SHOCK AND VIBRATION RESPONSE SPECTRA COURSE
Unit 6A. The Fourier Transform

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Introduction

Stationary vibration signals can be placed along a continuum in terms of the their qualitative characteristics.

A pure sine oscillation is at one end of the continuum. A form of broadband random vibration called white noise is at the other end.

Reasonable examples of each extreme occur in the physical world. Most signals, however, are somewhere in the middle of the continuum. An example is shown in Figure 1.

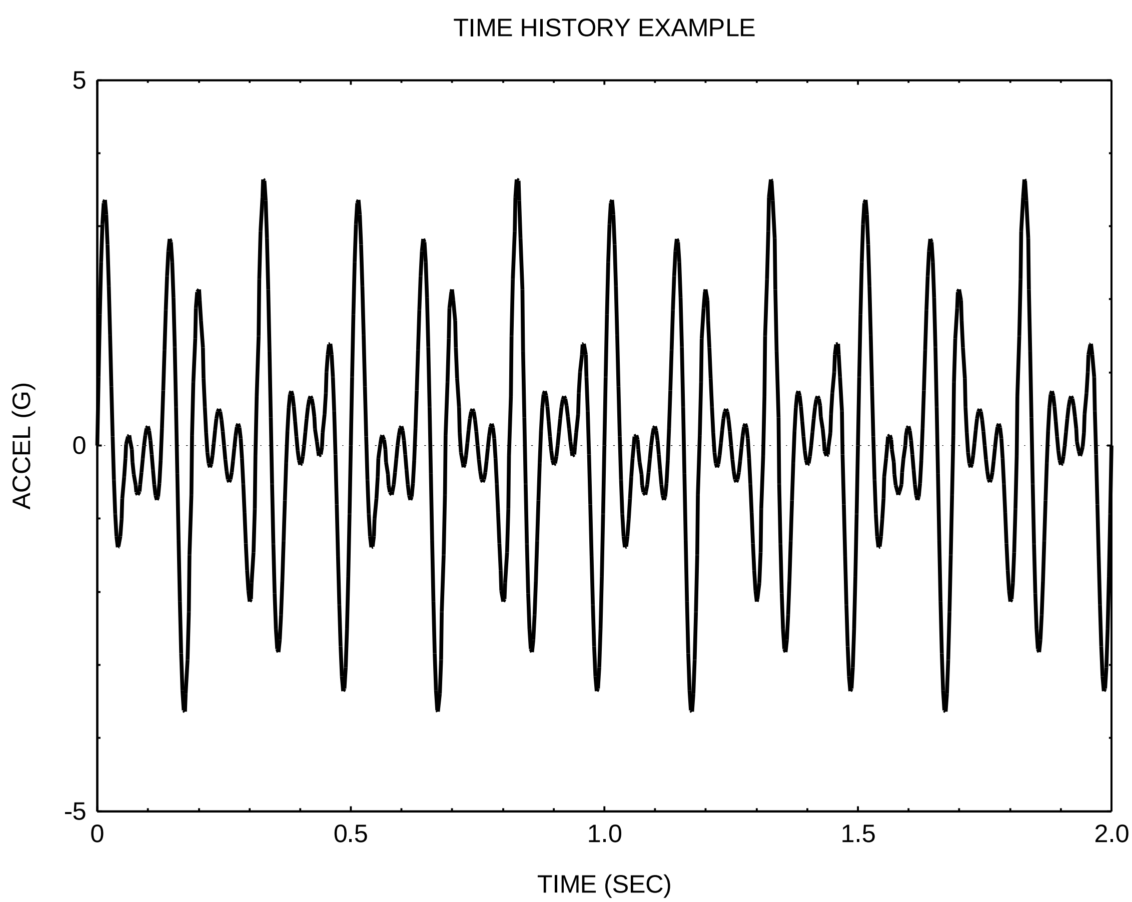

Figure 1.

The time history in Figure 1 appears to be the sum of several sine functions. What are the frequencies and amplitudes of the components? Resolving this question is the goal of this Unit.

At the risk of short-circuiting the process, the equation of the signal in Figure 1 is

$$y(t) = 1.0 \sin[2\pi(10)t] + 1.5 \sin[2\pi(16)t] + 1.2 \sin[2\pi(22)t] \tag{1}$$

The signal thus consists of three components with frequencies of 10, 16, and 22 Hz, respectively. The respective amplitudes are 1.0, 1.5, and 1.2 G.

In addition, each component could have had a phase angle. In this example, the phase angle was zero for each component.

Thus, we seek some sort of "spectral function" to display the frequency and amplitude data. Ideally, the spectral function would have the form shown in Figure 2.

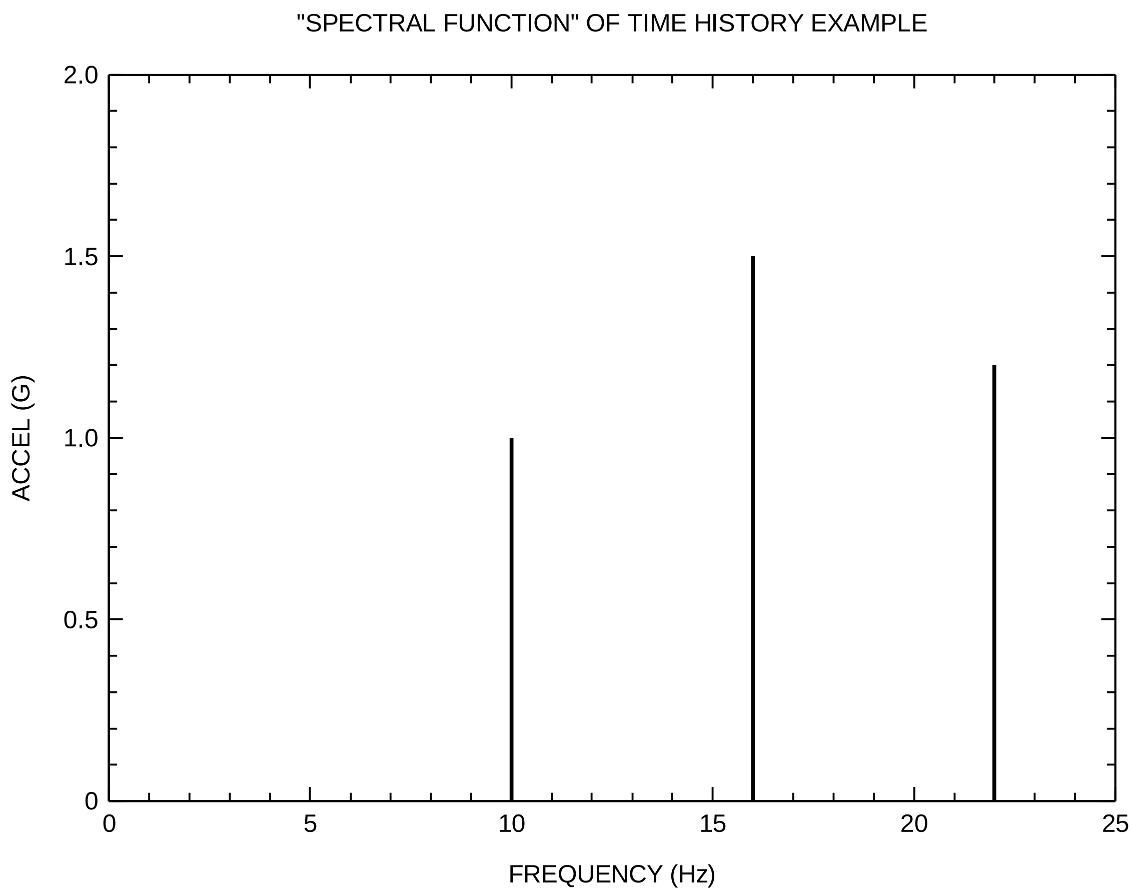

Figure 2.

Some engineers would claim that Figure 2 is the Fourier transform of the signal in equation (1). In some sense, this is true.

The Fourier transform, however, is….well…a methodology. Furthermore, it is a methodology which can be applied via many different forms and formulas. Here, the author is tempted to draw some analogy to philosophy or religion.

Mathematicians and engineers make such different use of Fourier transforms that a mathematician would likely be unable to comprehend an engineer's application and vice versa.

This Unit will attempt to bridge the gap. A method will eventually be derived to transform the time history in Figure 1 to the frequency domain "spectral function" in Figure 2. The desired "spectral function" will be shown to be based on the Fourier transform. Nevertheless, the phrase which most aptly describes this process is "some assembly required."

Continuous Fourier Transform

The Fourier transform is a method for representing a time history signal in terms of a frequency domain function.

The Fourier transform is a complex exponential transform which is related to the Laplace transform.

The Fourier transform is also referred to as a trigonometric transformation since the complex exponential function can be represented in terms of trigonometric functions. Specifically,

$$\exp[j\omega t] = \cos(\omega t) + j\sin(\omega t) \qquad (2a)$$

$$\exp[-j\omega t] = \cos(\omega t) - j\sin(\omega t) \qquad (2b)$$

where $j = \sqrt{-1}$

The Fourier transform X(f) for a continuous time series x(t) is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp[-j2\pi f t]dt \qquad (3)$$

where $-\infty < f < \infty$

Thus, the Fourier transform is continuous over an infinite frequency range.

The inverse transform is

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp[+j2\pi f t]df \tag{4}$$

Equations (3) and (4) are taken from Reference 1. Note that X(f) has dimensions of [amplitude-time].

Also note that X(f) is a complex function. It may be represented in terms of real and imaginary components, or in terms of magnitude and phase.

The conversion to magnitude and phase is made as follows for a complex variable V.

$$V = a + jb \tag{5}$$

$$\text{Magnitude } V = \sqrt{a^2 + b^2} \tag{6}$$

$$\text{Phase } V = \arctan(b / a) \tag{7}$$

Note that the inverse Fourier transform in equation (4) calculates the original time history in a complex form. The inverse Fourier transform will be entirely real if the original time history was real, however.

<u>Continuous Example</u>

Consider a sine function

$$x(t) = A\sin\left[2\pi \hat{f}\, t\right] \tag{8}$$

where

$$-\infty < t < \infty$$

The Fourier transform of the sine function is

$$X(f) = \left\{\frac{jA}{2}\right\}\left\{-\delta\left(f - \hat{f}\right) + \delta\left(-f - \hat{f}\right)\right\} \tag{9}$$

where δ is the Dirac delta function.

Note that

$$\delta\left(f - \hat{f}\right) = 0 \quad \text{for } f \neq \hat{f} \tag{10}$$

And

$$\int_{-\infty}^{\infty} \delta\left(f - \hat{f}\right)dt = 1 \tag{11}$$

The derivation is given in Appendix A. The Fourier transform is plotted in Figure 3.

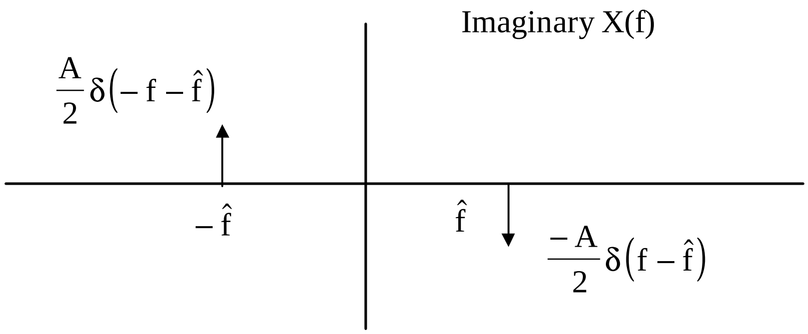

Figure 3. Fourier Transform of a Sine Function

The transform of a sine function is purely imaginary. The real component, which is zero, is not plotted.

On the other hand, the Fourier transform of a cosine function is

$$X(f)=\left\{\frac{A}{2}\right\}\left\{\delta\left(f-\hat{f}\right)+\delta\left(-f-\hat{f}\right)\right\} \tag{12}$$

The Fourier transform is plotted in Figure 4.

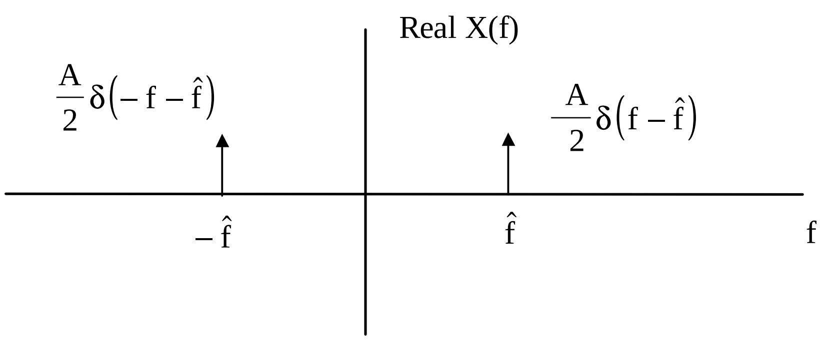

Figure 4. Fourier Transform of a Cosine Function

The transform of a cosine function is purely real. The imaginary component, which is zero, is not plotted.

Characteristics of the Continuous Fourier Transform

The plots in Figures 1 and 2 demonstrate two characteristics of the Fourier transforms of real time history functions:

1. The real Fourier transform is symmetric about the f = 0 line.
2. The imaginary Fourier transform is antisymmetric about the f = 0 line.

As an aside, the Dirac delta function is purely delightful from a mathematics point of view. Some mathematicians even promote it from a lowly function to a "distribution."

The Dirac delta distribution is of little or no use to the engineer in the test lab, however. A different approach is needed for engineers.

Discrete Fourier Transform

An accelerometer returns an analog signal. The analog signal could be displayed in a continuous form on a traditional oscilloscope.

Current practice, however, is to digitize the signal, which allows for post-processing on a digital computer. Thus, the Fourier transform equation must be modified to accommodate digital data. This is essentially the dividing line between mathematicians and engineers in regard Fourier transformation methodology. Nevertheless, further assembly is required to meet the engineering goal, which is still the "spectral function" in Figure 2.

The discrete Fourier Transform $\hat{F}_k$ for a digital time series x_n is

$$\hat{F}_k = \Delta t \sum_{n=0}^{N-1} \left\{ x_n \exp\left(-j \frac{2\pi}{N} nk \right) \right\}, \quad \text{for } k = 0, 1, \ldots, N-1 \tag{13}$$

where

N is the number of time domain samples,
n is the time domain sample index,
k is the frequency domain index,
Δt is the time step between adjacent points.

Note that $\hat{F}_k$ has dimensions of [amplitude-time].

The corresponding inverse transform is

$$x_n = \Delta f \sum_{n=0}^{N-1} \left\{ \hat{F}_k \exp\left(+j \frac{2\pi}{N} nk \right) \right\}, \text{ for } n = 0, 1, \ldots, N-1 \tag{14}$$

Note that the frequency increment Δf is equal to the time domain period T as follows

$$\Delta f = \frac{1}{T} \tag{15}$$

The frequency is obtained from the index parameter k as follows

$$\text{frequency (k)} = k\Delta f \tag{16}$$

The discrete Fourier transform in equation (13) requires further modification to meet the engineering goal set forth in Figure 2.

The following equation set is taken from Reference 2. As an alternate form, the Fourier transform F_k for a discrete time series x_n can be expressed as

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ x_n \exp\left(-j \frac{2\pi}{N} nk \right) \right\}, \text{ for } k = 0, 1, \ldots, N-1 \tag{17}$$

The corresponding inverse transform is

$$x_n = \sum_{k=0}^{N-1} \left\{ F_k \exp\left(+j \frac{2\pi}{N} nk \right) \right\}, \text{ for } n = 0, 1, \ldots, N-1 \tag{18}$$

Note that F_k has dimensions of [amplitude]. Thus, an important milestone is reached.

Discrete Example

The discrete Fourier transform of a sine wave is given in Figure 5.

A characteristic of the discrete Fourier transform is that the frequency domain is taken from 0 to (N-1)Δf. The line of symmetry is at a frequency of

$$\left[\frac{N-1}{2} \right] \Delta f \tag{19}$$

Nyquist Frequency

Note that the line of symmetry in Figure 5 marks the Nyquist frequency. The Nyquist frequency is equal to one-half of the sampling rate. Shannon's sampling theorem states that a sampled time signal must not contain components at frequencies above half the Nyquist frequency, from Reference 3.

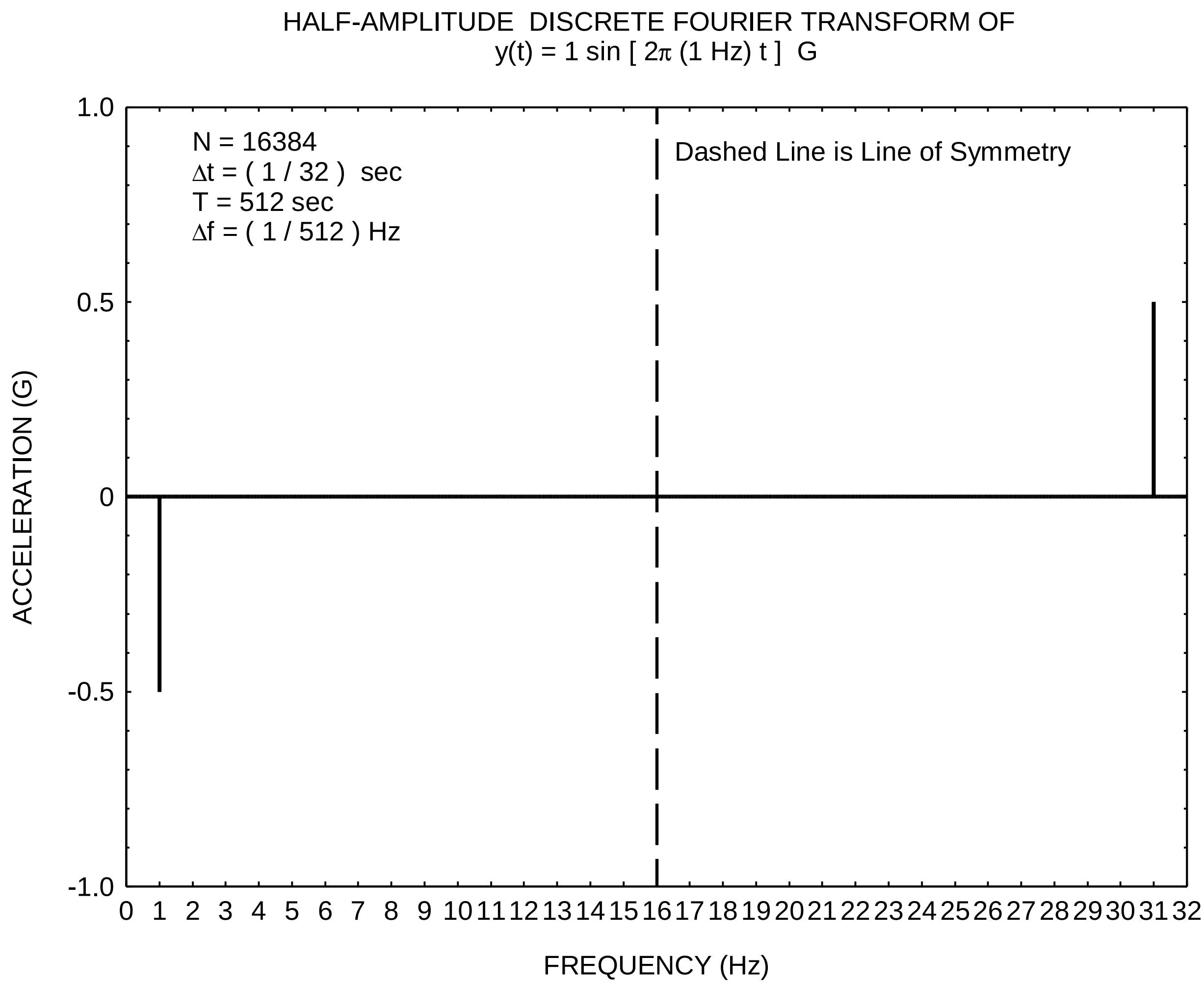

Figure 5. Fourier Transform of a Sine Wave

Note that the sine wave has a frequency of 1 Hz. The total number of cycles is 512, with a resulting period of 512 seconds. Again, the Fourier transform of a sine wave is imaginary and antisymmetric. The real component, which is zero, is not plotted.

<u>Spectrum Analyzer Approach</u>

Spectrum analyzer devices typically represent the Fourier transform in terms of magnitude and phase rather than real and imaginary components. Furthermore, spectrum analyzers typically only show one-half the total frequency band due to the symmetry relationship.

The spectrum analyzer amplitude may either represent the *half-amplitude* or *the full-amplitude* of the spectral components. Care must be taken to understand the particular convention of the spectrum analyzer. Note that the half-amplitude convention has been represented in the equations thus far, particularly equations (14) and (17).

The full-amplitude Fourier transform magnitude G_k would be calculated as

$$G_k = \begin{cases} \text{magnitude}\left\{ \left[\dfrac{1}{N}\right] \displaystyle\sum_{n=0}^{N-1} \{x_n\} \right\} \text{ for } k = 0 \\ 2 \text{ magnitude} \left\{ \left[\dfrac{2}{N}\right] \displaystyle\sum_{n=0}^{N-1} \left\{ x_n \exp\left(-j\dfrac{2\pi}{N} nk\right) \right\} \right\} \text{ for } k = 1, \ldots, \dfrac{N}{2} - 1 \end{cases}$$

with N as an even integer.

(20)

Note that k = 0 is a special case. The Fourier transform at this frequency is already at full-amplitude.

For example, a sine wave with an amplitude of 1 G and a frequency of 1 Hz would simply have a full-amplitude Fourier magnitude of 1 G at 1 Hz, as shown in Figure 6.

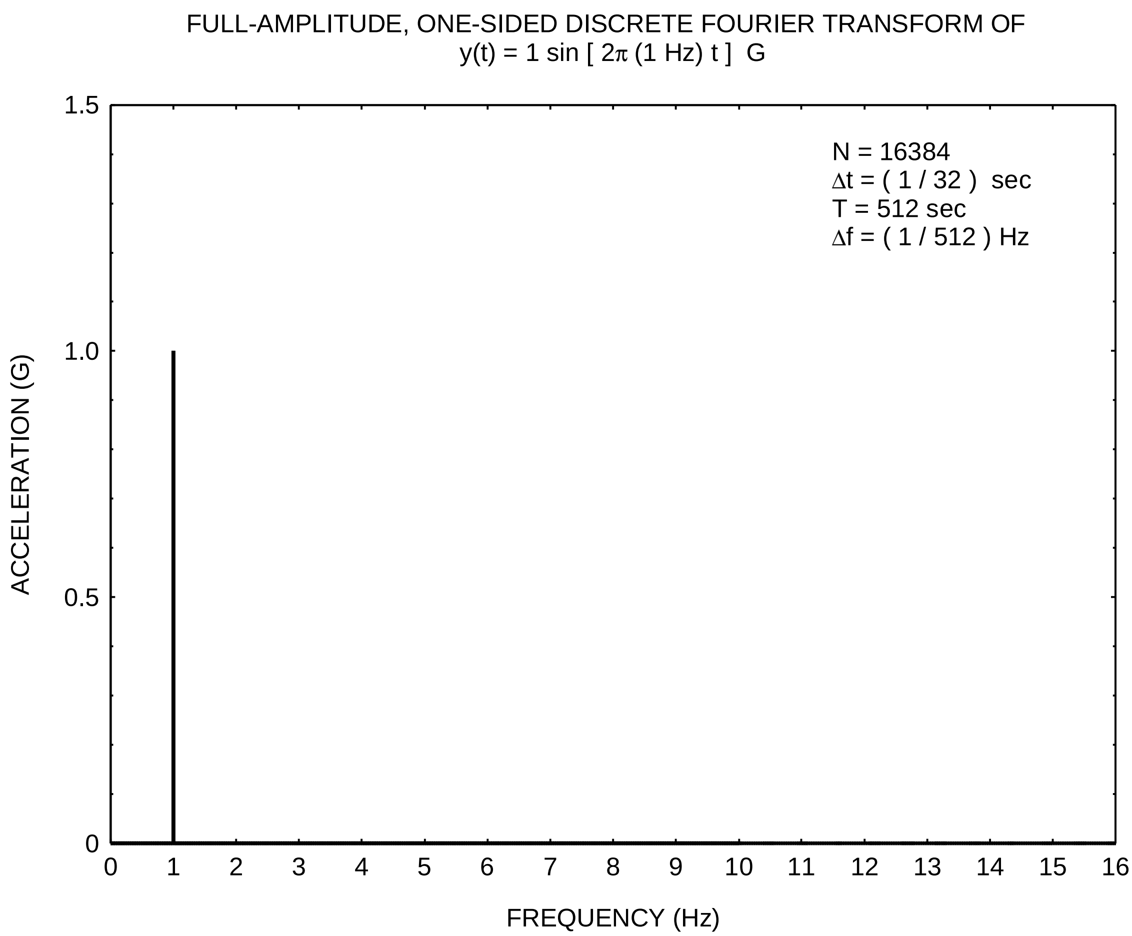

Figure 6.

Goal

The sine function considered in Figures 5 and 6 had a long duration of 512 seconds. The time history in Figure 1 has a duration of only 2 seconds, however. Note that the Fourier transform frequency resolution is the inverse of the duration, as given in equation (15). The frequency resolution is thus 0.5 Hz for a duration of 2 seconds.

The full-amplitude Fourier transform of the time history in Figure 1 is given in Figure 7. The "spectral function" goal is thus reasonably met, at least for this example. The course frequency resolution, however, gives the spectral lines a peak shape.

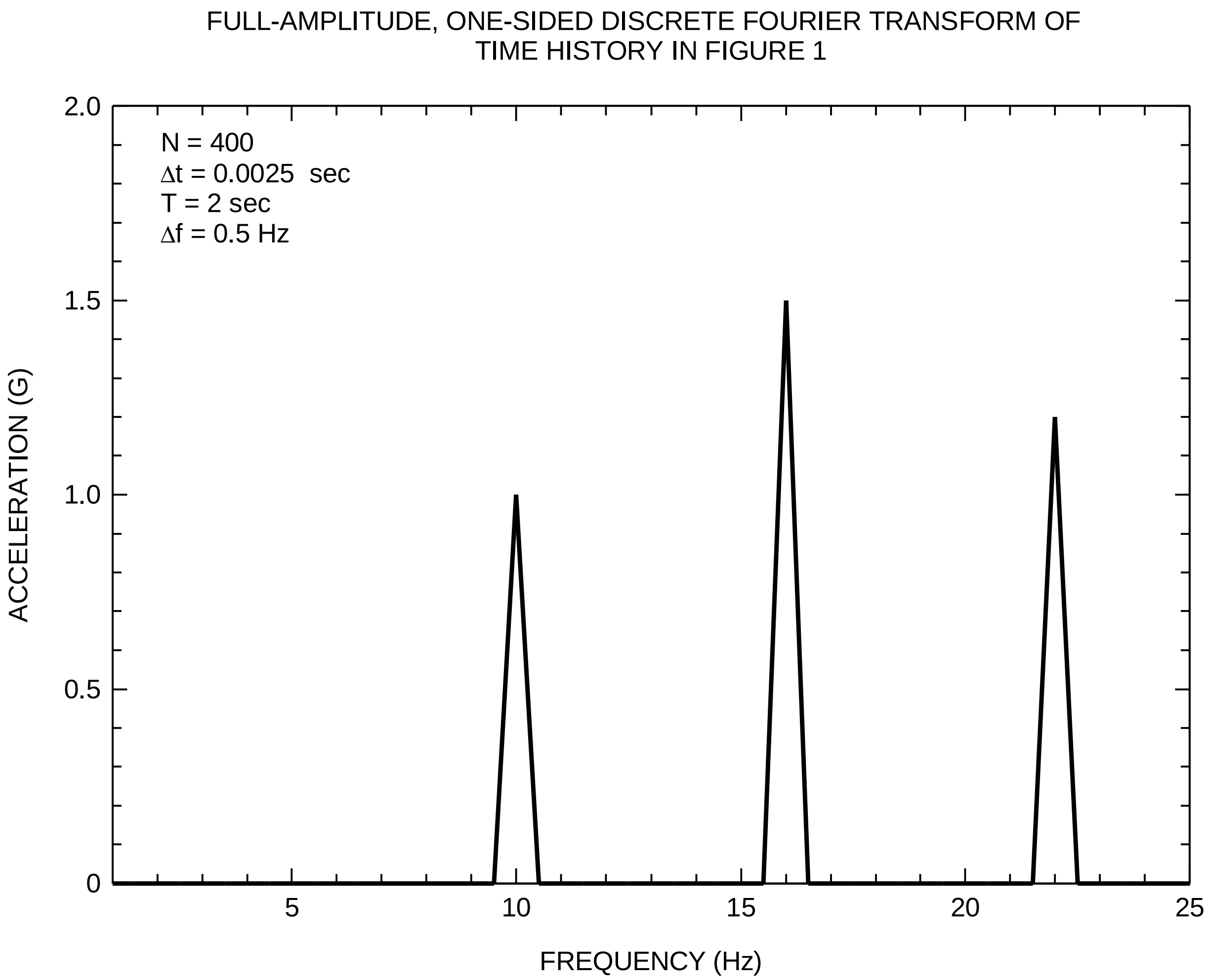

Figure 7.

The 10, 16, and 22 Hz sinusoidal frequencies are thus clearly apparent in Figure 7. The corresponding amplitudes are also correct per equation (1).

Note that this example is somewhat idealistic. The Fourier transform data in Figure 7 is defined at each 0.5 Hz frequency increment, beginning at 0. Thus, three of the spectral lines occur exactly at 10, 16, and 22 Hz.

What if the 10 Hz component in equation (1) were shifted to 9.75 Hz? The answer is that some of the energy would be shifted to 9.5 Hz and some to 10.0 Hz in the Fourier transform. This effect is one of several error sources in the Fourier transform. This error can be avoided by taking a longer duration.

Other error sources will be discussed in upcoming units. At length, the Fourier transform will be shown to be a marginal "spectral function" approach even using the full-amplitude equation (20). Nevertheless, more suitable tools can be built from the Fourier transform, as shown in upcoming units. Thus, "further assembly required."

Preview of Unit 6B

The discrete Fourier transform requires a tremendous amount of calculations. A Fast Fourier transform should be used if the number of time history samples is greater than 5000. The Fast Fourier transform is covered in Unit 6B.

Homework

1. Convert the following complex number into magnitude and phase: x = 5 + j 9

2. A time history has a duration of 20 seconds. What is the frequency resolution of the Fourier transform?

3. Recall file white.out from Unit 5. Take the Fourier transform using program fourier.exe. Plot both the half.out and full.out files. How does the amplitude vary with frequency?

4. Plot the half-sine time history hs.txt. Then take the Fourier transform. How does the amplitude vary with frequency?

References

1. W. Thomson, Theory of Vibration with Applications, 2nd Ed, Prentice-Hall, 1981.
2. GenRad TSL25 Time Series Language for 2500-Series Systems, Santa Clara, California, 1981.
3. R. Randall, Frequency Analysis 3rd edition, Bruel & Kjaer, 1987.
4. F. Harris, Trigonometric Transforms, Scientific-Atlanta, Technical Publication DSP-005, San Diego, CA.
5. T. Irvine, Statistical Degrees of Freedom, Vibrationdata Publications, 1998.

APPENDIX A

Consider a sine wave

$$x(t) = A \sin\left[2\pi \hat{f}\, t\right] \tag{A-1}$$

where

$-\infty < t < \infty$

The Fourier transform is calculated indirectly, by considering the inverse transform. Note that the sine wave is a special case in this regard.

Recall

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp\left[+j2\pi f t\right] df \tag{A-2}$$

Thus

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} X(f) \exp\left[+ j\, 2\pi f t\right] df \tag{A-3}$$

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} X(f) \left\{\cos\left[2\pi f t\right] + j \sin\left[2\pi f t\right]\right\} df \tag{A-4}$$

Let

$$X(f) = P(f) + j\, Q(f) \tag{A-5}$$

where

P(f) and Q(f) are both real coefficients

and

$-\infty < f < \infty$.

$$A \sin\left[2\pi \hat{f}\, t\right] = \int_{-\infty}^{\infty} \left\{P(f) + j\, Q(f)\right\} \left\{\cos\left[2\pi f t\right] + j \sin\left[2\pi f t\right]\right\} df \tag{A-6}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=\int_{-\infty}^{\infty}\left\{P(f)\cos\left[2\pi f\,t\right]-Q(f)\sin\left[2\pi f\,t\right]\right\}df$$
$$+\,j\int_{-\infty}^{\infty}\left\{P(f)\sin\left[2\pi f\,t\right]+Q(f)\cos\left[2\pi f\,t\right]\right\}df \qquad \text{(A-7)}$$

Equation (A-7) can be broken into two parts

$$A\sin\left[2\pi\hat{f}\,t\right]=\int_{-\infty}^{\infty}\left\{P(f)\cos\left[2\pi f\,t\right]-Q(f)\sin\left[2\pi f\,t\right]\right\}df \qquad \text{(A-8)}$$

$$0=j\int_{-\infty}^{\infty}\left\{P(f)\sin\left[2\pi f\,t\right]+Q(f)\cos\left[2\pi f\,t\right]\right\}df \qquad \text{(A-9)}$$

Consider equation (A-8)

$$A\sin\left[2\pi\hat{f}\,t\right]=\int_{-\infty}^{\infty}\left\{P(f)\cos\left[2\pi f\,t\right]-Q(f)\sin\left[2\pi f\,t\right]\right\}df \qquad \text{(A-10)}$$

Now assume

$$P(f)=0 \qquad \text{(A-11)}$$

With this assumption,

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}Q(f)\sin\left[2\pi f\,t\right]df \qquad \text{(A-12)}$$

Now let

$$Q(f)=q_1(f)+q_2(f) \qquad \text{(A-13)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)+q_2(f)\right]\sin\left[2\pi f\,t\right]df \qquad \text{(A-14)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)\right]\sin\left[2\pi f\,t\right]dt-\int_{-\infty}^{\infty}\left[q_2(f)\right]\sin\left[2\pi f\,t\right]df \qquad \text{(A-15)}$$

$$A\sin\left[2\pi\hat{f}\,t\right]=-\int_{-\infty}^{\infty}\left[q_1(f)\right]\sin\left[2\pi f\,t\right]dt+\int_{-\infty}^{\infty}\left[q_2(f)\right]\sin\left[-2\pi f\,t\right]df \qquad \text{(A-16)}$$

Equation (A-14) is satisfied by the pair of equations

$$q_1(f) = -\frac{A}{2}\,\delta\left(f - \hat{f}\right) \tag{A-17}$$

$$q_2(f) = \frac{A}{2}\,\delta\left(-f - \hat{f}\right) \tag{A-18}$$

where δ is the Dirac delta function.

By substitution,

$$Q(f) = \frac{-A}{2}\,\delta\left(f - \hat{f}\right) + \frac{A}{2}\,\delta\left(-f - \hat{f}\right) \tag{A-19}$$

Verification must be made that equation (A-9) is satisfied. Recall

$$0 = j\int_{-\infty}^{\infty}\left\{P(f)\sin\left[2\pi f t\right] + Q(f)\cos\left[2\pi f t\right]\right\}df \tag{A-20}$$

$$0 \overset{?}{=} j\int_{-\infty}^{\infty}\left\{0\sin\left[2\pi f t\right] + \left\{\frac{-A}{2}\,\delta\left(f - \hat{f}\right) + \frac{A}{2}\,\delta\left(-f - \hat{f}\right)\right\}\cos\left[2\pi f t\right]\right\}df \tag{A-21}$$

$$0 \overset{?}{=} j\left\{\frac{-A}{2}\cos\left[2\pi\hat{f}\ t\right] + \frac{A}{2}\cos\left[-2\pi\hat{f}\ t\right]\right\} \tag{A-22}$$

$$0 \overset{?}{=} j\left\{\frac{-A}{2}\cos\left[2\pi\hat{f}\ t\right] + \frac{A}{2}\cos\left[2\pi\hat{f}\ t\right]\right\} \tag{A-23}$$

$$0 = 0 \tag{A-24}$$

Recall the time domain function

$$x(t) = A\sin\left[2\pi\hat{f}\,t\right] \tag{A-25}$$

where

$$-\infty < t < \infty$$

The Fourier transform is thus

$$X(f) = \frac{-jA}{2}\,\delta\left(f - \hat{f}\right) + \frac{jA}{2}\,\delta\left(-f - \hat{f}\right) \tag{A-26}$$

$$X(f) = \left\{\frac{jA}{2}\right\}\left\{-\delta\left(f - \hat{f}\right) + \delta\left(-f - \hat{f}\right)\right\} \tag{A-27}$$