Chapter 10

COURSE II STRESS-STRAIN-DEFORMATION

10.1 Introduction

In Chapter 9 the concepts of structure, nodes and members were introduced. From the equilibrium of the nodes, member forces, axial force, bending moment, shear force and torque may be calculated and this is the objective of Course I for statically determinate truss, beam, frame and grid structures. Now in Course II the direction is towards structural design. The purpose is the determination of the capacity of a structure (truss, beam, frame or grid) to carry loads. The concept of stress (force per unit area) is introduced so that stress values may be related to results of experiments carried out in a testing laboratory on materials (steel, aluminum, carbon and glass fibres, concrete). Simplistically laboratory tests determine

(1) material strength properties
(2) material deformation characteristics

In Course II in the structural mechanics series, elementary stress and deformation characteristics are derived for members subjected to axial force, bending moment and shear force. Follow traditional theory this will be called strength of materials.

10.2 Strength of materials

10.2.1 Lecture 10.1 Stress components

Definition of stress-stress components-stress matrix

Read section 6.5 on stress components. Chapter 10 starts with basic deformations of stress and stress components and their transformations for rotation of coordinate axes. Whereas it may appear that deformation and strain are the natural phenomena to measure when studying the effects of forces on structural systems (here in described as truss, beams,

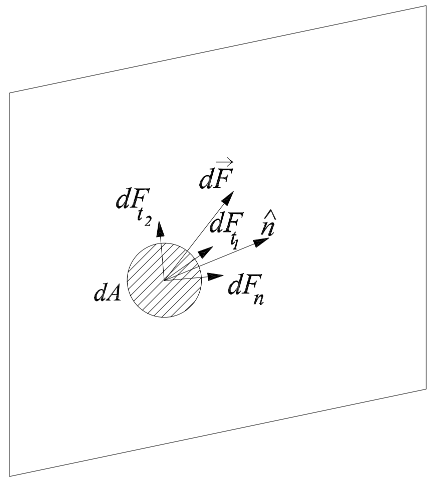

Figure 10.1: Force vector on elemental area

frames and grids), the strength of these systems is usually refered to in terms of the stresses determined from tests on samples of the materials. Stress, defined as force per unit area, allows a comparison of the capacity of structural members of different sizes and different material types. In Figure 10.1, $d\vec{F}$ is a force vector acting on an infinitesimal planar area dA. Then the stress σ, may be defined as,

$$\sigma = \lim_{dA \to 0} \frac{d|F|}{dA} \tag{10.1}$$

However since $d\vec{F}$ is a vector whose direction varies from point to point in the plane of the cross section, it is expedient to define normal and tangential components of the force and the corresponding stress components. Thus,

$$\text{normal stress direction } \hat{n} \quad \sigma_n = \lim_{dA \to 0} \frac{dF_n}{dA} \tag{10.2}$$

$$\text{shear stress direction } t_1 \; \sigma_{t_1} = \lim_{dA \to 0} \frac{dF_{t_1}}{dA} \tag{10.3}$$

$$\text{shear stress direction } t_2 \; \sigma_{t_2} = \lim_{dA \to 0} \frac{dF_{t_2}}{dA} \tag{10.4}$$

In this lecture course, generally only two dimensional stress systems will be considered and it will be usual to take a slice of the body of unit thickness bounded by planes parallel to the XY, YZ or ZX planes. A slice bounded by XY planes is shown in Figure 10.2. In Figure 10.2, the positive sense for each stress component on the element $(dx \times dy \times 1)$is shown. The shear stresses σ_{xy}, σ_{yx} are proven to be equal by taking moments of their corresponding forces about the origin O and equating to zero for equilibrium of the element,

$$-\sigma_{yx}(dx \times 1) \times dy + \sigma_{xy}(dy \times 1) \times dx = 0 \quad \text{hence} \quad \sigma_{xy} = \sigma_{yx} \tag{10.5}$$

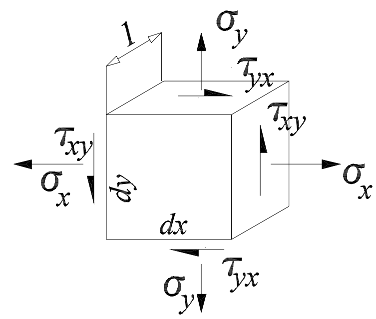

Figure 10.2: Stress components

The notation τ_{xy} will also be used to represent these shear stresses. The components of stress can be written as a matrix, and because of the equality in equation (10.5), the stress matrix is symmetric.

$$[\sigma] = \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{yx} & \sigma_y \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \tag{10.6}$$

10.2.2 Lecture 10.2 Stress transformation

Transformation of stress matrix components

It is important to establish the relationship between components of the stress matrix $[\sigma']$ in the X', Y' axes obtained from the X, Y axes by a rotation α about the Z axis through the origin O (see Figure 10.3). To establish the transformation consider the equilibrium of the small wedge of material bounded by the X, Y axes shown in Figure 10.3. The area dA has a normal $\hat{n}$ whose direction is obtained by a rotation θ from the X axis. Let $\{F_x, F_y\}^T$ be the components of the force per unit area acting on dA and consider the equilibrium of the wedge in the X and Y directions.

(1) $\sum F_x = 0$

$$F_x dA - \sigma_x(dA\cos\theta) - \tau_{yx}(dA\sin\theta) = 0$$

(2) $\sum F_y = 0$

$$F_y dA - \tau_x y(dA\cos\theta) - \sigma_y(dA\sin\theta) = 0$$

In matrix notation,

$$\left\{ \begin{array}{c} F_x \\ F_y \end{array} \right\} dA = \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix} \left\{ \begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right\} dA \tag{10.7}$$

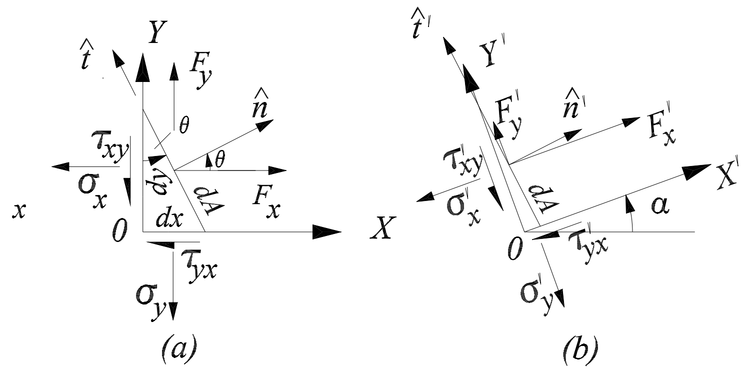

Figure 10.3: Stresses and forces on a wedge of material

However, the normal $\hat{n}$ to dA has direction cosines $(\cos\theta, \sin\theta)^T$, so that equation (10.7) can be written,

$$\{F\}dA = [\sigma]\{\hat{n}\}dA \tag{10.8}$$

Consider a second set of axes obtained from the first by a rotation through an angle α. Again a small wedge of material is considered, area dA, parallel to the infinitesimal area in the first case. In equation (10.8) the vector transformation, equation (1.5), can be applied to the two vectors F and $\hat{n}$ to transform components from the prime axes to the unprimed set, so that,

$$\{F\}dA = [L]^T\{F'\}dA \tag{10.9}$$

$$\{\hat{n}\} = [L]^T\{\hat{n}'\} \tag{10.10}$$

in which,

$$[L]^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{10.11}$$

Substituting these expressions in equation (10.9),

$$[L]^T\{F'\}dA = [\sigma][L]^T\{\hat{n}'\}dA \tag{10.12}$$

Then premultiply both sides of equation (10.12) by $[L]$, and noting $[L]^T = [L]^{-1}$, it follows that,

$$\{F'\} = [L][\sigma][L]^T\{\hat{n}'\} \tag{10.13}$$

However if the reasoning proceeds in the same way for the (X', Y') coordinate axes, with the stress components $[\sigma']$, it follows that in the prime set of coordinate axes,

$$\{F'\} = [\sigma']\{\hat{n}'\} \tag{10.14}$$

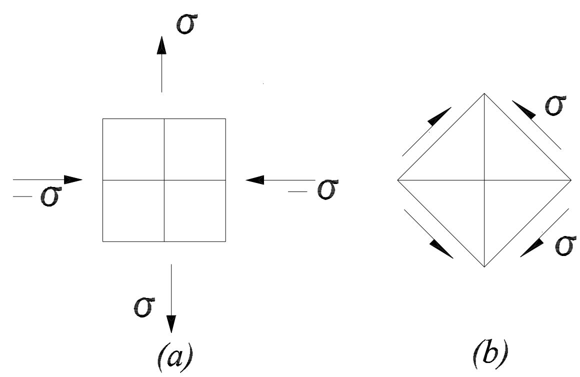

Figure 10.4: Direct and shear stress representation.

Since the expression for F' must be identical, and $\hat{n}$ is an arbitrarily chosen direction, it follows that,

$$[\sigma'] = [L][\sigma][L]^T \tag{10.15}$$

10.2.3 Lecture 3 Maximum-minimum stress

Principal stresses

For a stress state of stress $[\sigma]$ an angle $\bar{\alpha}$ can be found such that the shear stresses are equal to zero. Expand equation (10.15) for stress matrix $[\bar{\sigma}]$ with angle $\bar{\alpha}$ and set $\bar{\sigma}_{xy} = \bar{\sigma}_{yx} = 0$. The value of $\bar{\alpha}$ is found from,

$$\tan 2\bar{\alpha} = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y} \tag{10.16}$$

and the principal stresses,

$$\bar{\sigma}_x = 1/2(\sigma_x + \sigma_y) + R \tag{10.17}$$

$$\bar{\sigma}_y = 1/2(\sigma_x + \sigma_y) - R \tag{10.18}$$

In which,

$$R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \sigma_{xy}^2} \tag{10.19}$$

Shear stresses as a combination of direct stresses

Shear stress is the state in which $\sigma_x = \sigma_y = 0$, $\sigma_{xy} \neq 0$ This stress state may also be expressed by equal and opposite direct stresses acting in directions at 45^o to that of the planes on which the shear stresses are considered to act. Thus in Figure 10.4(b), $\sigma_x = -\sigma, \sigma_y = \sigma$, $\sigma_{xy} = 0$.

$$[\sigma] = \begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

Then using equation (10.15) with $\alpha = 45^o$, the stae of pure shear stress is obtained,

$$[\sigma'] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}$$

10.2.4 Tutorial 10.1

Read Section 6.5. The STATICS-2020 provides the command STRTRN (stress transformation) for the calculation of stress components in a rotated set of coordinate axes given the values in the XY axes, and in addition the principal axes and the principal stress matrix are calculated. The command is

STRTRN A B E F T=?

This command has the stress matrix A (2×2) as input together with a rotation angle given by $T=?$. The rotated stress matrix is in B. In addition the principal stresses have been calculated in E and in F the angle is given from the positive X axis to the maximum principal stress direction. The rotated stress components may be drawn on the rotated cube using the command,

PLTRTN B T=?

To plot the principal stresses, determine F and then setting T = F and replacing B by E repeat the call to PLTRTN.

T1.1 The stress state is given $\sigma_x = 10, \sigma_y = 10, \tau_{xy} = 5$.
1) calculate the stress state for a rotation of coordinates axes $\pm 20^o$.
2) Use the command PLTRTN as explained above to plot stresses on the principal axes for all three sets of stress components.

T1.2 The stress state is given $\sigma_x = 10, \sigma_y = -10.0, \tau_{xy} = 0.0$.
1) Calculate the stress state for a rotation of 45^o. What stress state has been obtained?

T1.3 Calculate the principal stresses for the stress state, $\sigma_x = 10, \sigma_y = 10, \tau_{xy} = -5.0$. Use STRTRN and plot results using PLTRTN E T=(F value).

10.2.5 Lecture 10.4 Deformation

Deformation of the continuum-strain

In this lecture the definition of strain and the transformation of the strain components (matrix) are developed. The definition of strain is deceptively simple. An element PQ, length l_0 suffers deformation so that its change in length is Δl. Then the strain in the direction PQ is defined,

$$\epsilon_{PQ} = \frac{\Delta}{l_0} \tag{10.20}$$

The deformations of the body, (considered herein, to be planar, in the XY coordinate system), are shown in Figure 10.5, in which PQ is displaced to $P'Q'$. The problem is to

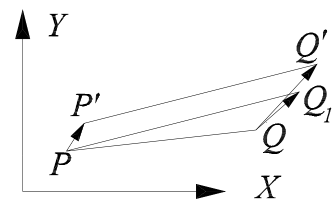

Figure 10.5: Deformation of line element PQ

separate the rigid body displacements (translation and rotation) from PQ to $P'Q'$, so that the strain ϵ_{PQ} may be determined in terms of the partial derivatives of the displacements (u, v) at P, Q being a neighbouring point to P. Deformations are infinitesimally small so that only first order of small quantities need be considered. The points P and Q are neighbouring points and in the limit, $Q \to P$.
Coordinates P: (x, y), Q: $(x + dx, y + dy)$
Body is deformed from PQ to $P'Q'$,
Coordinates P': (x + du, y+dv) Q': (x+dx+u+du, y+dy+v+dv)
PQ_1 is of length $P'Q'$, parallel to $P'Q'$. Then the vectors describing the deformation are given,

$$\Delta\vec{r} = \vec{PQ} = \begin{Bmatrix} dx \\ dy \end{Bmatrix}, \quad \vec{PQ_1} = \begin{Bmatrix} dx + du \\ dy + dv \end{Bmatrix}, \quad \Delta u = \vec{QQ_1} = \begin{Bmatrix} du \\ dv \end{Bmatrix} \tag{10.21}$$

Now (du, dv) are expressed in terms of first order partial derivatives of (u, v) with respect to (x, y) at P.

$$\begin{Bmatrix} du \\ dv \end{Bmatrix} = \begin{bmatrix} u_{,x} & u_{,y} \\ v_{,x} & v_{,y} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix} \tag{10.22}$$

or,

$$\Delta u = M \Delta r \tag{10.23}$$

Then the vector $\vec{PQ_1}$ is given,

$$\vec{PQ_1} = \Delta r' = \Delta r + \Delta u = [I + M]\Delta r \tag{10.24}$$

The square of the strained length is calculated from the inner or dot product,

$$\begin{aligned} \Delta r'^T \Delta r' &= \Delta r^T [I + M]^T [I + M] \Delta r \\ &= \Delta r^T [I + M^T + M + M^T M] \Delta r \end{aligned} \tag{10.25}$$

Now $\Delta r^T M^T \Delta r \equiv \Delta r^T M \Delta r$ and $M^T M$ is a second order small quantity so that from equation(10.25),

$$\Delta r'^T \Delta r' = \Delta r^T [I + 2M] \Delta r \tag{10.26}$$

The matrix $[M]$ is now decomposed into symmetric $[\epsilon]$ and antisymmetric $[\theta]$ components, That is,

$$[\epsilon] = \begin{bmatrix} u_{,x} & 1/2(u_{,y} + v_{,x}) \\ 1/2(u_{,y} + v_{,x}) & v_{,y} \end{bmatrix}; \quad [\theta] = \begin{bmatrix} 0 & -1/2(v_{,x} - u_{,y}) \\ 1/2(v_{,x} - u_{,y}) & 0 \end{bmatrix} \tag{10.27}$$

Making these substitutions,

$$\Delta r'^T \Delta r' = \Delta r^T [I + 2(\epsilon + \theta)] \Delta r \tag{10.28}$$

On multiplication , $\Delta r^T \theta \Delta r \equiv 0$ and represents rigid body rotation so that,

$$\Delta r'^T \Delta r' = \Delta r^T [I + 2\epsilon] \Delta r \tag{10.29}$$

Write the vector Δr in terms of its length l_o and the unit vector $\hat{t}$ in the direction PQ,

$$\Delta r = l_o \hat{t}; \quad \Delta r' = l \hat{t}' \tag{10.30}$$

Then making the substitutions, $\hat{t}^T \hat{t} = \hat{t}'^T \hat{t}' = 1$ etc. from equation (10.29),

$$l^2 = l_o^2 [1 + 2\hat{t}^T \epsilon \hat{t}] \tag{10.31}$$

and since the terms of ϵ are small quantities,

$$l \approx l_o [1 + \hat{t}^T \epsilon \hat{t}] \tag{10.32}$$

The strain on the direction PQ is thus given by,

$$\epsilon_{PQ} = \frac{l - l_o}{l_o} = \hat{t}^T \epsilon \hat{t} \tag{10.33}$$

The direct strains are obtained by setting $\hat{t}$ parallel to the X and Y coordinate axes respectively,

$$\epsilon_x = u_{,x}; \quad \epsilon_y = v_{,y} \tag{10.34}$$

The off-diagonal shear strains are,

$$\epsilon_{xy} = \epsilon_{yx} = 1/2(u_{,y} + v_{,x}) \tag{10.35}$$

are 1/2 the change in the right angle XPY at P due to shear distortion. The engineer's definition of strain γ_{xy} is twice ϵ_{xy} Although γ_{xy} is used in engineering constitutive relationships, ϵ_{xy} should be used when calculating strain terms in coordinate axes rotated from the XY axes.

10.2.6 Lecture 5 Transformation of strain

Transformation of strain components for a rotation of axes

The strain state at P is independent of the coordinate axes chosen to describe its components. Thus, for two sets of axes XY and $X'Y'$,

$$\epsilon_{PQ} = t^T \epsilon t = t'^T \epsilon' t' \tag{10.36}$$

The relationship between the unit vectors t, t' in the two coordinate systems is,

$$t = L^T t' \tag{10.37}$$

Making this substitution in equation(10.36)

$$\epsilon_{PQ} = t'^T L \epsilon L^T t' = t'^T \epsilon' t' \tag{10.38}$$

The direction $\hat{t}$ is arbitrary so that it follows that,

$$\epsilon' = L \epsilon L^T \tag{10.39}$$

This is the same transformation that was obtained for the stress matrix. Thus σ and ϵ possess the transformation properties under rotation of axes of second order tensor quantities.

Principal directions and principal strains

Write the rotation transformation matrix L as,

$$L = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

Then the individual components of ϵ' in equation (10.39)are,

$$\begin{aligned} \epsilon'_x &= c^2 \epsilon_x + s^2 \epsilon_y + 2\epsilon_{xy} \\ \epsilon'_y &= s^2 \epsilon_x + c^2 \epsilon_y - 2\epsilon_{xy} \\ \epsilon'_{xy} &= \epsilon'_{yx} = (c^2 - s^2) - cs(\epsilon_x - \epsilon_y) \end{aligned} \tag{10.40}$$

The principal strain directions are obtained by setting the shear strain to zero, $\epsilon'_{xy} = \epsilon'_{yx} = 0$, Thence,

$$\tan 2\bar{\alpha} = \frac{2\epsilon_{xy}}{(\epsilon_x - \epsilon_y)} \tag{10.41}$$

$$R = \sqrt{\frac{(\epsilon_x - \epsilon_y)^2}{4} + \epsilon_{xy}^2}$$

Then the principal strains are given,

$$\bar{\epsilon}_x = 1/2(\epsilon_x + \epsilon_y) + R; \quad \bar{\epsilon}_y = 1/2(\epsilon_x + \epsilon_y) - R \tag{10.42}$$

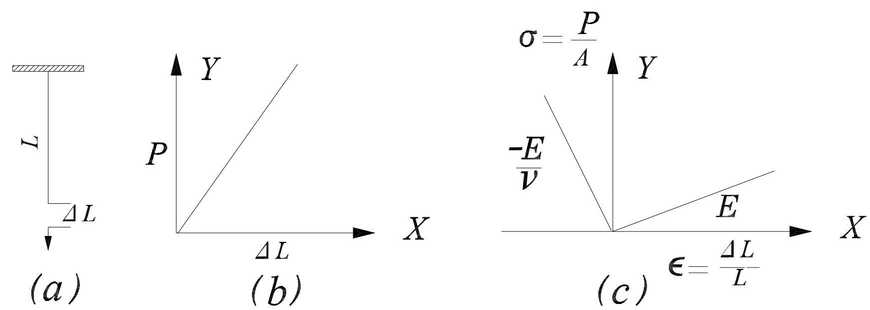

Figure 10.6: Elastic material behaviour-stretching of a wire

10.2.7 Lecture 6 Constitutive relationships

Constitutive Relationships, Hooke's Law, Elastic constants

When a wire of an elastic material, such as steel or glass fibre is loaded in tension, or compression (in this case a decrease in tension force), it undergoes deformation that is proportional to the change in the applied load. The deformation that occurs is both longitudinal (in the load direction) and lateral (at right angles to the load direction). Lateral strain is always in the opposite sense to the longitudinal strain, and if the material is homogeneous and isotropic the deformation is the same, irrespective of the direction of the applied load. These observations for a tensile load P are shown in Figure 10.7. The relationships in Figure 10.7(c) are nondimensional so that the stress σ is a force per unit area and the strain ϵ is the change in length per unit length. For direct strain the slope of the straight line σ verses ϵ_a defines the elastic constant E, known as the Young's modulus of elasticity of the material. For steel $E \approx 200 \times 10^3$mPa (30×10^6)lb per sq. inch. The plot of σ verses the transverse strain ϵ_T, is described using both E and a second constant, Poisson's ratio ν,

$$\frac{E\Delta l}{l} = \frac{P}{A}; \quad E\epsilon_a = \sigma; \quad \epsilon_a = \frac{\sigma}{E} \tag{10.43}$$

and for transverse strains,

$$\epsilon_T = -\nu\epsilon_a = -\frac{\nu\sigma}{E} \tag{10.44}$$

For a homogeneous, isotropic material the two elastic constants (E, ν) are sufficient to determine the deformations due to the stress state, (σ_x, σ_y) and if they act simultaneously at a point the total strains are obtained by the superposition of their separate effects.

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

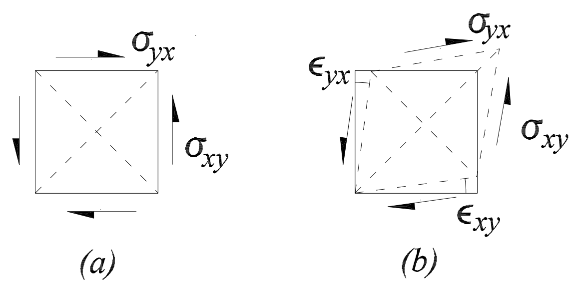

Figure 10.7: Shear stress and shear strain

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} \tag{10.45}$$

These two equations are combined using matrix notation,

$$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \end{array} \right\} = \frac{1}{E} \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \end{array} \right\} \tag{10.46}$$

10.2.8 Lecture 7 Shear stress-strain

Shear strain

Because there are three stress components $(\sigma_x, \sigma_y, \sigma_{xy})$ necessary to describe the state of stress at a point, a third constitutive relationship is necessary to relate shear strain ϵ_{xy} to the shear stress τ_{xy}. These quantities are shown in Figure 10.7. From Figure 10.7(b), the change in the right angle is equal to $2\epsilon_{xy}$, and in terms of σ_{xy},

$$2\epsilon_{xy} = \frac{\sigma_{xy}}{G} = \gamma_{xy} \tag{10.47}$$

The change in the right angle, γ_{xy} is twice the mathematical definition of shear strain ϵ_{xy}. The relationship between all three strain and stress components is called the material constitutive equation. In matrix form,

$$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & \frac{E}{G} \end{bmatrix} \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} \tag{10.48}$$

note:$\gamma_{xy} = 2\epsilon_{xy}, \quad \tau_{xy} = \sigma_{xy}$. A relationship is now established between the elastic constants E, G and ν. From Figure 10.4 the stress state is represented either by

(1) X-Y coordinates,

$$\text{stress} \quad [\sigma] = \begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix} \tag{10.49}$$

$$\text{strain} \quad [\epsilon] = \frac{\sigma(1+\nu)}{E} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.50}$$

Transform both of these quantities through 45^o,

$$[\sigma_{45}] \quad = \frac{\sigma}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sigma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ pure shear stress} \tag{10.51}$$

$$[\epsilon_{45}] \quad = \frac{\sigma(1+\nu)}{2E} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{\sigma(1+\nu)}{E} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ pure shear strain} \tag{10.52}$$

The transformation in equation(10.52) gives,

$$\epsilon_{xy,45} = \frac{(1+\nu)\sigma}{E} \tag{10.53}$$

whereas from the state of pure shear σ, equation (10.51)

$$\epsilon_{xy,45} = \frac{\sigma}{2E} \tag{10.54}$$

Since these results in equations (10.53) and (10.54) must be equal it follows that,

$$\frac{(1+\nu)}{E} = \frac{1}{2G}; \; mbox\text{hence} \quad G = \frac{E}{2(1+\nu)} \tag{10.55}$$

Note: the equation (10.48) is the flexibility constitutive equation, that is strains are given in terms of stresses. Inverting gives the corresponding stiffness constitutive equation,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{10.56}$$

The constitutive equations may be used to calculate deflections of elastic bodies subjected to known stress states, or to calculate applied forces acting on an elastic body subjected to applied displacements(deformations).

10.2.9 Tutorial 10.2

T2.1 A stress state at point P in a body has the values, $\sigma_x = 100mPa$,$\sigma_y = -100mPa$, and $\sigma_{xy} = 0.0$. Calculate the stress states for $\theta = \pm 45^o$. Use STRTRM command.
T2.2 For the stress state in **T2.1**, the material has the elastic constants, Young's modulus 200×10^3mPa, Poisson's ratio, $\nu = 0.3$.
(1) calculate the strains, $\epsilon_x, \epsilon_y, \gamma_{xy}$. Hence calculate the strains for $\theta = \pm 45^o$ using

STRTRN.
(2) For the stress states calculated in **T10.5**, calculate the shear strain ϵ_{xy}. Show that the strains are the same as in (1).
(3) For the material, Young's modulus 10×10^3 mPa, $\nu = 0.2$, the state of stress is given $\sigma_x = -40mPa$,$\sigma_y = -10mPa$, and $\tau_{xy} = 5mPa$.
(1) calculate the principal stress values.
(2) Calculate the strains, $\epsilon_x, \epsilon_y, \epsilon_{xy}$ and from these calculate the principal strains. Show that strains calculated from (1) are the same as from (2).
T2.3 A material has an ultimate tensile strength of 4mPa. The stresses at a point are $\sigma_x = -40mPa$, and $\tau_{xy} = 5mPa$. Determine the value of σ_y such that the maximum principal stress(tensile) is equal to the ultimate tensile stress.

10.2.10 Lecture 8 Concept of virtual work

Concepts of Work, Virtual Displacements, Contragredient Law

The fundamental concepts of the contragredient law are given ib Chapter 1 section 1.3. The student should this section 1.3. The principle of virtual displacements is an expression of the contragredient law. Here the principle is again enunciated in terms of force and moment vectors. The work done M_F by the force F given an displacement δ at a point on its line of action and in the direction of the force is,

$$W_F = F\delta \tag{10.57}$$

Similarly, the work done W_M by the moment M given a rotation β about the line of action of the moment,

$$W_M = M\beta \tag{10.58}$$

In both cases the work is expressed in units of force times distance. If now the force vector $\{F\}$, components, (F_x, F_y, F_z) is given a displacement $\{\delta\}$, components $(\delta_x, \delta_y, \delta_z)$ of a point on its line of action, the work done is the sum of the work done by all the components,

$$W_F = F_x\delta_x + F_y\delta_y + F_z\delta_z = F^T\delta = \delta^T F \tag{10.59}$$

For the moment vector M, components (M_x, M_y, M_z) given the rotation expressed by the vector β, components $(\beta_x, \beta_y, \beta_z)$ the work done will be

$$W_M = M_x\beta_x + M_y\beta_y + M_z\beta_z = M^T\beta = \beta^T M \tag{10.60}$$

A generalized force at the point P will be defined to consist of the force $\{F\}$ and the moment $\{M\}$ combined as the single vector, $\{R\}$

$$\{R\} = \left\{ \begin{array}{c} F \\ M \end{array} \right\} \tag{10.61}$$

The corresponding generalized displacement $\{r\}$ at P, will consist of the displacement vector $\{delta\}$ and rotation vector $\{\beta\}$,

$$\{r\} = \left\{ \begin{array}{c} \delta \\ \beta \end{array} \right\} \tag{10.62}$$

Then the work done by the generalized force $\{R\}$ for the generalized displacement $\{r\}$ is,

$$W = \{R^T\}\{r\} = \{r\}\{R\}^T \tag{10.63}$$

For the two dimensional situation discussed in these lectures, F, M, δ, β are two dimensional vectors and in each case the third dimension is simply deleted, being identically equal to zero. The equilibrium of a set of generalized forces $\{R\}$ acting on a body is considered. Any generalized force at a point P may be transformed to a statically equivalent generalized force at origin O. The successive applications of equations 1.5 and 1.6 to the components of the force achieves this result. Let the transformation matrix so obtained be designated $[b_i]$ for the force $\{R\}_i$. than at O for all forces,

$$\{R_o\} = \sum\{R_o\}_i = \sum[b_i]\{R_i\} \tag{10.64}$$

Work done,

$$\begin{aligned} W_o &= R_o^T \Delta r_o = \Delta r_o R_o^T \\ &= \Delta r_o (b_1 R_1 + b_1 R_1 + ... + b_n R_n) \\ &= (R_1^T b_1 + R_2^T b_2 + ... + R_n^T b_n) \Delta r_o \\ &= (R_1^T b_1 \Delta r_o + R_2^T b_2 \Delta r_o + ... + R_n^T b_n \Delta r_o) \end{aligned} \tag{10.65}$$

If the system of forces is in equilibrium, it follows that $R \equiv 0$, hence work done is equal to zero. That is,

$$\sum R_i^T \Delta r_i = 0 \tag{10.66}$$

This leads to the statement of the **principle of virtual displacements**:
If a system of forces is in equilibrium, the statically equivalent force system is equal to zero. Thence for a virtual (that is not necessarily the actual) displacement the work done is zero.

The converse to this principle states that, if the work done by a system of forces is zero for any arbitrary displacement, then the forces are in equilibrium.

Example-virtual displacements

Consider a virtual displacement consisting of an infinitesimal rotation β at C for $x \leq a$. Then for $W_C = 0$,

$$W_C = 0 = -M_C\,\beta - W(a-x)\,\beta + \frac{Wa}{L}(L-x)\,\beta$$

The moment M_C is calculated

$$M_C = \frac{Wa}{L}(L-x) - W(a-x) = \frac{W}{L}(L-a)x = \frac{Wbx}{L}$$

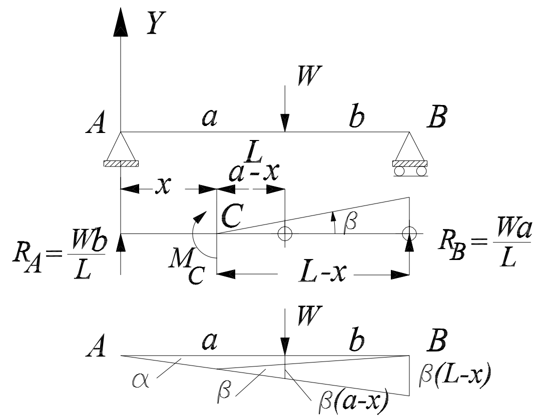

Figure 10.8: Virtual displacement-calculation of internal force in beam

For $x \geq a$,

$$W_C = 0 = -M_C\,\beta + \frac{Wa}{L}(L-x)\beta$$

$$M_C = \frac{Wa}{L}(L-x)$$

In Figure 10.8(b), an angle change of β radians occurs at the point $C(x)$. Calculate the deflection at the load point W, distance a from A.

$$M_x = \frac{dx}{L}W$$

Hence, using contragredience,

$$\delta_W = \frac{bx}{L}\beta = \beta\frac{bx}{L}$$

Prove this result from the geometry of the beam in Figure 10.8(b).

10.2.11 Tutorial 10.3

T3.1 For the beam shown in Figure 10.8 use virtual rotations and the principle of virtual displacements to calculate the reactions R_A , R_B
1) θ about A for R_B
2) θ about B for R_A

T3.2 Study the theory leading to the contragredient law, see Chapter 1 section 1.3. In Chapter 2, for analysis of trusses, the force transformation matrix for determinate trusses, relating member to nodal forces is given in equation (2.14),

$$S = bR$$

Using the contragredient law the relationship between member elongation and the nodal displacements is given,

$$r = b^T v$$

That is, the ith column of b_T,(the ith row of b, gives the nodal displacements for $v_i = 1$.

1) Run the Exercise A8 in the DATN.DAT file.

The b matrix is stored in the matrix D and the commands in the file store and print progressively, the 4th and 22nd rows of D. That is they gie the deflections of the nodes for Δl_4 and $\Delta l_{22} = 1.0$. Run this exercise using SUBMIT A8 and print out the file DATN.OUT. Thence draw the deflected shape for vertical deflections of the lower chord of the truss for both cases. Explain the results.

10.2.12 Lecture 9 Contragredience

The contragredient law has been proven in Chapter 1 section 1.3. The important principle of the contragredience will be generalized to form the **contragredient law** that is used to relate all corresponding force and displacement transformations. **The Contragredient Law**

The contragredient law forms the corner stone of modern structural mechanics. It develops the principle that equilibrium and displacement conditions are inextricably connected and if one is known, the other is automatically generated as its transpose. The contragredient law is concerned with statically equivalent force systems, P, Q and their kinematically equivalent displacements p, q. These displacements are such that if p are the displacements of the body in the directions of the P forces, those in the directions of Q forces will be q. Both P, Q can be generalized forces and p, q the corresponding generalized displacements. Now P is statically equivalent to Q if a linear transformation exists such that,

$$P = B^T Q \tag{10.67}$$

Let R_P, R_Q be the forces at the origin statically equivalent to P and Q.

$$R_P = b_P P, \quad R_Q = b_Q Q \tag{10.68}$$

thence for displacement Δ_O,

$$\begin{aligned} R_P^T \Delta_O &= P^T b_P^T \Delta_O = P^T p = Q^T B^T p \\ R_Q^T \Delta_O &= Q^T b_Q^T \Delta_O = Q^T q \end{aligned} \tag{10.69}$$

Now since P and Q are statically equivalent, $R_P \equiv R_Q$ hence,since P, Q are arbitrary,

$$q = B^T p \tag{10.70}$$

There is a second expression of the contragredient law in which kinematically equivalent displacements p, q are connected through the linear transformation,

$$q = Cp \tag{10.71}$$

It is now required to find the transformation which will ensure that the force systems P, Q corresponding to P, Q are statically equivalent. By the same reasoning as above, it can be shown that the required relationship is,

$$P = C^T Q \tag{10.72}$$

10.2.13 Lecture 10 Section properties

Section properties. Bending of prismatic beams

To start this lecture revise Chapter 9, lectures 5 and 6 and Chapter 3 sections 3.3.1 and 3.3.3. From these sections the bending moments and shear forces in statically determinate beams are calculated (see Chapter 9 Tutorial 5, T5.1 to T5.6). In the procedure for the structural design a beam for both strength and deflection considerations it is necessary to be able to calculate the internal stresses in the beam due to both bending moments and shear forces. In beam and frame structures these moments will be about one axis whereas for grids they will be about both axes in the plane of the cross section of the member. Bending moments generate longitudinal axial stresses in a beam whereas shear force gives rise to shear stresses and related tension and compression stresses.

In order to be able to calculate bending stresses it is necessary to first calculate the area properties of the beam cross section. These properties are the first moments of the area, (integrals of x, y over the cross section) and second moments of area I_x, I_y, I_{xy}, integrals of x^2, y^2, xy over the cross section area. The theory has been developed to locate the centroid of the cross section area and the second moment of area matrix in Chapter 6. Read sections 6.2 and 6.3. Note the method that has been used in which the properties of the triangular area are first established and then all regular figures whose boundaries are described by straight lines can have their properties calculated by the summation of the values of the appropriate triangles, see Figure 6.1 (a) and (b).

10.2.14 Lecture 11 Axial stresses in beams

From Chapters 3, 4 and 5 it will be remembered that a beam or frame element may be subjected to axial force F, and bending moments M_y (about y axis)and M_x (about the x axis) of the membeer cross section. Read section 6.2.3. Because there are three applied (generalized) forces, it should be possible to express the axial stress σ_z as a linear function of the coordinates to determine the constants a_0, a_1, a_2.

$$\sigma_z = a_0 + a_1 x + a_2 y$$

The expression for the stress calculation is derived in equation (6.39),

$$\sigma_z = \frac{F}{A} - [x\, y][I_{cc}]^{-1} \left\{ \begin{array}{c} M_y \\ M_x \end{array} \right\}$$

Because the longitudinal stress distribution is linear, so also the elastic strain distribution is linear. This leads to the bending hypothesis that plane cross sections of the beam before bending remain plane in the deformed state.

10.2.15 Tutorial 10.4

Read Chapter 6, sections 6.2.1 and 6.2.2.

T4.1 The rectangle with dimensions (b, d) shown in Figure 10.9 is divided into two sub-triangles 123, 243. The origin of the coordinates is at the centre of the rectangle. Using the expressions $X^T \Lambda X$, $X^T \Lambda Y$, and $Y^T \Lambda Y$, equations (6.11)and (6.12), calculate I_{00} for each triangle. Add the results and prove that for the rectangle

$$I_{00} = \frac{bd}{12} \begin{bmatrix} b^2 & 0 \\ 0 & d^2 \end{bmatrix}$$

T4.2 The thick I section dimensions (100,300,5,5), see Figure 6.8(1) and also the DATN.DAT file, problem(31) separator S1. Undertake the following exercises.

1) Run STATICS-2020, with SUBMIT S1 and print the second moment of area matrix E. Prove that the result is correct by calculating $I_{00} = I_C + A(\bar{X}\bar{X}^T)$ where, I_C is the inertia matrix for (47.5,290) rectangle about its centroid and $X^T = (25.75, 0.0)$. subtracting $2 \times I_{00}$ from the outer rectangle (100×300) Show that results obtained for I are the principal values and X, Y are the principal axes.

The command (STRESS A G F=? M=?,?) is used to calculate direct and bending stresses.

2) Calculate stresses for M=(1000,0)

3) Calculate stresses for M=(0,1000)

In both cases print the result and plot longitudinal stress on the cross section using the plot command,

PLTSTR A G A=40.0,30.,1.5 N=4

4) Apply moments M=(1000,1000) and show that the result is the sum of (1) and (2).

10.3 Lecture 12 Bending of thin walled beams

The basic theory for the calculation of bending stresses in thin walled sections is given in Section 6.3, see Figure 6.9 and equation(6.38) for the approximation made in the calculation of I', the second moment of area matrix of a thin rectangle $(t \ll l)$, calculated about its centroid, X' parallel to the long dimension the rectangle. The assumption is made that t^3 is small compared to l^3 and may be neglected. With this assumption section 6.3 shows how the second moment of area matrix is calculated, and the command
THSECT A M
is available for this purpose. Then the same commands as for thick sections are used to locate the centroid and calculate the second moment of area matrix for the whole section in the global coordinate axes. The use of these commands is shown in section 6.3 and see also DATN.DAT, exercises 37 to 40. The sections available for data generation are shown in Figure 6.10, and these are accessed by the command

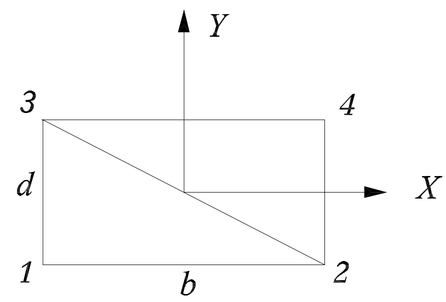

Figure 10.9: Rectangle-two sub triangles.

THINEX E=(1 to5) D=b, d, t_1, t_2.
Stresses are calculated, as for thin sections with the command
STRESS A G F=? M=?,?.

10.4 Tutorial 10.5

Exercises on thin wall stress calculations are given in the DATN.DAT file numbers 37-40, with separators T1 to T4. These exercises are taken from Figure 6.10. Remember the moment sequence, (M_y, M_z), as shown in Figure 6.3.

T5.1 The thin walled I section Figure 6.10, (1) is programmed in T1 and accessed by the SUBMIT T1 command.
1) Load case F=10000.0, M=0,0 Print and plot stress distribution.
2) Load case F=0.0, M=1000.0,0 Print and plot stress
3) Load case F=0.0, M=0,1000.0 Print and plot stress

T5.2 The thin walled equal angle section Figure 6.10, (4) is programmed in T4 and accessed by the SUBMIT T4 command.
1) Load case F=10000.0, M=0,0 Print and plot stress distribution.
2) Load case F=0.0, M=1000.0,0 Print and plot stress
3) Load case F=0.0, M=0,1000.0 Print and plot stress

T5.3 The thin walled channel section Figure 6.10, (3) is programmed in T3 and accessed by the SUBMIT T3 command.
1) Load case F=10000.0, M=0,0 Print and plot stress distribution.

2) Load case F=0.0, M=1000.0,0 Print and plot stress
3) Load case F=0.0, M=0,1000.0 Print and plot stress

T5.4 The thin walled Tee section Figure 6.10, (2) is programmed in T2 and accessed by the SUBMIT T2 command.
1) Load case F=10000.0, M=0,0 Print and plot stress distribution.
2) Load case F=0.0, M=1000.0,0 Print and plot stress
3) Load case F=0.0, M=0,1000.0 Print and plot stress

10.5 Lecture 13 Shear stress distribution

The theory for the calculation of shear flow in thin walled sections is given in Chapter 6, section 6.4. In this theory the thin walled cross section is considered to be composed of a number of individual thin rectangular elements. The elements then connect nodes of the cross section as shown in Figure 6.10(1-4). This is a basic finite element concept as already introduced in structural analysis of trusses, beams, and frames. The subdivision into elements allows integration of the area functions of each element to be calculated separately and the whole combined by considering the equilibrium of the nodes, to calculate the member shear flows. The basic idea for an element is shown in Figure 6.12. Read section 6.4 and from equations 6.46 to 6.50 derive the expression for the shear flow, $q_s = t\tau_s$. Remember t is constant in each rectangular element. Note then if q_{sI} and q_{sJ} are the shear flow values at ends $I - J$ of an element, integration of equation 6.50 gives the relationship between these two values. At any point s from I, q_s is then given by equation 6.56. Integrating this value from 0 to l(the element length), V_s is obtained, the total shear force produced by q_s on the element, see equation (6.58). Thence equations (6.59-6.60) and 6.65-6.67 show how the cross shear flows are determined from nodal equilibrium. The equilibrium equations such as equation (6.67) are easily solved.

10.6 Tutorial 10.6

Before commencing this tutorial study equations 6.45 to 6.58 and understand the finite element concept for the section elements in the derivation of equation (6.58) and the method of calculating V_s distributions and locating the coordinates of the shear centre (X_s, Y_s) measured from the centroid of the cross section. The purpose of these exercises is three fold:

1) To determine shear flow distributions for unit values of transverse shear forces $V_x = 1$ and $V_y = 1$.
2) To locate the shear centre and its significance.
3) To understand the nature of the effect of shear centre on the twist deformations of thin walled beams.

The example chosen are the same sections that were given in Tutorial 10.5 for the calculation of axial stresses due to bending moments.

T6.1 The thin walled I section Figure 6.10, (1) is programmed in T1A and accessed by the SUBMIT T1A command.

T6.2 The thin walled equal angle section Figure 6.10, (4) is programmed in T4A and accessed by the SUBMIT T4A command.

T6.3 The thin walled channel section Figure 6.10, (3) is programmed in T3A and accessed by the SUBMIT T3A command.

T6.4 The thin walled Tee section Figure 6.10, (2) is programmed in T2A and accessed by the SUBMIT T2A command.

For each exercise print the DATN.OUT file. This contains the coordinates of the centroid relative to the initial origin of coordinates, the shear centre relative to the centroid and the two shear flow distributions given q_I, q_c, q_J these values should then be plotted for each cross section. From the location of the shear centre for each cross section explain why a force applied through the shear centre produces no twist in the beam. Prove that the shear centre is also the centre of twist using the principal of virtual displacements.

10.7 Lecture 14 Deflection calculations

In this lecture the calculation of element and structure node deflection calculations is developed based on the contragredient principle. For discrete systems the contragredient law may be expressed,
Force transformation:

$$S = bR \tag{10.73}$$

Corresponding displacement transformation:

$$r = b^T v \tag{10.74}$$

If for an elastic system a linear transformation is established between (v, S) then nodal deflections can be calculated. That is suppose the relationship is known,

$$v = fS \tag{10.75}$$

in which (v, S) are the vectors containing all the member deformations and forces, and f is a matrix with member flexibility matrices as diagonal blocks.

$$f = \lceil f_1 \ddots f_i \ddots f_n \rfloor \tag{10.76}$$

Combining equations (10.73,-,10.76),

$$r = b^T f b R = FR \tag{10.77}$$

Then $F = b^T f b$ is the structure flexibility matrix. For a truss member (i) the member flexibility matrix is easily calculated to be,

$$f_i = \frac{l_i}{A_i E_i} \tag{10.78}$$

The truss flexibility can be calculated and hence nodal deflections. Using STATICS-2020, for determinate structures b is found by inversion of the nodal equilibrium equations, see equation(2.14). See also the theory for deflection calculations is section 2.2.5, and the command TRUSFL that is used to calculate F. See also section 2.2.6 for displacement of truss supports. Examples are given of truss deflection calculations in section 2.3.5 and 2.3.6. In the data file DATN.DAT exercises (1) to (9) have been programmed for the calculation of node deflections. The command PLTRUS A B C V N=3 is used to view the deflected truss shape.

10.8 Tutorial 10.7

In this tutorial truss deflections taken from (1) to (9) on DATN.DAT are studied. The trusses are shown under the Exercise menu in STATICS-2020, TRUSS1-TRUSS4 and TRUSS7 and in Chapter 2 Figures 2.8, 2.9, 2.10 and 2.13. This tutorial is intended to give exercises in the understanding of deflected shapes of structures. That is, to develop intuitive understanding, rather than simply numerical analysis expertize. Each truss is analysed using SUBMIT AN, where N is an integer 1 to 9, see either the Exercises menu or the figures in Chapter 2.

[**T7.1**]-to-[**T7-10**] The procedure for each truss is the same namely,

1)Run SUBMIT AN and then view the truss with the (R S N=2) option. Note members with tension, compression or zero member force. From this information draw an estimate of the deflected shape.

2) Now plot the deflected shape of the truss with the (S,V N=3) option and compare with your estimate describing the similarity or otherwise of the two drawings.

3) As an alternative to steps 1) and 2), first view the deflected shape and estimate the direction and position of the load vectors. Each truss exercise should be repeated until the student can obtain reasonable results. Finally a class exercise should be set with a new load case and the student submit his/her estimate without viewing the computed deflected shape or load as the case may be.

10.9 Lecture 15-Deflection calculations

For a bending element $(I - J)$ the member forces for bending in a plane (e.g. XY plane), are the end moments (M_i, M_j) and they produce deflections in the plane with end rotations (ϕ_i, ϕ_j) relative to the chord $I - J$ of the member as shown in Figure 3.12. The bending moment M at any point (ζ_1, ζ_2) along the member, due to these end moments will be linear and interpolated by the expression,

$$M = [\zeta_1 \; \zeta_2] \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} \tag{10.79}$$

The stress σ at any point in the cross section distance y from the centroid is,

$$\sigma = \frac{yM}{I} \tag{10.80}$$

and on an infinitesimal area dA of the cross section the force is,

$$S = \sigma dA = \frac{ydA}{I} M = [b]\, dA \; M = [b_1] M \tag{10.81}$$

Then the resulting strain ϵ and deformation v for a length dx of beam, are given,

$$\epsilon = \frac{\sigma}{E}, \quad v = \frac{S}{EdA}\, dx = [f_1]\, S \tag{10.82}$$

Then applying contragredience to the small slice dx, gives the relative rotation of the two sections dx apart due to M,

$$[f_1] S = [b_1]^T \frac{dx}{EdA} [b_1] M = [b]^T [f][b]\, dA\, dx M \tag{10.83}$$

In which $[f] = \frac{1}{E}$. Integrating over the whole cross section,

$$\theta = \int_A [b]^T [f][b]\, dA\, dx\, M \tag{10.84}$$

Substituting for $[b]$ and $[f]$,

$$\theta = \frac{dx}{EI} M \tag{10.85}$$

Now apply contragredience to (M_i, M_j) and (ϕ_i, ϕ_j), using equation (10.79) and redefining $[b]$ write equation (10.79) as,

$$M = [b] \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} \tag{10.86}$$

then the contribution to the end rotations, (ϕ_i, ϕ_j) due to θ is,

$$\left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} = [b]^T \theta = \frac{[b]^T\, dx}{EI} M = \frac{[b]^T [b]}{EI}\, dx \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} \tag{10.87}$$

Integrating $0 - l$ the beam length,

$$\left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} = \int \frac{[b]^T [b]}{EI}\, dx = \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} \tag{10.88}$$

If EI is constant,

$$\left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} = \frac{l}{6EI} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} \tag{10.89}$$

This is the flexibility expression for the beam element relating end rotations to end moments. Inverting this equation gives the stiffness expression relating end moments to end rotations,

$$\left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} = \frac{2EI}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} \tag{10.90}$$

Both these equations are used in beam analysis. For determinate beams only equation (10.89) is required for deflection calculations. Read section 3.4.1 and understand how distributed load w/unit length may be incorporated into deflection calculations using the v^* term for the additional rotation terms for the distributed load on the element. That is from equation (3.33)

$$\left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} = \frac{l}{6EI} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} + \frac{l^3}{24EI} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} w \tag{10.91}$$

These expressions are used in the tutorials for beam deflections.

10.10 Tutorial 10.8

T8.1 A cantilever beam, length l, is composed of a single member, nodes 1-2. A force W and a moment M can be applied to the free end, node 2. Show that the force transformation matrix is,

$$[b] = \begin{bmatrix} l & 1 \\ 0 & 1 \end{bmatrix}$$

Hence using equation (10.91),(3.37) and section 3.4.1, prove that the deflection and the rotation at the free end(node 2) of the cantilever, for the load W is given,

$$\{r\} = \left\{ \begin{array}{c} \delta \\ \theta \end{array} \right\}_2 = \left\{ \begin{array}{c} 2l \\ 3 \end{array} \right\} \frac{Wl^2}{6EI}$$

T8.2 The cantilever beam in **T8.1** has a uniformly distributed load w/unit length applied over the length l. Replace this load by $wl/2$ at nodes 1 and 2, and hence using $[b]$ as in **T8.3** and equation (10.91), show that

$$\{r\} = \left\{ \begin{array}{c} \delta \\ \theta \end{array} \right\}_2 = \left\{ \begin{array}{c} 3l \\ 4 \end{array} \right\} \frac{wl^3}{24EI}$$

T8.3 A simply supported beam, length l, has nodes 1,2,3. Node 2 divides the length into two members lengths $a, b, a+b = l$. Show that $[b]$ for a load at node 2 and the matrix of member flexibilities are given,

$$[b] = l \begin{bmatrix} 0 \\ ab \\ ab \\ 0 \end{bmatrix}; \qquad [f] = \frac{1}{6EI} \lceil a \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \; b \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rfloor$$

Thence prove that the deflection at the load point using equation(10.91) is,

$$\{\delta\} = \frac{a^2 b^2 (a+b)}{3EIl^2} W$$

and when the load is at the centre of the beam $a = b = l/2$,

$$\{\delta\} = -\frac{Wl^3}{48EI}$$

The following exercises use STATICS-2020, and are based on the data file DATN.DAT, see exercise numbers (18-22), separators B1 to B5.

T10.32 The beam in Figure 3.6(4) has a UDL of -1 on all elements. Run SUBMIT B4 and draw 1) the bending moment diagram using (PLTBEM A B C N N=2). From this diagram sketch your idea of the deflected shape. Rerun the problem and plot the deflected shape using (PLTBEM A B C R N=4)

T10.33 The beam in Figure 3.6(5) has a UDL of -1 on all elements. Run SUBMIT B5 and draw 1) the bending moment diagram using (PLTBEM A B C N N=2). From this diagram sketch your idea of the deflected shape. Rerun the problem and plot the deflected shape using (PLTBEM A B C R N=4)

10.11 Torsion-circular shafts

The prismatic member of circular cross section in Figure 10.10 is subjected to a torsion moment M_T which from conditions of equilibrium is the same for all cross sections. If α is the twist per unit length, then at a distance x from the fixed end, the total rotation θ is given,

$$\theta = \alpha x \tag{10.92}$$

The assumption is now made that deformations are symmetrical about the centroidal axis of the shaft and vary linearly with the radius r as shown in Figure 10.10. Thus for the yz axes chosen, the z displacement components of the point P are given,

$$w = r\theta = r\alpha x, \quad u = v = 0 \tag{10.93}$$

It should be noted that this assumption is valid for all points at radius r from O, with the $z - y$ axes thus being able to be located at any orientation in the cross section and the

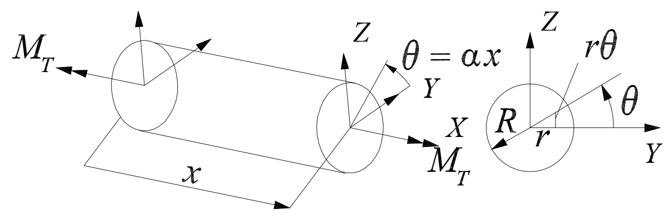

Figure 10.10: Torsion of Prismatic shaft-circular cross section

displacement w being always tangential to the circle of radius r. With this displacement w, the shear strain $2\epsilon_{xz}$ is given,

$$2\epsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = r\alpha \tag{10.94}$$

and the shear stress σ_{xz},

$$\sigma_{xz} = G(2\epsilon_{xz}) \tag{10.95}$$

The torsion moment is calculated by the integral,

$$M_T = \int_{area} r\sigma_{xz}\, dA = \int_{area} rG2\epsilon_{xz} dA \tag{10.96}$$

That is,

$$M_T = \int_{area} r^2\, dA\, \alpha = G\int_0^R \int_0^{2\pi} r^3 d\beta\, dr\, \alpha = \frac{G\pi R^4}{2}\alpha = GI_P\alpha \tag{10.97}$$

Thence,

$$\alpha = \frac{M_T}{GI_P} \tag{10.98}$$

the shear stress is calculated,

$$\sigma_{xz} = Gr\alpha = \frac{M_T r}{I_P} \tag{10.99}$$

The term I_P is the polar second moment of area of the circular cross section. For the length l of the member, it follows that,

$$\theta_l = \alpha l = \frac{l}{GI_P} M_T \tag{10.100}$$

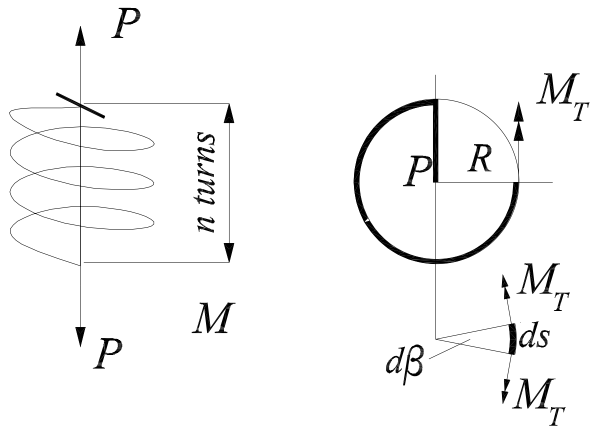

Figure 10.11: Close coiled helical spring

thus the flexibility of the member is written,

$$f_T = \frac{l}{GI_P} \tag{10.101}$$

inverting gives the member stiffness,

$$k_T = \frac{GI_P}{l} \tag{10.102}$$

Example Deflection of a closely coiled spring:
The spring is shown in Figure 10.11 and it consists of n turns of radius R of a wire of diameter r. In Figure 10.11 the small length ds of the coil which subtends an angle $d\beta$ at the axis of the spring coil has, from equation (10.101) and the relative rotation between its ends is given,

$$d\theta = \frac{ds}{GI_P} M_T \tag{10.103}$$

From the equilibrium of the portion of the spring (12) shown in Figure 10.11, the torsion in the spring is given,

$$M_T = -RP \tag{10.104}$$

Using the contragredient principle, the contribution to the deflection of P due to $d\theta$ is given,

$$\Delta r_P = -Rd\theta = \frac{R}{GI_P} M_T ds \tag{10.105}$$

Remember $I_P = \pi r^4/2$ and $ds = Rd\beta$, and integrating 0 to 2π on β and summing for n turns,

$$r_P = \frac{nR^3}{GI_P} \int_0^{2\pi} d\beta P = \frac{4nR^3 P}{Gr^4} \tag{10.106}$$