# Chapter 11

# COURSE III Indeterminate Structures

## 11.1 Introduction

In Course I, Chapter 9, the means by which member forces in statically determinate structures (trusses, beams, frames and grids) can be calculated is established by setting up the equilibrium equations of forces at the nodes,

$$[A]\{S\} = \{R\} \tag{11.1}$$

See the development of this equation in equations (2.13), (3.17), (4.7) and (5.7), for the various structure types. If the structure is indeterminate, then there are more columns in $[A]$ than rows and in order to find a solution additional equations of displacement compatibility are required. In fact there is now an infinite number of solutions possible depending on the member elastic properties. Read sections 2.2.7, 3.3.4, 4.3.1 and 5.3.1, in which the indeterminate structures are discussed. In the Course III each of the structure types will be studied separately, progressing from the relatively simple, indeterminate truss analysis through to beams, frames and grids.

## 11.2 Lecture 1 Indeterminate trusses

Read Chapter 2, section 2.2.7. Contragredience is used to first establish the relationship between nodal displacements $\{r\}$ and the member distortions $\{v\}$. See Course II for the development of the contragredient law. That is, from equation (11.1), contragredience gives,

$$\{v\} = [A]^T\{r\} \tag{11.2}$$

In Course II, the relationship between truss member force $S_i$ and truss member elongation $\Delta l_i$ was established,

$$S_i = \frac{EA_i}{l_i}\Delta l_i \tag{11.3}$$

Since the reaction forces are also considered with the member forces they are each given a stiffness of $10^{20}$(large number). For all members and reactions,

$$\{S\} = [k]\{v\} \tag{11.4}$$

Combining equations (11.1-11.4),

$$\{R\} = [A][k][A]^T\{r\} = [K]\{r\} \tag{11.5}$$

$$\{r\} = [K]^{-1}\{R\} = [F]\{R\} \tag{11.6}$$

Member forces may be determined,

$$\{S\} = [k][A]^T[F]\{R\} = [b]\{R\} \tag{11.7}$$

$$[b] = [k][A]^T[F] \tag{11.8}$$

These equations (11.4) to (11.8) are adequate for the analysis of structures given in this course. There are two basic commands for trusses that calculate $[k]$, $[F]$ and $[b]$, namely,

Truss member stiffness: TRMSTF A B C AR MS
Truss global stiffness: TRGSTF D MS K

See section 2.2.7. for the sequence of commands for indeterminate truss analysis. See also the Help menu in STATICS-2020.

## 11.3 Tutorial 11.1

Indeterminate trusses are given in exercises on DATN.DAT, numbers 10-15, in which both member forces and nodal deflections are calculated.

**T1.1** Problem A10, truss 8 in Figure 2.11. Note $[b]$ is located in matrix KA, and K has been inverted and so contains flexibility [F]. Print member forces and reactions. View and print deflected shape. (Load is 100 in Y direction on node 4).

**T1.2** Problem A13, truss 11 in Figure 2.11. Print member forces and reactions. View and print deflected shape of truss. (Load 100 on both nodes 4 and 6). Compare member forces with those in T1.1

**T1.3** Problem A14, truss 12 in Figure 2.12. This is a two span continuous truss. Vertical loads are applied to nodes 4,6,12. Print member forces and show tension and compression members. View and print deflected shape of truss.

## 11.4 Lectures 2-3 Indeterminate beams

For nodal forces applied to indeterminate beams the theory is the same as for truss structures equations (3.19) to (3.28). In the case of the beam member the stiffness matrix $k_i$ is now a $(2 \times 2)$ matrix whose magnitude was determined in Course II, equation (10.90) that is,

$$k_i = \frac{2EI_i}{l_i}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{11.9}$$

See section 3.3.4. There is however an added consideration for loads applied to members(between node loads). The state of zero node deflections $\{r\} = 0$ can be maintained only by the application of nodal forces to relevant node rotations. That is,

$$[A]\{S^*\} = \{R^*\} \tag{11.10}$$

The negative of these $\{R^*\}$ forces must be applied to release the nodes, so that now,

$$\{r\} = [K]^{-1}(\{R\} - [A]\{S^*\}) \tag{11.11}$$

because the distributed (or within member loads) can be calculated for each member and then added into $\{R\}$. Of course when calculating member forces, three cases of member loads are shown in Figure 11.1, UDL, linearly varying distributed load and a single point load. Each of these types generates its unique $\{S^*\}$ values and statically equivalent node forces, see Figure 11.1. The $\{S^*\}$ forces may be calculated for simple load cases on the beam. In STATICS-2020, only UDL over the whole length of a member has been programmed.

### 11.4.1 Uniformly distributed load

$\{R^*\} = -[A]\{S^*\}$ gives,

$$\{S\} = -\frac{wl^2}{12} \tag{11.12}$$

$$\{R\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} \frac{wl^2}{12} \tag{11.13}$$

The static replacement of $wl$ gives $wl/2$ at both nodes $I - J$ of the member.

### 11.4.2 Linearly varying distributed load

In this case

$$\{S^*\} = \frac{-wl^2}{60}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \tag{11.14}$$

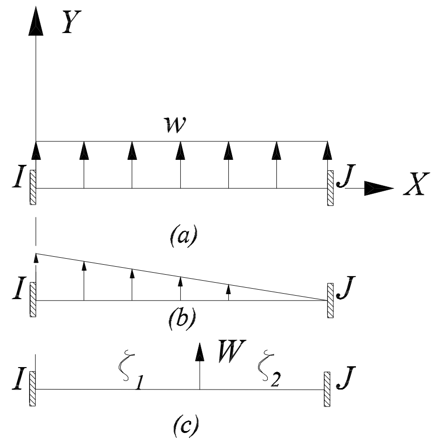

Figure 11.1: Member load types

and,

$$-[A]\{S^*\} = \begin{Bmatrix} 1 \\ 3l \\ -1 \\ -2l \end{Bmatrix} \frac{wl}{60} \tag{11.15}$$

To these values are added the static replacement forces $wl/3$ at $I$ and $wl/6$ at $J$.

### 11.4.3 Concentrated load W

The concentrated load is at the point that divides the element in the ratio $(\zeta_i, \zeta_2)$. Then

$$\{S^*\} = \frac{-Wl\zeta_1\zeta_2}{3} \begin{bmatrix} 1+2\zeta_1-\zeta_2 \\ 1-\zeta_1+2\zeta_2 \end{bmatrix} \tag{11.16}$$

and thence,

$$-[A]\{S^*\} = \begin{Bmatrix} \frac{-3}{l}(\zeta_1-\zeta_2) \\ 1+2\zeta_1-\zeta_2 \\ \frac{3}{l}(\zeta_1-\zeta_2) \\ 1-\zeta_1+2\zeta_2 \end{Bmatrix} \frac{Wl\zeta_1\zeta_2}{3} \tag{11.17}$$

For example when $\zeta_1 = \zeta_2 = 1/2$,

$$-[A]\{S^*\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} \frac{Wl}{8} \tag{11.18}$$

and the static replacement forces are $W\zeta_1$ at $I$ and $W\zeta_2$ at $J$. In STATICS-2020 only UDL is available and nodal loads and fixed end forces generated using,

BEAMLD A B E F C=? D=?

The member distributed force values are in the matrix array F and if present option $D = 1$ is used. In STATICS-2020, the commands used to calculate the inverted nodal stiffness matrix are,

Beam member stiffness: BMMSTF B IN MS

Beam global stiffness: MGSTF EQ MS K

See the command sequence for indeterminate beam analysis in section 3.5.2. For STATICS-2020 commands read section 3.3.5. Various statically indeterminate beams are shown in Figure 3.17. These have been programmed into DATN.DAT, exercises (23) to (26), separators B6 to B9. A single element fixed ended beam is given in (27)-B10. Using this example the deflection of a single element under UDL is obtained.

## 11.5 Tutorial 11.2

**T2.1** Run exercise B10 on DATN.DAT with SUBMIT B10. Plot both the bending moment diagram (PLTBEM A B C M N=2) and the deflected shape (PLTBEM A B C R N=4). Note the deflected shape is limited by the number of plotting segments per element. Print out moments and deflections and give the central deflection in terms of $wl^4/EI$. In the following exercises the problems start with the single span (4 element) propped cantilever and progress from 2 to 4 spans continuous. In all cases the load given is UDL over the whole length of beam.

**T2.2** Run exercise B6 on DATN.DAT with SUBMIT B6. Print reactions (S) and prove that they equilibrate the applied load ($\sum F = 0, \sum M = 0$). Display the bending moment diagram and from this sketch an estimated deflected shape. Compare that displayed with (PLTBEM A B C R N=4).

**T2.3** In exercise (24), separator B7, the overall length of the 3 spans is 24 units, with side spans of 6 each. Run SUBMIT B7 and draw bending moment diagram. Now edit the file DATN.DAT and make the spans equal. that is change second parameter in the BEAMEX command to 8. Now run STATICS-2020, SUBMIT B7 and compare bending moment diagram with the first case.

**T2.4** Exercise on four span continuous beam. The overall length is 32 with end spans 8(that is all spans of equal length). Run exercise with SUBMIT B8 and view bending moment and shear force diagrams. How can the bending moments in the end span be reduced by changing length of the end spans? Edit DATN.DAT file making end spans 80% of the centre spans and rerun. Compare the two sets of results.

## 11.6 Lectures 4-5 Indeterminate frames

Frame analysis is discussed in Chapter 4 and the member element , with member forces given in local (member) and global axes, shown in Figure 4.2. read Chapter 4, sections 4.2.1 and 4.2.2 and understand the member force transformations in equations 4.1 to 4.5. It should be noted that the basic matrix equations are the same in form for the stiffness analysis of both truss and beam structures. For frames, there are three member forces $(F, M_i, M_j)$ and the corresponding deformations $(\Delta l, \phi_i, \phi_j)$. Thus although the matrix equations are the same, the composition of the equilibrium equations, member and global stiffness matrix are different, see sections 4.2.2 and 4.3.1. The member stiffness matrix is a combination of the axial (truss type) and bending (beam type) element stiffnesses, from equation (4.12), for element $n$,

$$\left\{\begin{array}{c} F \\ M_i \\ M_j \end{array}\right\}_n = \frac{EI_n}{l_n}\begin{bmatrix} \frac{A_n}{l_n} & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}\left\{\begin{array}{c} \Delta l \\ \phi_i \\ \phi_j \end{array}\right\}_n = [k_n]\{v_n\} \tag{11.19}$$

The matrix of member stiffnesses $[k]$ thus consists of diagonal sub matrices $[k_n]$ and stiffness values of $10^20$ for the reactions terms that are again grouped after the member force values. As for beams and trusses, two commands are available to generate the global stiffness matrix $[K]$ and to invert to give the flexibility matrix $[F] = [K^{-1}]$.

member stiffness MS: FRMSTF B IN MS
global stiffness K:FRGSTF EQ MS K

The matrix EQ contains the nodal equilibrium equations and is a matrix giving (area, second moment of area, Young's modulus) for each member of the frame.

### 11.6.1 Member loads and nodal forces

Read section 4.3.2 indeterminate frames. The same concept for fixed end moments must be introduced for frame elements as for beam elements, see equation (4.18). There is an added provision that is necessary because frame elements may be at any orientation in the $XY$ plane, and not simply aligned with the $X$ axis as in beams. For frames, the distributed(member) load may be either in local $Y'$ direction of the member or in the global $Y$ direction. The first type of load simulates fluid pressure(such as wind loads), and the second the gravity forces acting on the mass of the element. the command for nodal load generation is

FRAMLD B E F C=? D=?

In which nodal data is stored in B, nodal forces in E and UDL member load data in F. The presence of these load types is flagged by C and D.
(0-absent), ($\neq 0$ present). For the distributed loads $D = \pm 1$, (+1 for local pressures), ($-1$ for global pressures). Loads are generated in the array LO, and the command generates and stores the fixed end moments to be used in the calculation of member forces with the

command and,

FRMFRC M V S AX

$M$, $V$, $S$, $AX$ are the member bending moments, shear forces, reactions and axial forces, respectively. A sample command sequence is given in section 4.3.3. Read also section 4.4. Frames for which data may be generated are shown in Figures 4.6, 4.8 and 4.9. See also the examples menu of STATICS-2020. On DATN.DAT the exercise numbers are (43-45), (47-51) and 67. The command for data generation is

FRAMEX E=N D=a,b,c,d

Use of this command can be followed by reading section 4.3.3. or the command sequences on DATN.DAT. Note, $N = (1 - 8)$, 4 or 11 and if negative the supports of the specified frame are pinned rather than fixed.

## 11.7 Tutorial 11.3

**T3.1** The single bay frame in exercise 43 has both distributed member loads and a horizontal force on node 2. 1) In the FRAMLD command, change C=0. Run STATICS-2020. View and draw bending moment diagram
2) In FRAMLD command change C=1, D=0. Run and print S and determine shears in each column.
3) Print axial forces AX
4) Plot deflected shape

**T3.2** Repeat the exercise T3.1 for the exercise 44. Compare results with the single bay frame in T3.1.

**T3.3** The single bay frame in exercise 47 has different column heights, (10,15) respectively, and a uniformly distributed load acts on the inclined top chord. Compare results for deflection and bending moments with results in T3.1. Plot both deflected shape and bending moment diagram.

**T3.4** Single bay gable frame-exercise 48.
1)The gable frame(5) in Figure 4.6(see also exercise menu) has a distributed load on the top left hand inclined member acting vertically. Analyze the frame and draw the bending moment diagram and the deflected shape.
2) Prove that the reactions equilibrate the vertical load.
3) Apply a horizontal force of 100 at the top of the eaves level node (2) and analyze frame drawing bending moment diagram and deflected shape.
4) Repeat (3) with load now at apex node (3) of the frame.

**T3.5** Two span gable frame given in exercise 49
1) Analyze for the vertically applied UDL as given in (47). Draw bending moment diagram and deflected shape.

2) Now apply load of 100 at eave's level node (2) and repeat the analysis as in 1).
3) Compare results in [T3.5] with those in [T3.4] and discuss.

**T3.6** Circular arch segment exercise The circular arch segment shown in Figure 4.9 is given in exercise 67 and is subjected to a radial pressure on all elements.
1) Plot bending moments is arch. What are the support fixity conditions?
2) Plot axial thrust in the arch and compare force with that in a thin circular hoop subjected to the same external pressure.
3) With bases fixed rework 1) and compare bending moment diagrams.

## 11.8 Lectures 5-6 Grid Structures

Indeterminate analysis of grid structures follows the same pattern as for frames in the previous lecture. In Chapter 5 it was shown that a planar structure in Te $XZ$ plane and loaded by forces in the $Y$ direction at right angles to the plane has for nodal forces, moments $(M_x, M_z)$ and transverse force $F_y$. Read sections 5.1 and 5.2. The plan view of a grid is shown in Figure 5.2, member forces and their transformation in Figure 5.3 and nodal moments and force in Figure 5.4 Analysis of grid structures is very similar to that frame structures.

| structure type | nodal forces | member forces |
|---|---|---|
| frame structure | $F_x, F_y, M_z$ | $(M_i, M_j, F)$ |
| grid structure | $M_x, M_z, F_y$ | $(M_i, M_j, M_T)$ |

The transformation of member forces to global force components is given in equations (5.1) to (5.5), and symbolically the resulting nodal equilibrium equations are given in equation (5.7). If the number of rows $<$ number of columns, then the indeterminate analysis must be used. Read section 5.3.1. For the member the stiffness relationship must be determined for the member distortions $(\Delta\theta_T, \phi_i, \phi_j)$ in which $\Delta\theta_T$, is the twist of end $J$ clockwise , relative to end $I$ when viewed from end $I$. For this the stiffness constant $k_T$ for twist must be determined.

$$\left\{ \begin{array}{c} M_T \\ M_i \\ M_j \end{array} \right\}_n = \frac{EI_n}{l_n} \begin{bmatrix} \dfrac{k_n}{I_n} & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} \left\{ \begin{array}{c} \Delta\theta_T \\ \phi_i \\ \phi_j \end{array} \right\}_n = [k_n]\{v_n\} \tag{11.20}$$

The torsion constant is a simple expression for a prismatic member of circular cross section, for which $k_T = (\pi D^4/32)$ for a solid section. Indeterminate analysis follows the steps setout in equations (5.13) to (5.17) and also in equations (11.4) to (11.8). The commands for indeterminate analysis are similar for frame analysis. The load vector is setup with the command,

GRIDLD B E F C=? D=?

The matrix E may contain either concentrated node moments or force, C=1 load present, C=0 no concentrated moments or load. The array F contains member distributed loads, D= loads present D=0, no loads. Fixed end forces are calculated for the nonzero values in F. The two commands for the member and global stiffness calculations are as for frame analysis, GRMSTF and GRGSTF respectively. These are the same commands as in frame analysis however the data they process is different. The IN values are different see Chapter 5 and the EQ matrix has been setup with the different command,

GRIDEQ A B C

## 11.9 Tutorial 11.4

The indeterminate grid structures that have their data generated automatically by the GRIDEX command are in Figures 5.7 (2), Figures (5.9-5.11). See DATN.DAT, problems 53 and 55-57.

**T4.1** The grid analysis in Figure 5.7 (2) is set out in exercise 53, separator G2 in DATN.DAT file. The dimensions used are a=b=c=10 and a load of 900 is applied in the $Y$ direction on node 3. From the output determine the following,

1) Equilibrium of load and reactions.
2) Deflections of node 2 and 3.
3) Shears in members (1) and (3).
4) Bending moments in members (1) and (3).
5) Twist moments in members 1,2,3.

**T4.2** The grid shown in Figure 5.9 exercise 55, G4 represents a series of beams 1 to 4 supported at nodes 1-4 and connected by a beam continuous with them through nodes 5-8. The problem will be how much load is shared between the beams 1-4, for example for a concentrated load applied at any of the nodes 5 to 8. In exercise a load of 800 is applied on node 8. All beams have the same stiffnesses (20,10) whereas the transverse beams have the values (5,2.5).

1) Draw vertical deflected shape of nodes 5,6,7,8.
2) From the shear forces in members 1-4, determine how much load is carried by each cantilever.
3) Check that the reactions and loads are in equilibrium.

**T4.3** Repeat the exercise [T4.2] first changing the load to be applied at node 7.

**T4.4** The grid in Figure 5.10 (6) exercise 56, separator G5 on DATN.DAT file is a typical bridge deck system. Main girders 4-11, 5-12 and cross girders, 1-2-3, 8-9-10, 15-16-17. The problem is to determine the load sharing between the girders because of the cross girder stiffness. In this example the girders 8-9-10 will be the most active in this regard. A load of 900 is applied in the $Y$ direction at node 8.

1) Draw vertical deflected shape of nodes 5,6,7,8.

2) Check that the reactions equilibrate applied load.

3) From the reactions, (1-9)-(2-10), (3-11) and (4-12) ascertain how the load is shared between the longitudinal girders (1-9),(2-12), (3-11) and (4-12).

4) Determine maximum moment as a function of $(Wl)/4$.

**T4.5** The grid shown in Figure 5.11 (9) G6 separator on DATN.DAT is typical of many highway bridges in that the girders are at an angle of $20^o$ to the right span. Although the clear span given in Figure 5.11 is 30 m, and thus the same as for G5, the skew span is however now equal to $30 \cos 20 = 31.925m$ The same load of 900 is applied in the $Y$ direction at node 8. Analyze this grid and,

1) Draw vertical deflected shape of nodes 5,6,7,8.

2) Check that the reactions equilibrate applied load.

3) From the reactions, (1-9)-(2-10), (3-11) and (4-12) ascertain how the load is shared between the longitudinal girders (1-9),(2-12), (3-11) and (4-12).

4) Determine maximum moment as a function of $(Wl)/4$.

5) In all cases (1)-(4) compare with the corresponding results for [T4.4].

**T4.6** Repeat exercise [T4.4], changing the load in DATN.DAT to be applied at node 7.

**T4.7** Repeat exercise T4.5, changing the load application point in DATN.DAT to be node 7. Compare all corresponding values with those obtained in [T4.6].

## 11.10 Lectures 7-8 Geometric stiffness Stability

This topic has been considered in Chapter 7 and the lectures in this Course III, refer to the various sections therein. Read section 7.1. The basic concept of geometric stiffness is developed for a single element in section 7.1.1, Figure 7.1. In Figure 7.1, the equilibrium equation is written, 1) in the undeformed position and 2) the deformed position. It is seen that for small deflections an additional term linear in the node displacement, and depending on the axial force in the member $(P/l)$ is introduced. This term is present only when the equilibrium equation is written in the deformed position and it has the property that it can make the total horizontal stiffness at the node tends to zero, see Figure 11.2. As shown in Figure 11.1, $(k - P/l)$ may be made to approach the horizontal axis and this coincides with $P_{cr}$,

$$P_{cr} = kl \tag{11.21}$$

Read section 7.1.1 and note to zero is linear in both $k$ and $l$. The purpose of this lecture is to extend the idea in section 7.1.1 and Figure 11.2 to structures for which structural properties (stiffness and/or flexibility) are distributed rather than concentrated, as is the spring at the node in Figure 11.1. To do this a simply supported beam is studied. A simply supported beam of uniform cross section, possessing a plane of symmetry(or principal plane) $XY$ and loaded in this plane is shown in Figure 7.2 has an initial imperfect shape characterized by a function $y_0(x)$. The problem to calculate the deflected shape is solved in section 7.1.2 in which the differential equation of the deflected shape is set up in equation (7.6). Read section 7.1.2, the solution for the critical or buckling load is,

$$y = A\sin kl\,; \quad k^2 = \frac{P}{EI} \tag{11.22}$$

and the constant $A$ is evaluated to satisfy the boundary condition $y = 0$ at $x = l$. This leads to the critical load being expressed as,

$$P_{cr} = \frac{n^2\pi^2 EI}{l^2} \tag{11.23}$$

There are a infinite number of solutions starting with the lowest $n = 1$, and this is called the Euler load (after Leonard Euler who first solved this problem). The equation (11.23) is useful in its own right but also as a means of checking numerical techniques for approximate determination of critical loads of structures for which analytical solutions are not possible. The remainder of this lecture course is devoted to this numerical calculation of critical loads. To calculate the linear effects of axial forces on beam members for small displacements from the equilibrium position it is necessary to study beam deflections in greater detail than in Course II. The theory applies to deflections in either of the principal planes of a prismatic beam member and here assumed to be the $XY$ plane.

## 11.11 Tutorial 11.5

**T5.1** Using equations 7.13 and 7.9 sketch the mode shapes for $n = 1, 3, 5, ...$ and also $n = 2, 4, 6, ....$ Explain the concept of symmetric and antisymmetric shapes referring to equation(7.9).

## 11.12 Beam deflections

Read sections 7.1.3 and 7.1.4 In this section the calculation of beam deflections is studied following Course II, section 10.9. Note the use of $[b]$ and $[\bar{b}]$ in equation (7.6), in which $[b]$ refers to the loaded node $W$ and $[\bar{b}]$ to the end moments $(M_i, M_j)$. For two such load and displacement groups, partition the matrices accordingly.

$$\begin{Bmatrix} r \\ \bar{r} \end{Bmatrix} = \begin{bmatrix} b^T \\ \bar{b}^T \end{bmatrix} [f]\,[\,b\;\bar{b}\,] \begin{Bmatrix} R \\ \bar{R} \end{Bmatrix} = \begin{bmatrix} b^T f b & b^T f \bar{b} \\ \bar{b}^T f b & \bar{b}^T f \bar{b} \end{bmatrix} \begin{Bmatrix} R \\ \bar{R} \end{Bmatrix} \tag{11.24}$$

Thence with $\bar{R} = 0$,

$$\{\bar{r}\} = [\bar{b}^T][f][b]\{R\} \tag{11.25}$$

here,

$$\{\bar{r}\} = (\phi_i\ \phi_j)^T; \quad R = W \tag{11.26}$$

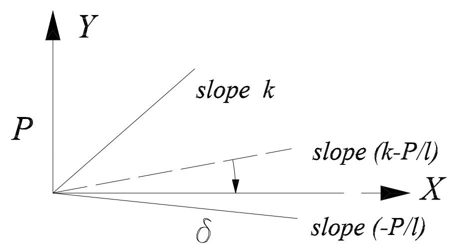


Figure 11.2: Elastic, geometric and tangent stiffnesses

Study and prove examples I and II in Chapter 7. In these sections two basic deflection problems are solved in preparation for the study of the effect of axial forces on deflection calculations.

1) Given a load $W$ applied at a point $(\zeta_1\, \zeta_2)$ on a beam calculate the values of the end rotations, $(\phi_1\, \phi_2)$. Then apply this to various load cases, section 7.1.3.

2) Given end moments $(M_i\, M_j)$ calculate the magnitude of the deflection $v_x$ at a point $x$ of the beam, section 7.1.4

3) From (2) calculate $v_x$ in terms of end rotations, equation (7.27).

Read section 7.1.4.

Now equation is used with $\{R\} = (M_i\ M_j)^T$, $\bar{R} = W$ and $v_x = v_\zeta$. Prove the equation (7.25) Use the stiffness matrix of the beam to calculate $v_x$ in terms of end rotations $(\phi_i\, \phi_j)$, see equation (7.27).

## 11.13 Tutorial 11.6

**T6.1** A simply supported beam, length $l$ has 3 nodes 1,2,3 and two members lengths $(a, b), a + b = l$. The left hand member is loaded with a linearly varying distributed load, w/unit length at node 1 and 0 at node 2. Use the equation

$\bar{r} = [\bar{b}]^T([f][b]w + v*)$

to calculate the vertical deflection of node 2. (use section 7.1.3 and equation (7.19) in the calculations.

## 11.14 Lecture 10

Read section 7.1.5. From the theory developed in section 7.1.4, the end rotations may be calculated including the bending moment induced by the axial force $P$(its first order approximation). See equation (7.28) and understand the use of equation (7.29). It is seen in equation (7.31), the axial force $P$, then produces a linear approximation to the relationship between the increment in the nodal end rotations $\Delta\phi_P$ and the end rotations $(\phi_i\ \phi_j)$ equation (7.31) and combined with the elastic deflections, the resulting equation(7.32), Study equations (7.33-7.36). Equation (7.36) may also be obtained directly from equation(7.32) by collecting terms of $(\phi_i\ \phi_j)$ and multiplying by the beam stiffness matrix. Now develop the eigenvalue equation (7.37) with $P = \lambda$ and hence obtain equation (7.39), the approximation to the Euler load of 12. This approximation is 21% in error and is considered too inaccurate.

## 11.15 Lecture 11-rotation of member chord

In lecture 10 it was shown that if the deflected shape is approximated by a single member whose shape is quadratic in form the approximation to the sine curve in equation (7.12) is rather poor and hence the large error in the Euler load approximation. The situation is improved by providing additional nodes along the beam and thus introducing more degrees of freedom that give a better approximation to the sine curve. In effect the idea combines the rigid body rotation of a member as given in Figure 7.1 and section 7.1.1. Read section 7.1.6, in which the rotation of the member is combined with the deflection relative to the chord in equation (7.42). Using the displacement transformation matrix in equation (7.44) the expression for the member geometric stiffness is given in equation (7.46),

$$\{\Delta R_G\}_i = P[A]_i[k_G]_i[A]_i^T \tag{11.27}$$

This is the same form as the elastic stiffness and so can be generated at the same time. Read section 7.1.6 and develop the transformations that lead to equation (7.56). Hence complete the section arriving at the critical (Euler) load approximation using one internal node,

$$P_{cr} = 9.945$$

The error is now only 0.7% which is sufficiently accurate as the quantity $EI$ is probably not known to this accuracy. See Figure 7.6 for the calculated node shape.

## 11.16 Lecture 13

The command sequences for the calculation of structure critical loads are now discussed. These are available for beam and frame structures and follow a similar procedure as for elastic stiffness. The axial force values, must be input for members, positive compression to agree with sign convention in equation(7.84). Read section 7.3 for beam geometric stiffness and section 7.4 for frame geometric stiffness.

| **beam command** | | | | **frame command** | | | |
|---|---|---|---|---|---|---|---|
| LOADR | B1 | R=? | C=? | LOADR | B1 | R=? | C=? |
| (member axial forces in B1) | | | | (member axial forces in B1) | | | |
| BMMSTF | B | B1 | MG | FRMSTF | B | B1 | MG |
| BMGSTF | GEQ | MG | KG | FRGSTF | GEQ | MG | KG |

Following the calculation of $K_G$, the trial mode shape R0 must be input. Then the command CRITLD performs the iteration and produces the converged mode shape in R0 and eigenvalue in LA. The sequences for beams and frames are:

| **Determinate structure** | | | | | **indeterminate structure** | | | | |
|---|---|---|---|---|---|---|---|---|---|
| LOADR | R0 | R=? | C=? | | LOADR | R01 | R=? | C=? | |
| (approximate mode shape in R0) | | | | | (approximate mode shape in R0) | | | | |
| CRITLD | FL | KG | R0 | LA | CRITLD | K | KG | R0 | LA |

## 11.17 Lecture 12 Eigenvalue calculation

In lecture 11 the section 7.1.6 was studied and the Example III worked to produce the improved approximation to the Euler load for the simply supported beam. It should be noticed that the numerical method may be extended to a variety of problems. For example if in Figure 7.5 the element (2) carried a different axial force,to the element (1) this can be easily taken into consideration by changing the member $P$ force for (2) and hence the $[k_G]$ matrix in equation (7.49). Note also that for the small size of eigenvalue problems encountered in the examples studied the equation (7.53) can be rearranged as in equation (7.54) so that the eigenvalue problem in equation (7.55) is solved for the value $1/\lambda = 1/P$, so that the smallest $P$ gives the largest eigenvalue. Iteration to $1/\lambda$ thus yields the Euler load. Before studying more detailed calculation of eigenvalues and buckling modes for beam and frame structures the theory for the iteration of the eigenvalue to the largest root is discussed. The method used is suitable in that it applies to equation (7.55) when the matrix $[A]$ is un symmetric. The theory is given in section 7.2 and the equation (7.58), $\lambda$ is now substituted for $1/P$. The essence of the theory is given in equations (7.63) to (7.67) in which it is shown that the eigenvectors of the matrices $[A]$ and $A^T$ are orthogonal, and may be normalized so that their lengths are equal to unity. That is,

$$\begin{aligned} \{X_r^T\}\{X_s'\} &= 0 \quad (r \neq s) \\ \{X_r^T\}\{X_r'\} &= 1 \end{aligned} \tag{11.28}$$

See the proof in section 7.2. These orthogonality conditions result in the dominance of the first mode shape that corresponds to the largest $\lambda$( smallest $P$), when the inner product as in equation (7.72).

## 11.18 Tutorial 11.7

Exercises in beam buckling in which the effects of loaded lengths, part tension and part compression,and variation of member stiffnesses can be studied using exercise 29, B12 which is a simply supported beam with 4 elements and 3 internal nodes. In B12 the axial force is the same in all 4 members $(1,1,1,1)$. Then the choice of the starting vector on the iteration R0 $= (0,0,0,0,1,0,0,0,0,0)^T$ is suitable. However if the axial forces in the members are $(1,1,-1,-1)$ this starting vector is incorrect because it contains no antisymmetric component and an incorrect result is obtained. Now the vector R0 $=$ $(0,0,1,0,0,0,0,0,0,0)$ will produce correct results. Six exercises are given to illustrate the effects of changing one of the parameters at a time. All these exercises are run with SUBMIT B12 command.

**T7.1** Run exercise 29 and prove that the critical load ($P_{cr} = \pi^2 EI/l^2$ $l = 10, E = 100, I =$ 1 is approximated by 9.875, ($\pi^2 = 9.870$). Draw the mode shape.

**T7.2** Edit DATN.DAT file and copy exercise 29 to DATN1.DAT. Now edit DATN1.DAT, changing B1 to $(1,1,-1,-1)$ and RO to $(0,0,1,0,0,0,0,0,0,0)$. Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.3** Edit DATN1.DAT file and change B1 to $(1,1,2,2)$ and RO as in T7.2.Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.4** Edit DATN1.DAT file and change B1 to $(1,1,0,0)$. and RO as in T7.2. Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.5** Edit DATN1.DAT file and change IN to $(10,1,1/2,1/2,1)$ and RO as in T7.1. Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.6** Edit DATN1.DAT file and change IN to $(10,1,2,2,1)$ and RO as in T7.1. Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.7** Edit DATN1.DAT file and change IN to $(10,1,1,2,2)$ and RO as to that in T7.2. Run STATICS-2020 using DATN1.DAT as the input file. Print the value of critical load (LA) and draw mode shape.

**T7.8** Make a comparison of all values calculated in T7.1 to T7.7 and annotate your results with reasons for the different critical load values.