# MODERN STRUCTURAL MECHANICS for UNDERGRADUATE STUDENTS

J.L. Meek
Civil Engineering Department
University of Queensland
St. Lucia 4072
Queensland
Australia

February 2004

# Contents

**1 BASIC CONCEPTS** **9**
1.1 Introduction and scope . . . . . . 9
1.2 Basic transformations . . . . . . 11
1.2.1 Force and displacement vectors-contragredience . . . . . . 11
1.2.2 Rotation of coordinate axes . . . . . . 12
1.2.3 Transfer of force components and moment in $X - Y$ plane . . . . . . 13
1.3 Contragredient principle . . . . . . 14
1.3.1 Contragredient transformations . . . . . . 14
1.3.2 Joint equilibrium equations . . . . . . 15
1.3.3 Node deflections . . . . . . 16
1.4 Units of length, mass and force . . . . . . 17
1.5 Concepts of structure . . . . . . 17
1.6 Strength of materials . . . . . . 18
1.7 Teaching module-basic principles . . . . . . 18
1.7.1 Exercises in units . . . . . . 19
1.7.2 Exercises in transformations . . . . . . 20

**2 TRUSS ANALYSIS** **23**
2.1 Introduction . . . . . . 23
2.2 Theory for planar truss analysis . . . . . . 24
2.2.1 Member forces, reactions and equilibrium equations . . . . . . 24
2.2.2 Reactions . . . . . . 27
2.2.3 Determinate trusses-solution of the equilibrium equations . . . . . . 29
2.2.4 Plot commands for viewing truss geometry . . . . . . 31
2.2.5 Calculation of truss deflections . . . . . . 32
2.2.6 Displacements of supports . . . . . . 33
2.2.7 Statically indeterminate planar trusses . . . . . . 34
2.2.8 Example of statically indeterminate truss. . . . . . . 36
2.3 Truss study module . . . . . . 37
2.3.1 Study programme in truss analysis . . . . . . 37
2.3.2 Statically determinate truss analysis exercises . . . . . . 38
2.3.3 Exercises for determinate truss analysis . . . . . . 39

2.3.4 Influence lines . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 41
2.3.5 Calculation of truss node deflections . . . . . . . . . . . . . . . . . . 42
2.3.6 Truss exercises for deflection calculations . . . . . . . . . . . . . . . 42
2.4 Statically indeterminate truss analysis . . . . . . . . . . . . . . . . . . . . 43
2.5 Exercises in indeterminate truss analysis . . . . . . . . . . . . . . . . . . . 44

**3 BEAM ANALYSIS 49**
3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 49
3.2 Rigid body reactions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 49
3.2.1 Command sequence . . . . . . . . . . . . . . . . . . . . . . . . . . . 49
3.2.2 Transformation theory . . . . . . . . . . . . . . . . . . . . . . . . . . 50
3.2.3 Retaining wall reactions-foundation pressure . . . . . . . . . . . . . 52
3.3 Analysis of beams for bending moment and shear force . . . . . . . . . . . 54
3.3.1 Theory for beam elements . . . . . . . . . . . . . . . . . . . . . . . . 54
3.3.2 Beam reactions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 56
3.3.3 Distributed loading on beam element . . . . . . . . . . . . . . . . . . 58
3.3.4 Statically indeterminate beams . . . . . . . . . . . . . . . . . . . . . 60
3.3.5 Commands available for indeterminate beam analysis . . . . . . . . 62
3.4 Calculation of deflections . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63
3.4.1 Beam deflections-theory . . . . . . . . . . . . . . . . . . . . . . . . . 63
3.4.2 Plot of beam deflections . . . . . . . . . . . . . . . . . . . . . . . . . 64
3.4.3 Example: Calculation of beam deflections in non-principal axes . . . 65
3.5 Summary of beam projects . . . . . . . . . . . . . . . . . . . . . . . . . . . 67
3.5.1 Beam bending moments, shear force and reactions . . . . . . . . . . 67
3.5.2 Indeterminate beam analysis, examples 7-10 Figure (3.16). . . . . . 67
3.6 Rigid body reaction module . . . . . . . . . . . . . . . . . . . . . . . . . . 68
3.6.1 Rigid body reaction exercises . . . . . . . . . . . . . . . . . . . . . . 69
3.7 Statically determinate beam module . . . . . . . . . . . . . . . . . . . . . . 70
3.7.1 Statically determinate beam exercises . . . . . . . . . . . . . . . . . 72
3.8 Statically indeterminate beam module . . . . . . . . . . . . . . . . . . . . . 74
3.9 Determinate beam deflections . . . . . . . . . . . . . . . . . . . . . . . . . 76
3.10 Multiple Load Cases-Section Selection . . . . . . . . . . . . . . . . . . . . 77

**4 PLANE FRAME ANALYSIS 79**
4.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 79
4.2 Determinate frame structures . . . . . . . . . . . . . . . . . . . . . . . . . 79
4.2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 79
4.2.2 Member forces − nodal equations of equilibrium . . . . . . . . . . . 80
4.3 Node and member loads . . . . . . . . . . . . . . . . . . . . . . . . . . . . 83
4.3.1 Statically determinate structure deflections . . . . . . . . . . . . . . 84
4.3.2 Statically indeterminate frames . . . . . . . . . . . . . . . . . . . . . 85
4.3.3 Nodal forces, statically indeterminate frames . . . . . . . . . . . . . 87
4.3.4 Frame analysis . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 87

4.3.5 Notes . . . 88
4.4 Frame analysis module . . . 89
4.4.1 FRAME ANALYSIS TUTORIAL EXERCISES . . . 89

**5 PLANE GRID ANALYSIS** **95**
5.1 Introduction . . . 95
5.2 Member forces– member node forces . . . 96
5.3 Node and member loads . . . 100
5.3.1 Statically indeterminate grids . . . 101
5.3.2 Nodal forces, statically indeterminate grids . . . 103
5.3.3 Grid analysis using examples 1 to 7 in software . . . 104
5.4 Grid analysis module . . . 104
5.4.1 GRID ANALYSIS TUTORIAL EXERCISES . . . 105
5.4.2 Exercises in grid analysis . . . 106

**6 SECTION PROPERTIES** **111**
6.1 Introduction . . . 111
6.2 Section properties of areas bounded by straight lines . . . 111
6.2.1 Interpolation of space within a triangle . . . 112
6.2.2 Principal axes, principal directions . . . 114
6.2.3 Calculation of axial stresses in the beam . . . 117
6.2.4 Commands for calculating section properties . . . 118
6.2.5 Section property exercises . . . 120
6.3 Shear stress in beams . . . 122
6.4 Thin walled cross section . . . 122
6.4.1 Thin section property exercises . . . 125
6.5 Shear flow in thin walled beams . . . 125
6.6 Stress tensor-rotation of axes-principal axes . . . 132
6.6.1 Two dimensional stress state . . . 132
6.6.2 Three dimensional stress state . . . 134
6.7 Example of cross section properties . . . 134
6.7.1 Symmetric I section . . . 135
6.7.2 Thick angle section . . . 136
6.7.3 Thin section angle . . . 136
6.7.4 Example: section properties of rectangle with semi-circular cut out . 137
6.7.5 Example of shear flow distribution in thin walled sections . . . 138
6.8 Section properties and stresses in beams . . . 140
6.8.1 Hollow section . . . 140
6.9 Section properties module . . . 141
6.10 Section property exercises . . . 142
6.10.1 Stress calculations . . . 143

**7 AXIAL FORCE EFFECTS BUCKLING TENSION** **145**
7.1 Introduction . . . 145
7.1.1 Single bar-spring system with axial compressive load . . . 146
7.1.2 Column buckling-Euler load . . . 147
7.1.3 Deflection of beams . . . 148
7.1.4 Calculation of beam deflection due to end moments $M_i, M_j$ . . . 149
7.1.5 Effect of axial force on end rotations . . . 150
7.1.6 Rotation of a member, change of nodal force components . . . 152
7.2 Matrix Iteration, calculation of buckling load . . . 156
7.3 Commands to calculate beam critical load . . . 158
7.4 Frame buckling . . . 160
7.5 Tension structures . . . 162
7.5.1 Introduction . . . 162
7.5.2 Analysis of a plane cable . . . 163
7.5.2.1 Cable force . . . 165
7.5.3 Elastic and geometric stiffness . . . 167
7.5.3.1 Example of geometric stiffness . . . 170
7.5.4 Three dimensional cables . . . 171
7.5.4.1 Three dimensional cable net stiffnesses . . . 172
7.5.4.2 Geometric stiffness . . . 173
7.5.5 Iterative calculation . . . 174

**8 NATURAL FREQUENCIES OF STRUCTURES** **177**
8.1 Introduction-Oscillating Systems . . . 177
8.1.1 Oscillation of a Simpe Pendulum . . . 177
8.1.2 Simple spring-lumped mass systems . . . 179
8.1.3 Beams with distributed mass . . . 183
8.2 Calculation of natural frequenciesof structures . . . 185
8.2.1 Statically indeterminate beams . . . 189
8.3 Frame structures-natural frequencies and mode shapes . . . 190
8.4 Natural frequency exercise module . . . 191

**9 COURSE I STATICS** **195**
9.1 Structural mechanics course outlines . . . 195
9.2 Computational structural analysis-Statics . . . 196
9.2.1 Lecture 9.1 Matrix-Vector Transformations . . . 196
9.2.2 Tutorial 1 . . . 197
9.2.3 Lecture 2 Equilibrium of forces . . . 199
9.2.4 Tutorial 2 . . . 199
9.2.5 Lecture 3 Truss structures . . . 200
9.2.6 Tutorial 3 . . . 201

9.2.7 Lecture 4 Beam Free Body Diagrams . . . . . . . . . . . . . . . . . . . 203
9.2.8 Tutorial 4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 204
9.2.9 Lecture 5 Bending moment-shear force . . . . . . . . . . . . . . . . . . 207
9.2.10 Lecture 6 Beam discretization . . . . . . . . . . . . . . . . . . . . . . 209
9.2.11 Tutorial 5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 211
9.2.12 Lectures 7−8 Determinate frames . . . . . . . . . . . . . . . . . . . . 212
9.2.13 Tutorial 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 213
9.2.14 Lectures 9-10 Determinate grids . . . . . . . . . . . . . . . . . . . . . 214
9.2.15 Tutorial 7 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 216

**10 COURSE II STRESS-STRAIN-DEFORMATION 219**
10.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 219
10.2 Strength of materials . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 219
10.2.1 Lecture 10.1 Stress components . . . . . . . . . . . . . . . . . . . . . 219
10.2.2 Lecture 10.2 Stress transformation . . . . . . . . . . . . . . . . . . . 221
10.2.3 Lecture 3 Maximum-minimum stress . . . . . . . . . . . . . . . . . . . 223
10.2.4 Tutorial 10.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 224
10.2.5 Lecture 10.4 Deformation . . . . . . . . . . . . . . . . . . . . . . . . . 224
10.2.6 Lecture 5 Transformation of strain . . . . . . . . . . . . . . . . . . . 227
10.2.7 Lecture 6 Constitutive relationships . . . . . . . . . . . . . . . . . . . 228
10.2.8 Lecture 7 Shear stress-strain . . . . . . . . . . . . . . . . . . . . . . . 229
10.2.9 Tutorial 10.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 230
10.2.10 Lecture 8 Concept of virtual work . . . . . . . . . . . . . . . . . . . . 231
10.2.11 Tutorial 10.3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 233
10.2.12 Lecture 9 Contragredience . . . . . . . . . . . . . . . . . . . . . . . . 234
10.2.13 Lecture 10 Section properties . . . . . . . . . . . . . . . . . . . . . . 235
10.2.14 Lecture 11 Axial stresses in beams . . . . . . . . . . . . . . . . . . . 235
10.2.15 Tutorial 10.4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 236
10.3 Lecture 12 Bending of thin walled beams . . . . . . . . . . . . . . . . . . 236
10.4 Tutorial 10.5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 237
10.5 Lecture 13 Shear stress distribution . . . . . . . . . . . . . . . . . . . . . 238
10.6 Tutorial 10.6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 238
10.7 Lecture 14 Deflection calculations . . . . . . . . . . . . . . . . . . . . . . 239
10.8 Tutorial 10.7 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 240
10.9 Lecture 15-Deflection calculations . . . . . . . . . . . . . . . . . . . . . . 240
10.10 Tutorial 10.8 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 242
10.11 Torsion-circular shafts . . . . . . . . . . . . . . . . . . . . . . . . . . . . 243

**11 COURSE III Indeterminate Structures 247**
11.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 247
11.2 Lecture 1 Indeterminate trusses . . . . . . . . . . . . . . . . . . . . . . . 247
11.3 Tutorial 11.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 248
11.4 Lectures 2-3 Indeterminate beams . . . . . . . . . . . . . . . . . . . . . . 249

11.4.1 Uniformly distributed load . . . . . 249
11.4.2 Linearly varying distributed load . . . . . 249
11.4.3 Concentrated load W . . . . . 250
11.5 Tutorial 11.2 . . . . . 251
11.6 Lectures 4-5 Indeterminate frames . . . . . 252
11.6.1 Member loads and nodal forces . . . . . 252
11.7 Tutorial 11.3 . . . . . 253
11.8 Lectures 5-6 Grid Structures . . . . . 254
11.9 Tutorial 11.4 . . . . . 255
11.10Lectures 7-8 Geometric stiffness Stability . . . . . 256
11.11Tutorial 11.5 . . . . . 257
11.12Beam deflections . . . . . 257
11.13Tutorial 11.6 . . . . . 258
11.14Lecture 10 . . . . . 259
11.15Lecture 11-rotation of member chord . . . . . 259
11.16Lecture 13 . . . . . 259
11.17Lecture 12 Eigenvalue calculation . . . . . 260
11.18Tutorial 11.7 . . . . . 261

**A** **263**
A.1 Trusses . . . . . 263
A.2 Rigid Bodies . . . . . 264
A.3 Beams . . . . . 264
A.4 Section Properties . . . . . 264
A.5 Plane Frames . . . . . 265
A.6 Grid Structures . . . . . 266
A.7 Beams-Load Summation . . . . . 266
A.8 Sundary Examples-Critical Loads . . . . . 266
A.9 Beginners Truss . . . . . 267

# Chapter 1

# BASIC CONCEPTS

## 1.1 Introduction and scope

This text presents a set of structural mechanics study modules that will prepare engineering undergraduate students for the computer environment in which they will be expected to participate, both in the later years of their course and then in their careers in the work force. It has three main objectives:

(1) To present the theory in a modern form using the most simple principles.
(2) To give the student's study programme in a calculation and error free environment.
(3) To give the student the ability to undertake a wide range of challenging projects with a simple software package.

The software package STATICS-2020 is supplied so that the students have in one program all the tools necessary to complete the course of study. The purpose of the STATICS-2020 suite of programmes is to assist the undergraduate student in understanding the principles involved in topics of modern structural mechanics in a calculaton free enviroment. Within the software is a extensive set of examples for the various topics so that the burden of data preparation is reduced as is the setting and marking of assignments by the instructor. In addition it will be found that the student can use the suite of programs in simple but nevertheless challenging design applications. A data file DATN.DAT is provided with a preprogrammed suite of exercises each of which may be accessed through a single SUBMIT command. In this first release the topics listed below have been included. Some of these topics take the student into at least a second or third year level in structural mechanics.

1. Calculation of member forces and reactions of statically <u>determinate</u> planar trusses.
2. Calculation of member forces and reactions of statically <u>indeterminate</u> planar trusses.
3. Check on the equilibrium of a node of a truss subjected to its loads, reactions and internal member forces.

4. Calculation of reactions for planar rigid bodies subjected to applied forces and moments.
5. Analysis of statically determinate beams for shear forces and bending moments.
6. Analysis of statically indeterminate beams for shear forces and bending moments.
7. Analysis of statically determinate plane frames.
8. Analysis of statically indeterminate plane frames.
9. Analysis of statically determinate plane grids.
10. Analysis of statically indeterminate plane grids.
11. Calculation of cross section properties. Area, centroid, second moment of area and all coordinates referenced to the centroidal axes. Principal axes location.
12. Calculation of properties of thin walled open sections. Area, centroid, second moment of area and all coordinates referenced to the centroidal axes. Principal axes location.
13. Calculation of stress for axial load and bending moments about non-principal axes of a cross section defined in topics (11) and (12).
14. Calculation of shear flow distributions in thin walled cross sections (12) and location of the shear centre.
15. Calculation of truss flexibility matrix for trusses and thence truss deflections.
16. Calculation of beam flexibility matrix for beams and thence beam deflections.
17. Calculation of flexibility matrix for frames and thence frame deflections.
18. Calculation of flexibility matrix for grids and thence grid deflections.
19. Calculation of buckling loads of beams and frames.
20. Calculation of natural frequencies of beams, frames and grids.

The study of indeterminate structures is included in items (2), (6), (8) and (10). This is intentional although the topics may be studied in a different order. The purpose of including topics here that were once considered advanced, is to show that only a very simple extension of the ideas used in determinate analysis is required to undertake indeterminate analysis once the student is introduced to the gontragredient law and the software is available to perform the numerical claculations involved. Again it must be emphasised that this is not a black box approach as the student must be able to understand and execute the necessary steps. Calculation of flexibility matrices and structure deflections is left until topics 15 to 18, because this will lead to more advanced study of structure vibration and buckling stability phenemona. Several commands allow the student to view simple plots of the results, member forces and deflections of the various problems.

It will be seen from the theory developed in this text that the basic ideas of structural analysis are built from the equilibrium of the forces acting either on individual members or on the nodes of an assemblage of members. The contragredient law is introduced and then used to obtain the corresponding displacement transformation matrices between related quantities. For statically indeterminate structures the stiffness method of analysis is developed to calculate firstly the global stiffness matrix and from this the member force transformation matrix. For all determinate structures contragredience is used to obtain the member distortions to node displacement transformation and thence the structure flexibility matrix and node deflections are obtained. The software for truss, beam, frame and grid analyses first sets up the joint (node) equilibrium equations for the problem being analysed. It then checks for instability, determinancy or indeterminancy of these equations. If determinate, the analysis can proceed directly to the solution. If indeterminate the program prompts with a message to this effect and the stiffness method of analysis may be used via the application of the contragredient law with the commands that are supplied for its application. In this case elastic stiffness properties of individual members must be supplied. In the process of solving an indeterminate problem, the structure flexibility matrix is obtained and is used to calculate deflections.

The purpose of the arrangement of the chapters on the various topics of the following text is to provide the student with a concise presentation of the appropriate theory which can be expanded upon in any lecture course of modern structural mechanics. It must be mentioned at the outset the STATICS-2020 is not simply a computerized version of existing undergraduate texts on the subject, but rather a break with the traditional approach. From the very beginning, force and displacement quantities are expressed in terms of their vector components and elementary matrix theory used in the appropriate transformations of these quantities. The matrix operation commands in the software may be used to give the students necessary back ground knowledge of elementary matrix theory. It will be assumed that the student has been given an introduction to forces, moments, displacements and rotations although some discussion is given in Chapters 1 and 9. The basic ideas of truss, beam, frame and grid structures should be introduced concurrently with the subject matter theory in the text. the Chapters 9-11 give three course outlines to follow in a raeching programme.

## 1.2 Basic transformations

### 1.2.1 Force and displacement vectors-contragredience

Some simple ideas of force and displacement comp[onents are introduced. There are two basic transformations, here given for the planar situation, (but which also apply in general for three dimensional vectors), that are to be used in setting up the equations of equilibrium

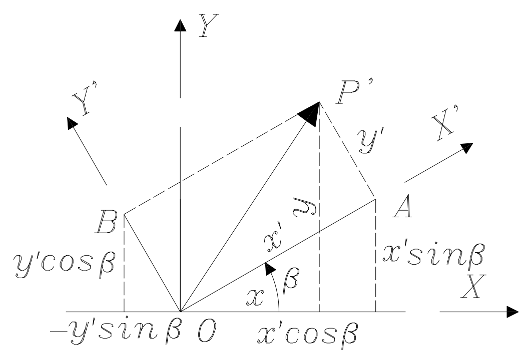


Figure 1.1: Rotation of coordinate axes.

of any system. The first relates to the effect on components of a vector, expressed in a rectangular cartesian system of coordinates, of a rotation of the coordinate axes. The second to the transformation of components of a vector from one point in the plane to another point in the same plane. These transformations are developed here because they can be used not only in setting up equilibrium equations but also because they illustrate the contragredient law.

### 1.2.2 Rotation of coordinate axes

In Figure 1.1, consider the position vector components $(x, y)$ and rotate the coordinate axes through the angle $\beta$ as shown in Figure 1.1 to positions $OA(X'), OB(Y')$. Then from Figure 1.1 the vector $OA$, has components, $(x'\cos\beta, x'\sin\beta)$ and $OB$, components $(-y'\sin\beta, y'\cos\beta)$. However in the $(X, Y)$ coordinate axes, the vector $OP$ has components $(x, y)$, so that,

$$\left\{\begin{matrix} x \\ y \end{matrix}\right\} = \begin{bmatrix} x'\cos\beta \\ x'\sin\beta \end{bmatrix} + \begin{bmatrix} -y'\sin\beta \\ y'\cos\beta \end{bmatrix} \tag{1.1}$$

That is, in matrix form,

$$\left\{\begin{matrix} x \\ y \end{matrix}\right\} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \left\{\begin{matrix} x' \\ y' \end{matrix}\right\} \tag{1.2}$$

This is an orthogonal transformation so that its transpose is also its inverse, thence the reverse transformation is given,

$$\left\{\begin{matrix} x' \\ y' \end{matrix}\right\} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \left\{\begin{matrix} x \\ y \end{matrix}\right\} \tag{1.3}$$

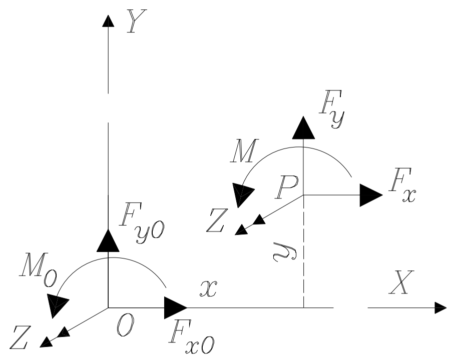


Figure 1.2: Transfer of force from P to origin.

These two transformations will be written as in equations (1.4), and (1.5).

$$\{x'\} = [L]\{x\} \tag{1.4}$$

and,

$$\{x\} = [L]^T\{x'\} \tag{1.5}$$

Notice that the rows of $[L]$ give the unit vectors in the $X', Y'$ directions expressed in the $X, Y$ coordinate system.

### 1.2.3 Transfer of force components and moment in $X - Y$ plane

The second transformation to be considered here concerns the transfer of force components from one point ina plane to a second point in the same plane. The point $P$ in Figure 1.2 has force components $(F_x, F_y)$ acting at $(x, y)$ in the $X-Y$ plane and a moment $M$ about the $Z$ axis. The statically equivalent force components $(F_x, F_y)_o$ and moment $M_o$ act at the origin $O$. In Figure 1.2, it is seen that $M, M_o$ are represented as components of moment vectors about the $Z$ axis. The positive sense is given by the right hand screw rule. That is, place the eye at the point and look along the $Z$ axis. Then the positive moment is in a clockwise direction. By simple statics, taking moments about the origin $O$ of the forces at $P$, the equivalent values at $O$ are,

$$\begin{Bmatrix} F_x \\ F_y \\ M \end{Bmatrix}_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & x & 1 \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ M \end{Bmatrix}_P \tag{1.6}$$

This transformation is written in equation (1.7) as,

$$\{F_o\} = [T]\{F_P\} \tag{1.7}$$

Consider now the interesting and apparently unrelated problem of having displacements $(r_x, r_y)_o$ and rotation $\theta_o$ specified at $O$ and requiring the kinematically equivalent quantities at $P$. For small rotations and displacements the corresponding displacements and rotation at $P$ can be easily deduced by considering one component at a time,

$$\begin{Bmatrix} r_x \\ r_y \\ \theta \end{Bmatrix}_P = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r_x \\ r_y \\ \theta \end{Bmatrix}_o \tag{1.8}$$

That is,

$$\{r_P\} = [T]^T\{r_o\} \tag{1.9}$$

These two equations (1.7) and (1.9) express the relationship between force and displacement components of the two equivalent sets and this relationship is an expression of the contragredient law (see also equations (1.4) and (1.5) that are also expressions of the same law.

## 1.3 Contragredient principle

### 1.3.1 Contragredient transformations

In developing the contragredient law for discrete force and displacement systems the first concept to be considered is that of statically equivalent force systems. Two force systems represented by the vectors $\{P\}$ and $\{Q\}$ are statically equivalent if the components are identical when both systems are transfered to a common coordinate axes. Then consider these two force systems connected by the linear transformation,

$$\{Q\} = [B]\{P\} \tag{1.10}$$

In certain circumstances this may be a reversible transformation, that is $\{P\} = [B]^{-1}\{Q\}$ as in equations (2.14), (2.15). This is however an unnecessary restriction, and it will be assumed that in general the dimensions of $\{P\}$ and $\{Q\}$ will be unequal. For example, in the truss shown in Figure (1.3) there is a relationship connecting member forces $(F_1, F_2, F_3) = (Q^T)$ and $(R_1, R_2) = \{P\}^T$. However there will be solutions for which the conditions of joint equilibrium are satisfied $\sum F_{x_i} = \sum F_{y_i} = 0$ with external loads being zero. A solution of this type corresponds to a temperature change or lack of fit in the three bar members. Associated with each of the force systems $\{P\}$ and $\{Q\}$ there will be sets of displacements, $\{p\}$ and $\{q\}$. These are compatible displacements, so that if the displacements in the directions of $\{Q\}$ are $\{q\}$, those in the directions of $\{P\}$ are $\{p\}$. From equation (1.10) it is possible to determine the transformation that relates $\{p\}$ and $\{q\}$ and this is an expression of the contragredient law. If the two force systems are given compatible displacements the work done in each system is the same, since both force and displacement systems have identical components when transformed to the common reference origin. That is, the equations of the work done are expressed by the dot products,

$$\{P\}^T\{p\} = \{Q\}^T\{q\} \tag{1.11}$$

However from equation (1.10) it follows,

$$\{Q\}^T = \{P\}^T[B]^T \tag{1.12}$$

and substitution in equation (1.11)

$$\{P\}^T\{p\} = \{P\}^T[B]^T\{q\} \tag{1.13}$$

The concept that $\{P\}$ is arbitary is now evoked, so that terms are equated one by one. It follows that,

$$\{p\} = [B]^T\{q\} \tag{1.14}$$

The contragredient law appears in the second form in which $\{P\}$ and $\{Q\}$ are statically equivalent force systems and the associated sets of displacements $\{p\}$ and $\{q\}$, are related by the linear transformation,

$$\{q\} = [C]\{p\} \tag{1.15}$$

Then using the same reasoning for equality of the work done, for compatible displacements and the statically equivalent force systems,

$$\{p\}^T\{P\} = \{q\}^T\{Q\} = \{p\}^T[C]^T\{Q\} \tag{1.16}$$

Again evoking the arbitariness of $\{p\}$ it follows that,

$$\{P\} = [C]^T\{Q\} \tag{1.17}$$

Two applications of the contragredient law are given for the formation of joint equilibrium equations and the calculation of node deflections of a structure.

### 1.3.2 Joint equilibrium equations

Suppose that any structure is subdivided in some way into members and that member forces and structure reactions are contained in the vector $\{S\}$. The applied nodal forces are in $\{R\}$ and the joint equilibrium equations are written

$$[A]\{S\} = \{R\} \tag{1.18}$$

Then if the node displacements corresponding to $\{R\}$ are $\{r\}$ and the member distortions and reaction displacements corresponding to $\{S\}$ are in $\{v\}$, using the contragredient law from equation (1.10)it follows that

$$\{v\} = [A]^T\{r\} \tag{1.19}$$

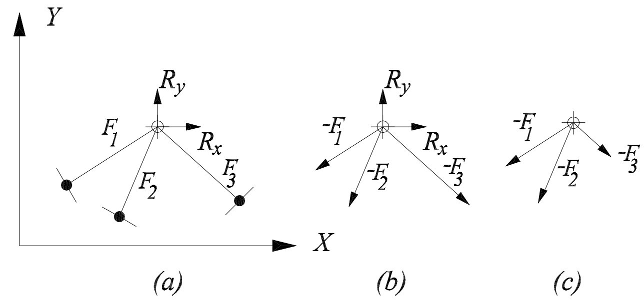


Figure 1.3: Force systems-node and member forces.

This application can be used in the calculation of member forces in statically indeterminate structures. On the other hand suppose that the relationship between $\{v\}$ and $\{r\}$ can be obtained from the kinematics of the structure displacements. That is suppose the transformation between $\{r\}$ and $\{v\}$ is given,

$$\{v\} = [a]\{r\} \tag{1.20}$$

Then the joint equilibrium equations are given by the contragredient law,

$$[a]^T\{S\} = \{R\} \tag{1.21}$$

It follows that $[a]^T = [A]$ and of course $[a] = [A]^T$. Both these expressions are found to be usefull in analysis of structures.

### 1.3.3 Node deflections

For statically determinate structures the transformation to obtain member forces from node forces is obtained from $[A]^{-1}$

$$\{S\} = [A]^{-1}\{R\} = [b]\{R\} \tag{1.22}$$

Then using the contragredient law the nodal displacements are obtained from the member distortions and reaction displacements,

$$\{r\} = [b]^T\{v\} \tag{1.23}$$

The power of this equation should not be under estimated as it provides the best means for calculating structure deflections in all cases where $[b]$ is known or is easily calculated. Many examples are given in the text.

## 1.4 Units of length, mass and force

All structures must have their dimensions, mass, material properties and applied force expressed in the same consistent set of units. There are two main systems of units currently in use, the imperial system and the metric system. The main difference between the two systems is as how the unit of force is defined. In the imperial system, the unit of force is defined in terms of the earth's gravitational field. That is, one pound force accelerates a mass of one pound at 32(g) feet / $sec^2$. In the metric system the unit of force is the Newton which is the force to accelerate a mass of one kilogram at 1 metre/$sec^2$. Since the acceleration due to gravity is approximately 9.8m/$sec^2$, the Newton is approximately $0.1 \times 1$ kilogram force and is thus relatively small compared with the pound force. The basics of the two systems are set out in Table 1.1.

Table 1.1

| Imperial system | |
|---|---|
| length | inch (in), foot (ft) |
| mass | pound (lb), ton(=2240 lb) |
| force | pound (lb), ton(=2240 lb), kip(=1000 lb ) |
| moment | inch pound (in lb), foot pound (ft lb), inch kip (in k) |
| stress | pounds/$inch^2$, kips/$inch^2$, tons/$foot^2$ |

| Metric system | |
|---|---|
| length | millimetre(mm), metre(m) |
| mass | kilogram (kg) |
| force | Newton (N), kiloNewton (kN), megaNewton (mN) |
| moment | Newton millimetres (N mm), Newton metres (N m) |
| stress | Pascal(Newton/$metre^2$), megaPascal(megaNewton/$metre^2$) |

In the example exercises given in STATICS-2020 of statically determinate trusses, the dimensions are relative. That is, for practical cases in either system the dimensions given in the exercises may be scaled to give realistic structure dimensions. For example for the trusses in Figure 2.11, the panel size is 10. This may be 10 feet with a truss span of 60 feet. If the metric system is used the truss span would be 60 metres which may require scaling by 1/3 to make a realistic span.

## 1.5 Concepts of structure

A structure is considered to be composed of a number of members identified by unique node numbers connected together at nodes to form a stable system. For the analysis of a structure three quantities are required to be specified for the generation of the node equilibrium equations, namely,

1. Node numbers and node coordinates.

2. Member numbers and the member node numbers that identify how the members are connected to the nodes.

3. Reaction points and directions of application.

For statically determinate structures the equilibrium equations may be inverted directly. If indeterminate, member stiffness properties are necessary to develop the stiffness analysis of the structure. All these points will be developed in detail for each structure type in its appropriate chapter. Two dimensional structures are conveniently classified according to the primary forces acting on their members as follows:

| Structure type | Member force type |
|---|---|
| Truss | axial force |
| Beam | bending moment and shear force |
| Frame | axial force, bending moment and shear force |
| Grid | bending moment, shear force and torsion |

Although a general purpose, commercial sotware package for structural analysis may treat all these various catagories alike as three dimensional structures with the appropriate components zero, it is important for the understanding of structural behaviour to treat each type separately. Thus each type is given a separate chapter with an accompanying set of assignments.

## 1.6 Strength of materials

Strength of materials is that branch of mechanics that has as its concern the determination of stresses in structure members as well as many solutions from the theory of elasticity. The stresses in structural members can be related to tests on the materials for the design of the structure to carry safely its service loads. For truss members this is a relatively simple problem since for axial force, the member stress is simply the axial force divided by the area of the cross section. For beams subjected to bending, twisting moments and shear this is a more difficult problem and a chapter is devoted to some of the various situations encountered. For example there are thick and thin walled sections for which shear stresses are calculated differently and torsion requires the solution of a second order differential equation for cross sections other than circular. A study of member deformation properties is also necessary for the determination of member stiffness matrices.

## 1.7 Teaching module-basic principles

The exercises in Chapter 1 give the student understanding of the two systems of units, imperial and metric and to gain experience in the fundamental transformations equations (1.6) and (1.8). The imperial system is in use in the U.S.A while many other countries use one or more versions of the metric system. The examples of the force transformations will

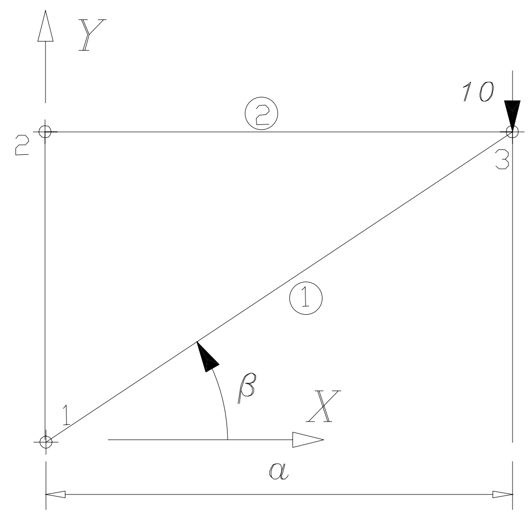


Figure 1.4: Force on end of angle bracket.

give a preliminary understanding of the fundamentals before embarking on the computer analysis of trusses, beams, frames and grids. Exercises (BA1) and (BA2) are on unit conversion and are to give an understanding of the relative magnitudes of the various quantities.

### 1.7.1 Exercises in units

**(BA1)** Units of force. If acceleration due to gravity (g) is given as $9.8m/sec^2$ or $32ft/sec^2$ and 1 kg(mass) = 2.24 pounds(mass),

1. Express 1 Newton in term of pounds force.
2. What is the value of 1 kN in pounds force?
3. What is the value of 1 pound force in Newtons?

**(BA2)** A rectangular, solid area has a breadth of 5 inches (127mm) and depth of 20 inches (304.8mm).

1. Calculate the area, in
   (1) $inches^2$ (2) $mm^2$ (3) $m^2$.
2. Calculate the second moment of the area $= bd^3/12$, in
   (4) $mm^4$ (5) $m^4$ (6) $inches^4$.
3. Comment on the magnitude of the resuts in (1) to (6).

### 1.7.2 Exercises in transformations

**(BA3)** In Figure 1.1, three forces act at the point O with magnitudes and directions as follows,

(1) $F_1 = 10$ units, $\beta_1 = 30^{\emptyset}$
(2) $F_2 = 20$ units, $\beta_2 = 45^{\emptyset}$
(3) $F_3 = 10$ units, $\beta_3 = 120^{\emptyset}$

Calculate the $X - Y$ components of these three forces and answer the folllowing:

**(a)** Are the forces in equilibrium?

**(b)** If not, calculate the $X - Y$ components of the force to close the force polygon and hence equilibrate these forces.

**(c)** From the $X - Y$ components calculate the magnitude and direction of the resultant. ($R = \sqrt{R_x^2 + R_y^2}$; $\sin\theta = R_y/R$).

**(BA4)** The two forces act at a point;

(1) $F_1 = 20$ units, $\beta_1 = 30^{\emptyset}$
(2) $F_2 = 10$ units, $\beta_2 = 50^{\emptyset}$

Calculate the $X - Y$ components of these forces and hence find the force necessary to keep the forces in equilibrium. Calculate its magnitude and direction. Hence show that the three force vectors form a closed triangle (triangle of forces).

**(BA5)** The force $F_1 = 20$ units acts an angle of $\beta_1 = 30^{\emptyset}$ to the $X$ axis. Calculate the magnitude of the force necessary to equilibrate $F_1$ using components. What is its direction?

**(BA6)** A force $F = 10$ units acts the negative $Y$ direction at node 3 of the simple truss-bracket shown in Figure 1.4. The member (1) of the bracket is inclined at an angle $\beta^{\emptyset}$ to the horizontal. Calculate the value of $F_1$, force in member (1), for the following values of $\beta = 15^{\emptyset}, 30^{\emptyset}, 45^{\emptyset}, 60^{\emptyset}$?. Plot $F_1$ verses $\beta$.

**(BA7)** The pole shown in Figure 1.5 has loads, at node (3) of 0.2kN horizontal and 0.5kN vertical. Use equation (1.6) to calculate the forces, (moment, axial force and shear), in the pole at the following sections;

1. In the vertical portion just below node (2).
2. At the base of the pole, node (1).

**(BA8)** The pole in Figure 1.5 is supported by a concrete block in the ground below node (1). This block moves so that at (1), there is a vertical downwards displacement of 5mm and an clockwise rotation of 0.01 radians. Use equation (1.8) to calculate the displacements of nodes (2) and (3). Is the displacement at node (3) excessive? If so suggest a limit to be placed on the rotation. ( Hint: limit the horizontal displacement to 1/400 of the height).

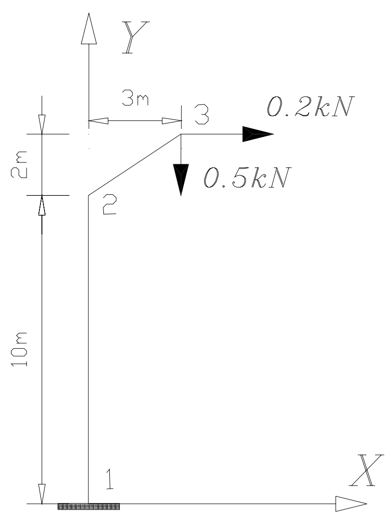


Figure 1.5: Transformation of forces on pole.