal, but measured data never really is. What is the dominant frequency?

5. The program maxfind.exe generates descriptive statistics for a time history with two columns: time(sec) and amplitude(units). Copy this file to same directory as the sine.txt file. Then type:

   maxfind sine.txt

   What is the average, standard deviation, RMS, and kurtosis values of this signal?

Note that the maxfind.exe program assumes amplitude units of G, but this is irrelevant. Substitute in/sec for G in your notes.

6. Pogo is a type of sinusoidal vibration that occurs in certain rocket vehicles due to combustion instability. Read tutorial pogo.pdf to learn more about this effect. This tutorial is not concerned with descriptive statistics per se, but it gives a real-world example of sinusoidal vibration.