SHOCK AND VIBRATION RESPONSE SPECTRA COURSE
Unit 7A. Power Spectral Density Function

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Introduction

A Fourier transform by itself is a poor format for representing random vibration because the Fourier magnitude depends on the number of spectral lines, as shown in previous units.

The power spectral density function, which can be calculated from a Fourier transform, overcomes this limitation. Again, some assembly is required.

Note that the power spectral density function represents the magnitude, but it discards the phase angle. The magnitude is typically represented as G^2/Hz. The G is actually GRMS.

Calculation Method

Power spectral density functions may be calculated via three methods:

1. Measuring the RMS value of the amplitude in successive frequency bands, where the signal in each band has been bandpass filtered.
2. Taking the Fourier transform of the autocorrelation function. This is the Wierner-Khintchine approach.
3. Taking the limit of the Fourier transform X(f) times its complex conjugate divided by its period T as the period approaches infinity. Symbolically, the power spectral density function $X_{PSD}(f)$ is

$$X_{PSD}(f) = \lim_{T \to \infty} \frac{X(f) X^*(f)}{T} \tag{1}$$

These methods are summarized in Reference 1. Only the third method is considered in this unit.

Fourier Transform Method

Equation (1) assumes that the Fourier transform has a dimension of [amplitude-time].

The following equations are taken from Reference 2.

The discrete Fourier transform [amplitude-time] is

$$X(k) = \Delta t \sum_{n=0}^{N-1} x(n) \exp(-j \frac{2\pi}{N} nk) \text{ for } k = 0, 1, ..., N-1 \tag{2a}$$

Note that the index k can be related to the frequency

$$\text{frequency } (k) = k \Delta f \tag{2b}$$

The inverse transform is

$$x(n) = \Delta f \sum_{k=0}^{N-1} X(k) \exp(+j \frac{2\pi}{N} nk) \text{ for } n = 0, 1, ..., N-1 \tag{3}$$

These equations give the Fourier transform values X(k) at the N discrete frequencies $k\Delta f$ and give the time series x(n) at the N discrete time points $n\ \Delta t$. The total period of the signal is thus

$$T = N\Delta t \tag{4}$$

where

N is number of samples in the time function and in the Fourier transform
T is the record length of the time function
Δt is the time sample separation

Consider a sine wave with a frequency such that one period is equal to the record length. This frequency is thus the smallest sine wave frequency which can be resolved. This frequency Δf is the inverse of the record length.

$$\Delta f = 1/T \tag{5}$$

This frequency is also the frequency increment for the Fourier transform.

Alternate Fourier Transform Method

The Fourier transform with dimension of [amplitude-time] is rather awkward.

Fortunately, the power spectral density can be calculated from a Fourier transform with dimension of [amplitude]. The corresponding formula is

$$X_{PSD}(f) = \lim_{\Delta f \to 0} \frac{F(f) F^*(f)}{\Delta f} \tag{6}$$

The Fourier transform $F(k)$ for the discrete time series $x(n)$ is

$$F(k) = \frac{1}{N}\sum_{n=0}^{N-1}\left\{x(n)\exp\left(-j\frac{2\pi}{N}nk\right)\right\}, \text{ for } k = 0, 1, \ldots, N-1 \tag{7a}$$

Note that the index k can be related to the frequency

$$\text{frequency}(k) = k\,\Delta f \tag{7b}$$

The corresponding inverse transform is

$$x(n) = \sum_{k=0}^{N-1}\left\{F(k)\exp\left(+j\frac{2\pi}{N}nk\right)\right\}, \text{ for } n = 0, 1, \ldots, N-1 \tag{8}$$

<u>One-sided Fourier Transform Approach</u>

The power spectral density functions in equations (1) and (6) were both double-sided. The power spectral density amplitude would be symmetric about the Nyquist frequency.

A one-sided, or single-sided, power spectral density function is desired.

Let $\hat{X}_{PSD}(f)$ be the one-sided power spectral density function.

$$\hat{X}_{PSD}(f) = \lim_{\Delta f \to 0} \frac{G(f)G^*(f)}{\Delta f} \tag{9}$$

The one-sided Fourier transform G(k) is

$$G(k) = \begin{cases} \text{magnitude}\left\{ \left[\frac{1}{N}\right] \sum_{n=0}^{N-1} \{x(n)\} \right\} \text{ for } k = 0 \\ 2 \text{ magnitude}\left\{ \left[\frac{1}{N}\right] \sum_{n=0}^{N-1} \left\{ x(n) \exp\left(-j \frac{2\pi}{N} nk \right) \right\} \right\} \text{ for } k = 1, \ldots, \frac{N}{2} - 1 \end{cases}$$

with

N as an even integer

Frequency (k) = $k\Delta f$

(10)

Implementation

Calculation of a power spectral density requires that the user select the Δf value from a list of options. The Δf value is linked to the number of degrees of freedom.

Statistical degrees of freedom

The reliability of the power spectral density data is proportional to the degrees of freedom.

The statistical degree of freedom parameter is defined from References 3 and 4 as follows:

$$\text{dof} = 2BT \quad (11)$$

where dof is the number of statistical degrees of freedom and B is the bandwidth of an ideal rectangular filter. This filter is equivalent to taking the time signal "as is," with no tapering applied. Note that the bandwidth B equals Δf, again assuming an ideal rectangular filter.

The 2 coefficient in equation (11) results from the fact that a single-sided power spectral density is calculated from a double-sided Fourier transform. The symmetries of the Fourier transform allow this double-sided to single-sided conversion.

For a single time history record, the period is T and the bandwidth B is the reciprocal so that the BT product is unity, which is equal to 2 statistical degrees of freedom from the definition in equation (11).

A given time history is thus worth 2 degrees of freedoms, which is poor accuracy per Chi-Square theory, as well as per experimental data per Reference 3. Note that the Chi-Square theory is discussed in Reference 5.

Breakthrough

The breakthrough is that a given time history record can be subdivided into small records, each yielding 2 degrees of freedom, as discussed in Reference 4 for example. The total degrees of freedom value is then equal to twice the number of individual records. The penalty, however, is that the frequency resolution widens as the record is subdivided. Narrow peaks could thus become smeared as the resolution is widened.

An example of this subdivision process is shown in Table 1. The process is summarized in equations (12) through (16).

Table 1. Example: 4096 samples taken over 16 seconds, rectangular filter.

Number of Records NR	Number of Time Samples per Record	Period of Each Record T_i (sec)	Frequency Resolution $B_i=1/T_i$ (Hz)	dof per Record $=2B_i\ T_I$	Total dof
1	4096	16.	0.0625	2	2
2	2048	8.	0.125	2	4
4	1024	4.	0.25	2	6
8	512	2.	0.5	2	16
16	256	1.	1.	2	32
32	128	.5	2.	2	64
64	64	.25	4.	2	128

Notes:

1. The subscript "i" is used to denote "individual" in Table 1.
2. The rows in the table could be continued until a single sample per record remained.

Also note that:

$$\text{Total dof} = 2\ NR \quad (12)$$

$$NR = T / T_i \quad (13)$$

$$B_i = 1 / T_i \quad (14)$$

$$NR = B_i\ T \quad (15)$$

$$\text{Total dof} = 2\ B_i\ T \quad (16)$$

Window

A window is typically applied to each time segment during the power spectral density calculation, as discussed in References 3, 4, and 6. The purpose of the window is to reduce a type of error called leakage. One of the most common windows is the Hanning window, or the cosine squared window. This window tapers the data so that the amplitude envelope decreases to zero at both the beginning and end of the time segment. The Hanning window w(t) can be defined as

$$w(t) = \begin{cases} 1 - \cos^2\left[\pi \frac{t}{T}\right], & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases} \tag{21}$$

The window operation reduces the leakage error but also has the effect of reducing the statistical degrees-of-freedom.

Also, a normalization factor of $\sqrt{8/3}$ is applied to the Hanned data to compensate for the lost energy, from Reference 7.

Overlap

The lost degrees-of-freedom can be recovered by overlapping the time segments, each of which is subjected to a Hanning window. Nearly 90% of the degrees-of-freedom are recovered with a 50% overlap, according to Reference 3.

The concept of windows and overlapping is represented in Figure 1.

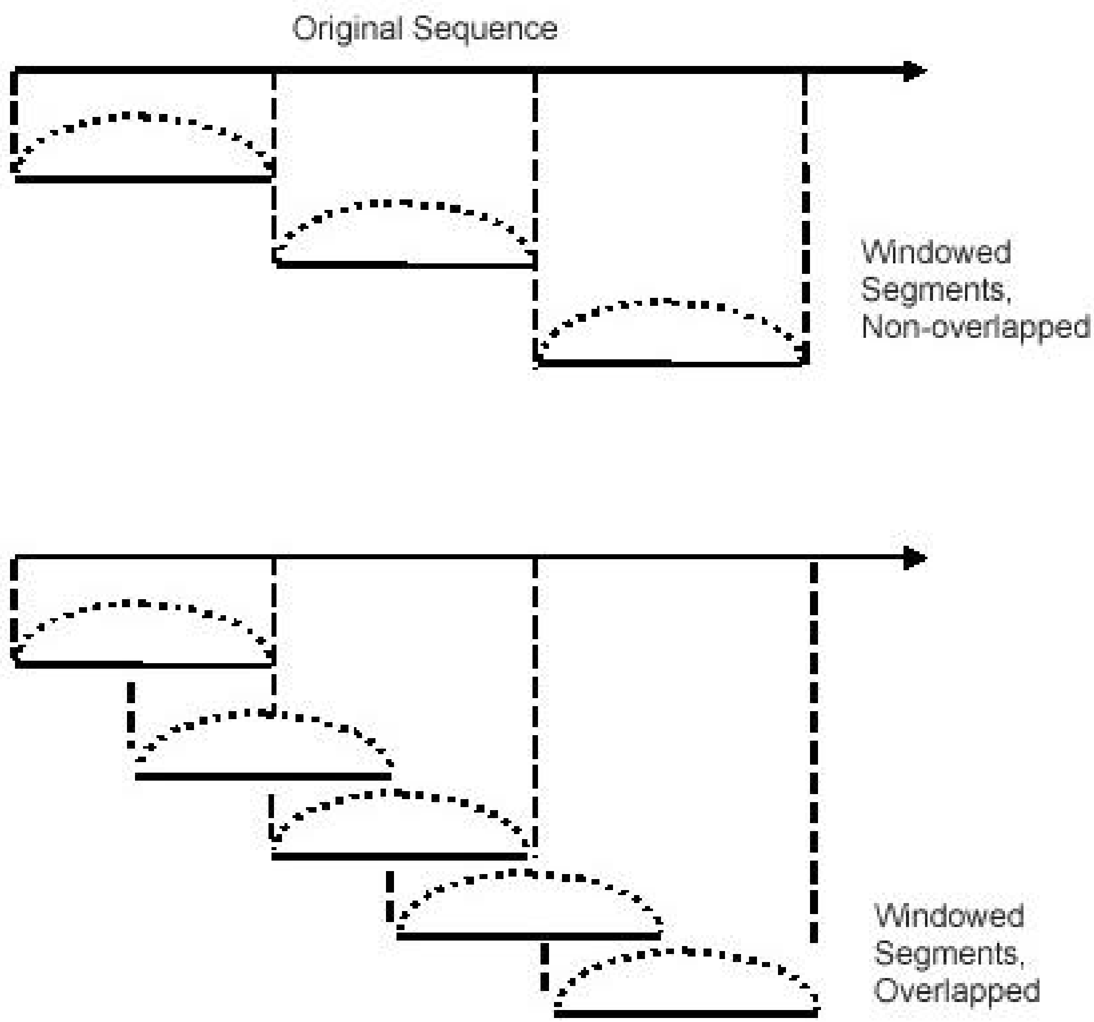
Original Sequence
Windowed
Segments,
Non-overlapped
Windowed
Segments,
Overlapped

Figure 1.

Fast Fourier Transform

Three variations of the discrete Fourier transform have been given in this report. The solution to any of these transforms requires a great deal of processing steps for a given time history. Fast Fourier transform methods have been developed, however, to greatly reduce the required steps. These methods typically require that the number of time history data points be equal to 2^N, where N is some integer. The derivation method is via a butterfly algorithm, as shown, for example, in Reference 8.

Records with sample numbers which are not equal to an integer power of 2 can still be processed via the fast Fourier transform method. Such a record must either be truncated or padded with zeroes so that its length becomes an integer power of 2.

Summary

Time history data is subdivided into segments to increase the statistical-degrees-of-freedom by broadening the frequency bandwidth. Next, a window is applied to each segment to taper the ends of the data. Finally, overlapping is used to recover degrees-of-freedom lost during the window operations. The effect of these steps is to increase the accuracy of the power spectral density data. Nevertheless, there are some tradeoffs as shown in the following examples.

Homework

1. Use program generate.exe to synthesize a white noise time history with 1 G standard deviation, 10 second duration, and 1000 samples per second.

2. Use program poweri.exe to calculate the power spectral density. Choose 256 samples per second, which corresponds to 78 dof and $\Delta f = 3.9$ Hz. Select the mean removal and Hanning window options. Plot the output file a.out, preferably in log-log format.

3. Repeat step 3 for 128 samples per second, which corresponds to 156 dof and $\Delta f = 7.8$ Hz.

4. Compare the power spectral density curves from steps 2 and 3. Do the curves have a similar or different amplitude?

References

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