Chapter 3

BEAM ANALYSIS

3.1 Introduction

The analysis of beam structures is an exercise in the second transformation of forces introduced in Chapter 1, section 1.2.2, equation (1.6). This transformation is also the basis for the calculation of the reactions of planar rigid bodies supported in a statically determinate manner. This topic will be studied as a prelude to beam analysis.

3.2 Rigid body reactions

3.2.1 Command sequence

A planar rigid body in the $X-Y$ plane is supported with three reactions so that it is stable and determinate is subjected to force and moment components see Figures 3.1 an 3.2.
The command:

REACTN A B C

sets up statically equivalent forces at the origin for loads applied to the planar rigid body and the (3×3) matrix that transforms the three specified reaction components to their equivalent forces at the origin. The input matrix A contains all information concerning reaction force locations. The statically equivalent forces are generated in B and the reaction transformation matrix is in C. Then the reactions are calculated by two additional matrix commands and displayed on the screen by the print command as follows:

INVERT C T=2
MULT C B R
PRINT R

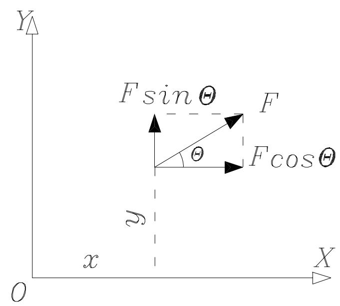


Figure 3.1: components F_x, F_y at x, y

3.2.2 Transformation theory

The necessary theory to set up the transformations is given below. For this purpose a force is described by, (see Figure 3.1),

1. Its magnitude.
2. The angle of its application (measured counterclockwise positive from the X axis).
3. Its point of application, in (X, Y) coordinates.

The transformation to equivalent force and moment components at the origin of any force F, inclined at an angle θ to the X axis is given by equation (3.1), see Figure 3.1,

$$\{F_o\} = \left\{ \begin{array}{c} F_x \\ F_y \\ M_z \end{array} \right\} = \left[\begin{array}{c} \cos\theta \\ \sin\theta \\ -x\cos\theta + y\sin\theta \end{array} \right] F \tag{3.1}$$

and for a unit value of a force reaction the corresponding column in [C] is given by,

$$\left[\begin{array}{c} \cos\theta \\ \sin\theta \\ -x\cos\theta + y\sin\theta \end{array} \right] \tag{3.2}$$

If a moment is applied, or a reaction component is a moment, the transformation is,

$$\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] \tag{3.3}$$

for both cases. Finally, the conditions of equilibrium are expressed as,

$$\sum\{F_{io}\} + [C]\{R\} = 0 \tag{3.4}$$

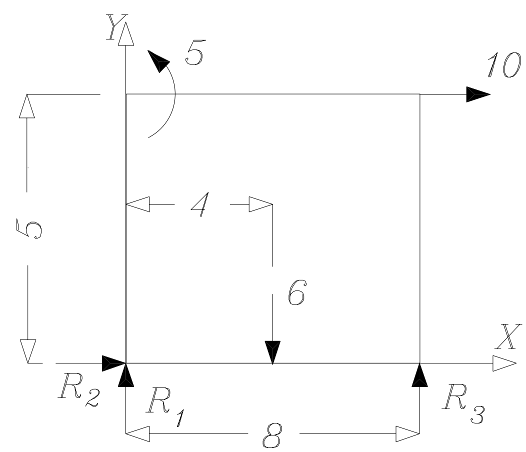


Figure 3.2: Rigid body reactions

These equations are solved for the reactions $\{R\}$ so that,

$$\{R\} = -[C]^{-1} \sum \{F_{io}\} \tag{3.5}$$

In the [A] matrix the first column gives the applied force at the (x, y) coordinates given in the corresponding (3,4) columns of the same row. The angle of a force to the $x-$axis, measured positive counter-clockwise sense is given in column 5 for the corresponding force. For an applied moment about the z-axis, only the moment, sign counter clockwise positive is required in the second column, all other columns in the row being zero. To calculate the reactions for a set of forces and reactions all relevant information is first stored in the matrix $[A]$. Consider for example the rigid body shown in Figure 3.2. The data required is stored in the (6×5) matrix given in the table below. The last three rows being the reaction data.

10	0	8	5	0
6	0	4	0	−90
0	5	0	0	0
1	0	0	0	90
1	0	0	0	0
1	0	8	0	90

The sequence of operations to calculate the reactions is given below.

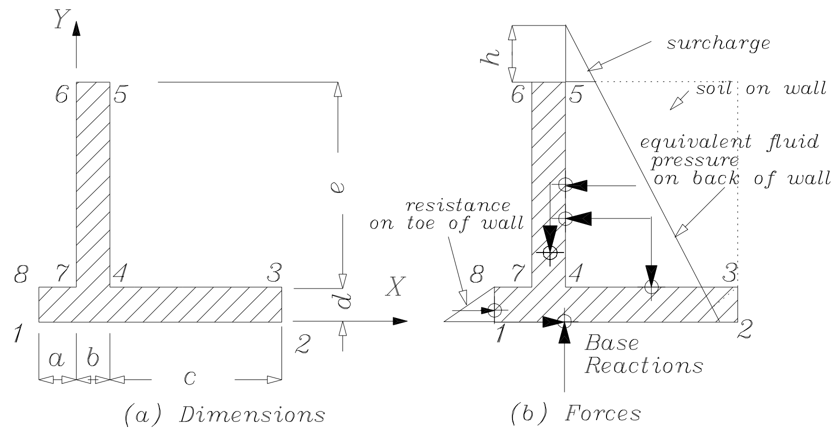


Figure 3.3: Retaining wall, rigid body, base reactions

LOADR	A	R=6	C=5
	(DATA)		
REACTN	A	B	C
INVERT	C	T=2	
MULT	C	B	R
PRINT	R		
RESULT:	R_1	=	−2.625
	R_2	=	−10.00
	R_3	=	8.625

The forces and the reactions acting on the body may be displayed with the command,

PLTRBD A R

Applied forces are shown in red, reactions in blue.

3.2.3 Retaining wall reactions-foundation pressure

As an elementary example of the use of rigid body reaction theory in section (3.2.1) the stress distribution on the base of a contilever retaining wall subjected to vertical gravity loads and assumed horizontal pressure of an equivalent fluid is calculated. The chosen dimensions in Figure 3.3 are defined by the five user input parameters (a, b, c, d, e). The command

RETWAL A1 D=?,?,?,?,?

generates the (8×2) matrix A1 of (X, Y) coordinates measured from the toe of the wall, node 1 shown in Figure 3.3(a). The loading parameters on the wall are given by,

H = surcharge

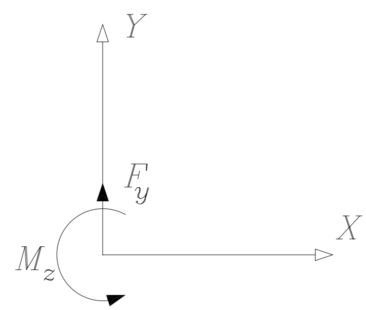


Figure 3.4: Positive nodal forces on beam node.

w1 = density in Kg of concrete in wall
w2 = density of soil above the rear nib of the wall
p1 = equivalent fluid pressure in rear faces of wall
p2 = equivalent fliud resisting pressure on front nib of wall

First the area $1, 2, ..., 8$ and its centroidal coordinates relative to node 1 are determined using the two section commands, PERIM and PROPER. In PROPER command D is a dummy array for the second moment of area and is not used here. Thus the next step involves the two commands,

PERIM A1 M =1 N=8 S=1
PROPER A1 B1 C1 D

Area is in B1 and coordinates (XC, YC) of the centre of mass of a unit length of the wall are in C1. the coordinates in A1 are now referenced to the centroid. The next step is to calculate the various loads that act on the wall that must be equilibrated by the vertical and horizontal forces on the base. These are first calculated by considering three reactive components (H, M, V) at node 1 and then transfering to the point in the base for which $M = 0$. these forces are calculated by first loading in the five load parameters give above with the LOADR command.

LOADR L R=1 C=5
(H, w1, w2, p1, p2)

This command sets up the (8×5) matrix described in section (3.2.2). All forces are referenced relative to node 1. Then the command REACTN,

REACTN A1 B1 C1

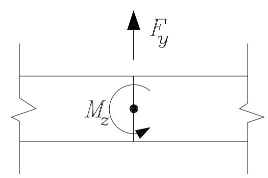


Figure 3.5: Node in beam structure.

generates the statically equivalent forces at node 1, and the two commands,

INVERT C1 T=2
MULT C1 B1 R

calculate the reactions at node 1 necessary to equilibrate the active forces acting on the wall. Finally the command,

SHFTRN A1 A R S

locates the centre of pressure from 1 on the base in $S(3)$, and gives the assumed linear stress distribution in $S(1), S(2)$. If the distance in $S(3)$ is less than 1/3 the base width the stress distribution is triangular and is caculated on the assumption that no tensile forces act on the base. An example of the above analysis is given in DATN.DAT, number 16 with the separator marker R2.

3.3 Analysis of beams for bending moment and shear force

3.3.1 Theory for beam elements

The basic equilibrium equation for a beam node is the same form for the beam as for the truss. That is,

$$\text{member force} - \text{reaction} = \text{applied force} \tag{3.6}$$

Now however, the member forces are different from those for the simple truss bar member, because the transverse loads on the beam must be equilibrated by shear forces in the beam, which in turn produce bending moments in the elements of the beam. Thus, the equilibrium equations considered for the beam node will be, see Figure 3.4,

$$\sum F_y = 0 \tag{3.7}$$

$$\sum M_z = 0 \tag{3.8}$$

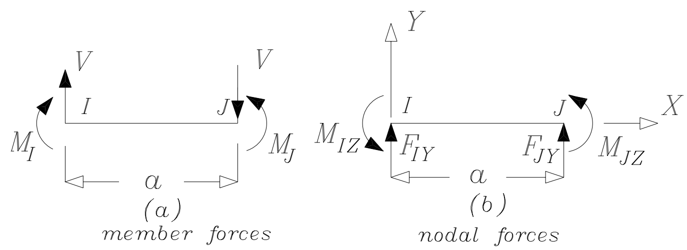


Figure 3.6: Beam member forces-local (global) components.

Also, the node for the beam is not easily identified as the "pin" joint of the truss. The beam node is simply an infinitely thin, rigid slice of beam at the node section, as shown in Figure 3.5.

The equations $\sum F_x = 0$ will be assumed to be satisfied automatically because no axial forces are applied to the beam. In this theory then, the nodal force variables will be,

$$\left\{ \begin{array}{c} F_y \\ M_z \end{array} \right\} \tag{3.9}$$

Moments acting on the node will be considered positive, taken in an anticlockwise direction, see Figure 3.4. On the beam element the sign convention will be that bending moments causing tension on the lower, (negative Y), fibre will taken as positive (see Figure 3.6(a)). The basic member forces on the beam element, shown in Figure 3.6(a) will be the end moments (M_I, M_J), that in turn generate a shear force V, that is expressed in terms of these two moments. The relationship between the end moments and the shear on the beam element are calculated by considering the equilibrium of the element in Figure 3.6(a). That is, taking moments about end I (anticlockwise positive),

$$\sum M_z = 0 \qquad -Va - M_I + M_J = 0 \tag{3.10}$$

This equation gives the expression for the shear force to be,

$$V = \frac{1}{a}(-M_I + M_J) \tag{3.11}$$

Thence the three quantities, (V, M_I, M_J), are together written as,

$$\left\{ \begin{array}{c} V \\ M_I \\ M_J \end{array} \right\} = \left[\begin{array}{cc} -\frac{1}{a} & \frac{1}{a} \\ 1 & 0 \\ 0 & 1 \end{array} \right] \left\{ \begin{array}{c} M_I \\ M_J \end{array} \right\} \tag{3.12}$$

The nodal forces are shown in Figure 3.6(b). Using the Figure 3.6(b), together with equation (3.12), it follows that all the forces on the ends of the member in global coordinates

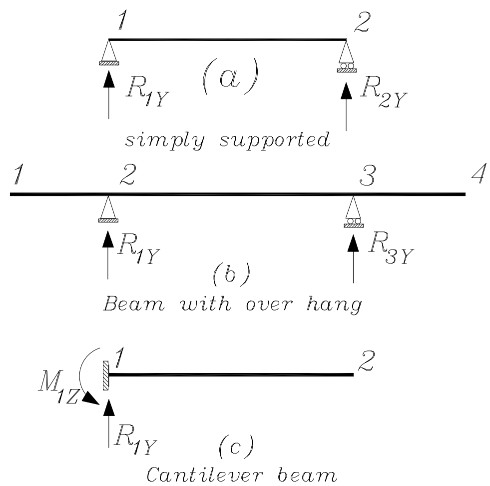


Figure 3.7: Beam reaction types.

are given,

$$\begin{Bmatrix} F_{Iy} \\ M_{Iz} \\ F_{Jy} \\ M_{Jz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} V \\ M_I \\ M_J \end{Bmatrix} = \begin{bmatrix} -\frac{1}{a} & \frac{1}{a} \\ -1 & 0 \\ \frac{1}{a} & -\frac{1}{a} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} M_I \\ M_J \end{Bmatrix} \tag{3.13}$$

It is seen that only (M_I, M_J) contribute to the joint equilibrium equations. The components on the left hand side of equation (3.13), are the member forces which now enter directly into the equilibrium equations (3.7).

3.3.2 Beam reactions

Reactions can be either a transverse force R_{cY}, or a moment M_{cz} and for statically determinate beams are 2 in number unless the beam has internal hinges. Typical simple beams and their reactions are shown in Figure 3.7. If the beam is continuous then it can be made statically determinate by introducing the appropriate number of intermediate internal hinges. Examples of such possibilities are shown in Figure 3.8. For the cases shown in Figure 3.7, the relationship holds,

$$2\mathrm{J} = 2\mathrm{M} + 2, \qquad \text{or} \quad \mathrm{J} = \mathrm{M} + 1 \tag{3.14}$$

For the cases in Figure 3.8, if NH is the number of hinges and NR the number of reactions,

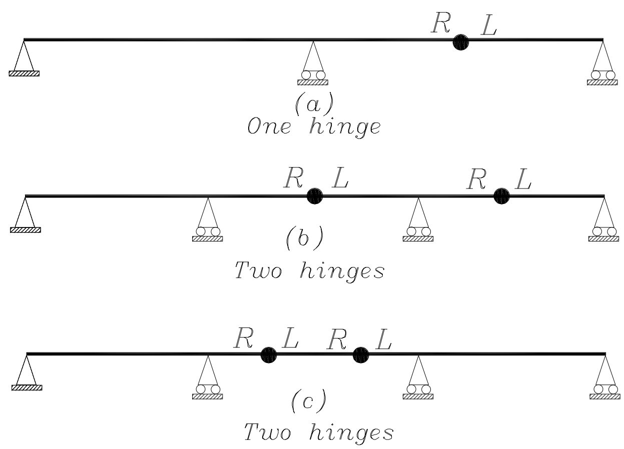


Figure 3.8: Statically determinate beams with intermediate hinges.

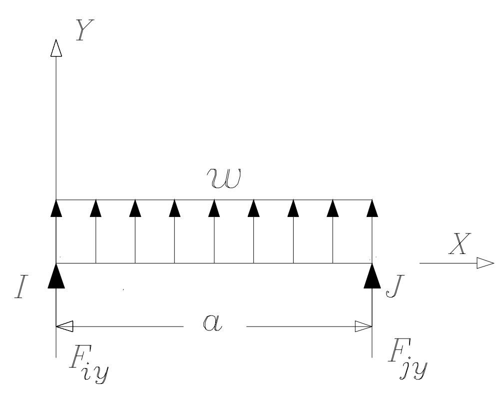


Figure 3.9: Uniformly distributed load on beam element.

then the relationship becomes,

$$2\mathrm{J} = 2\mathrm{M} + \mathrm{NR} - \mathrm{NH} \tag{3.15}$$

For the internal hinge, the moments in the elements to left and right of the hinge are independent, so that at the hinge there are now three equilibrium equations, for

$$\begin{Bmatrix} F_y \\ M_L \\ M_R \end{Bmatrix} \tag{3.16}$$

From the beam member force components, equation(3.13), the internal hinge conditions and the reaction types and locations the nodal equilibrium equations are established as

$$[A]\{S\} = \{R\} \tag{3.17}$$

and for determinate structures the force transformation matrix is the inverse of $[A]$, that is,

$$\{S\} = [A]^{-1}\{R\} \tag{3.18}$$

3.3.3 Distributed loading on beam element

Beam elements may also be subjected to uniformly distributed load as shown in Figure 3.9. The equivalent nodal forces at I and J are simply one half of the total load wa for the element of length a. That is, the nodal forces for w positive upwards are:

$$\frac{wa}{2} \quad \text{and the end shears are} \quad V_I = \frac{-wa}{2} \quad V_J = \frac{wa}{2} \tag{3.19}$$

and the centre bending moment is given,

$$M_C = \frac{-wa^2}{8} \tag{3.20}$$

See Figure 3.10 for the bending moment within the element and the end shears from the distributed load only. In the beam analysis commands, both concentrated nodal forces and distributed loads on members can be used. Member bending moments are then calculated for each member, with the three values,

moments M_L M_C M_R
shears V_L V_R

The STATICS-2020 software provides three routines for statically determinate beam analysis. Six examples, (see Figure 3.15), are available using the command,

BEAMEX E=? L=?

In this command the beam span L is given by the user input. The basic command for beam analysis is,

BEAMEQ A B C [D]

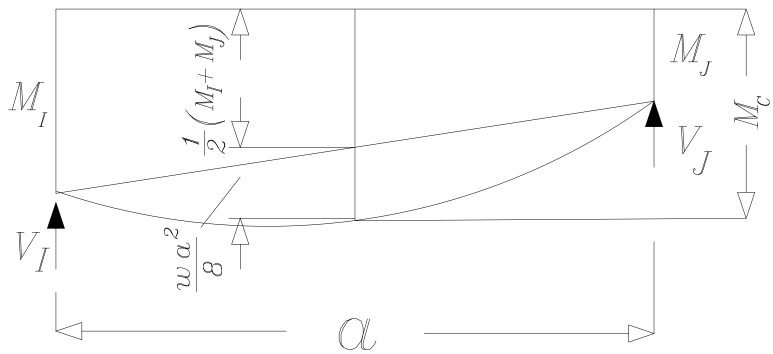


Figure 3.10: Bending moment and end shears due to U.D.L on element.

This routine sets up the equilibrium equations in the program defined array EQ using the following values,

A	=	row matrix of node x coordinates
B	=	element node numbers $(I - J)$ one row per element
C	=	reaction matrix (node-reaction-type 1=force or 2=moment) one row per reaction component
D	=	row matrix of hinge node numbers if present (optional)

If the beam is determinate the equilibrium equations are inverted within the command BEAMEQ. The command BEAMLD, sets up the nodal load vector, using the matrices A and B, and two additional matrices E and F defined as follows,
E stores concentrated load data by rows with the following information:

node load type(1 = force, 2 = moment) magnitude

F stores any distributed load values, each value being the load intensity on the corresponding element and zero values must be included. Thus the command is,

BEAMLD A B E F C=? D=?

E and F should be present in the command irrespective of whether or not the load group is present. C=? gives the flag to indicate concentrated load. C=1 present, C=0 no concentrated loads. D=? gives the flag to indicate distributed loads on members. D=1, present, D=0 no distributed loads. Loads generated by the BEAMLD command are stored in the program defined array LO. Having setup the load vector, member bending moments, shear forces and beam reactions are calculated the command,

BEAMMO M V S

Moments (M_I, M_C, M_J) are stored member by member in M, shears (V_I, V_J) are stored in V and reactions R_{cy} or M_z are stored in S. The sequence of STATICS-2020 commands to analyse a beam structure takes the form:

LOADR	A	R=1	C=?			nodal coordinates
LOADI	B	R=?	C=2			$(I - J)$ for each member
LOADI	C	R=?	C=2			reaction, (node, reaction type)
BEAMEQ	A	B	C			equilibrium matrix EQ and (inverse in EQ)
LOADR	E	R=?	C=3			node, concentrated load, concentrated moment(if present)
LOADR	F	R=1	C=?			distributed loads (if present)
BEAMLD	A	B	E	F	C=?	D=? Sets up load vector LO
BEAMMO	M	V	S			Calculate M, V S

These values of M, V, S are calculated from the LO load vector generated by BEAMLD. The values may be obtained by using the PRINT command,

PRINT array name (M,V or S)

The beam geometry, the bending moment and the shear force diagrams may be plotted using the PLTBEM command as follows:

PLTBEM A C (M or V) N=1,2 or 3

If N=1, then only the beam geometry is plotted and (M or V) is not required. If N=2 and M is present the bending moment diagram is plotted and if N=3 and V is present the shear force diagram is plotted.

3.3.4 Statically indeterminate beams

As for planar trusses, when the equilibrium matrix $[A]$ is set up for the beam nodes, that is,

$$[A]\{S\} = \{R\} \tag{3.21}$$

Then the beam is indeterminate when the number of rows (equations) is less that the number of columns (member forces + reactions). When this occurs the stiffness method of analysis is used to determine the $[b]$ matrix. For an individual member subjected to end moments M_i, M_j, shown positive in Figure 3.11, the stiffness relationship between these moments and the end rotations (ϕ_i, ϕ_j) relative to the chord is given by,

$$\left\{ \begin{array}{c} M_i \\ M_j \end{array} \right\} = \frac{2EI}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} \tag{3.22}$$

Reaction stiffness is set to a large number, e.g. 10^{20}. For all members, including reactions, the member stiffness matrices can be grouped in $[k]$ as diagonal submatrices so that,

$$\{S\} = [k]\{v\} \tag{3.23}$$

Using equation (3.21), contragredience gives the displacement transformation to be,

$$\{v\} = [A]^T\{r\} \tag{3.24}$$

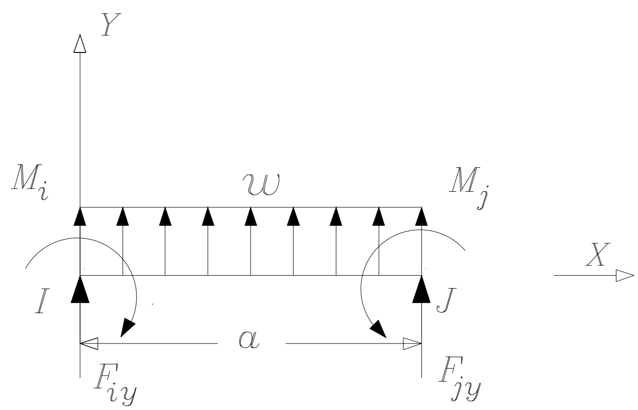


Figure 3.11: U.D.L. with fixed end moments on beam element

Combining equations (3.21), (3.23) and (3.24), determines the structure stiffness matrix, $[K]$

$$[A][k][A]^T\{r\} = \{R\} \quad \text{or} \quad [K]\{r\} = \{R\} \tag{3.25}$$

If distributed loads, (in the present analysis uniformly distributed on the whole element), are present then in the case of zero node displacements

$$\{r\} = 0 \tag{3.26}$$

there will be fixed end forces $\{S^*\}$ on these members with U.D.L. Then,

$$[A]\{S^*\} = \{R^*\} \tag{3.27}$$

The negative of $[R^*]$ must be applied, so that displacements calculated from the initial position, $\{r\} = 0$ are given by,

$$\{r\} = [K]^{-1}\{\{R\} - [A]\{S^*\}\} \tag{3.28}$$

The force transformation matrix for the indeterminate structure is given by,

$$\begin{aligned}\{S\} &= [k][A]^T[K]^{-1}\{R\} \\ &= [b]\{R\}\end{aligned} \tag{3.29}$$

so that,

$$[b] = [k][A]^T[K]^{-1} \tag{3.30}$$

This matrix is calculated by software commands in STATICS-2020 and from equation (3.29) member forces calculated remembering that the fixed end moments must be included. This is done automatically by the BEAMMO command. That is member forces, moments and reactions are given,

$$\{S\}_{total} = [b]\{R\} + \{S\}^* \tag{3.31}$$

For uniformly distributed load over the whole member S^* is given,

$$\{S^*\} = \frac{wa}{12}\left\{\begin{array}{c} -6 \\ -a \\ 6 \\ -a \end{array}\right\} \tag{3.32}$$

3.3.5 Commands available for indeterminate beam analysis

In the command BEAMLD for the calculation node loads, that is,

BEAMLD A B E F C=? D=?

the distributed load magnitudes have been stored in the array F and are used in conjunction with D=1, and fixed end moment release forces are also generated. For indeterminate analysis by the stiffness method, it is necesssary to read in beam member inertias and Young's modulus as follows,

LOADR IN R=? C=1

R= (1 + number of members), the first value being the Young's modulus, which for force calculations can simply be taken equal to unity. Then member stiffness matrices are calculated in the matrix MS with the command,

BMMSTF B IN MS

The nodal stiffness matrix is calculated with the command BMGSTF which also generates $[b] = [k][A]^T[K]^{-1}$ stored in the program defined matrix KA. The command is identical now for the indeterminate truss analysis. The matrix A in equation (3.30) is the program defined array EQ. That is,

BMGSTF EQ MS K

When the command,

BEAMMO M V S

is given, it uses KA together with the load vector and any distributed loads to calculate the member bending moments (M), member shear forces (V) and beam reactions (S).
To calculate nodal deflections R, multiply the load vector LO by the flexibility matrix K, that is

MULT K LO R

The beam may be plotted with the command,

PLTBEM A B C (M, V or R) N=1,2,3 or 4

When only the plot of the beam dimensions, nodes, members and reaction types and locations is required, N=1 is used and M or V are omitted.
To plot the bending moment diagram, use M with N=2
and to plot the shear force diagram use V with N=3

To plot the deflected shape use R with N=4
. Bending moments are calculated at element end and mid points and with the sign convention given in Figure 3.6.

3.4 Calculation of deflections

3.4.1 Beam deflections-theory

Because the force variables at each beam node are the transverse force F_Y and the moment M_z about the z axis, beam deflections at each node will be, transverse displacement and rotation respectively.

The same equations as in section 2.2.5 for truss deflection calculations, still apply. However now the member flexibility matrix for the end moments (M_i, M_j) is given by,

$$[f_i] = \frac{l_i}{6EI_i}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.33}$$

This relationship requires a linear variation of M between member nodes $(i - j)$. Then,

$$\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{l_i}{6EI_i}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \tag{3.34}$$

Now if distributed load, w_i is also present on the member, there will be additional rotations, v_i^* given by,

$$v_i^* = \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{-l_i^3}{24EI_i}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} w_i \tag{3.35}$$

Thus for the node forces,

$$\{S\} = [b]\{R\} \tag{3.36}$$

Now the nodal deflections $\{r\}$ are given using the contragredient law,

$$\{r\} = [b]^T\{v_s + v^*\} \tag{3.37}$$

That is,

$$\{r\} = [b]^T\{[f][b]\{R\} + v^*\} \tag{3.38}$$

Finally,

$$\{r\} = [b]^T[f][b]\{R\} + [b]^T\{v^*\} \tag{3.39}$$

The beam flexibility matrix for the nodal forces is given,

$$[F] = [b]^T[f][b] \tag{3.40}$$

The above theory is true for both determinate and indeterminate sructures. Here it is only used for determinate structures because in the process of calculating [b] for indeterminate structures the flexibility matrix has been calculated in K and this may be used for calculating deflections.

In STATICS-2020, both concentrated loads on nodes and uniformly distributed loads on members are considered, and two commands are introduced to calculate the flexibility matrix and the nodal deflections. The load vector is calculated in the beam analysis program and so the user must start using the BEAMEQ, BEAMLD commands. Following this the beam element moments of inertia I_i are read into a row array of (NMBS + 1) columns, labelled herein IN, with IN(1) being used to store Young's modulus of elasticity. The deflection calculations must be flagged to indicate the presence of distributed load which requires the calculation of v^*. The command sequence is as follows:

1. Calculate beam flexibility matrix FL using the command

BEAMFL IN FL

2. Calculate nodal deflections,

BEAMDF FL R IN D=?

FL	=	flexibility matrix
R	=	node displacements
IN	=	location where $l^3/24EI$ terms have been stored
D	=	0 no distributed loads
	=	1 distributed loads present

The deflected shape of the beam may be plotted using the PLTBEM command with the option N=4 as follows,

PLTBEM A B C R N=4

The calculation of deflections in the $X-Y$ coordinates for an unsymmetric beam cross section, (See Chapter 6 for details), can be made by using,

$$\frac{(I_x I_y - I_{xy}^2)}{I_x} \tag{3.41}$$

instead of I_y. Then to obtain the out of plane x deflections scale those calculated for y by $(-I_{xy}/I_x)$.

3.4.2 Plot of beam deflections

Deflections of a member are defined by the nodal deflections (y_I, y_J) and the end rotations relative to the chord, (ϕ_I, ϕ_J), shown in their directions in Figure 3.12. The deflected curve is cubic and can be expressed in terms of these four end values. The expression for y_x is calculated to be,

$$y_x = [\zeta_1 \quad -L(\zeta_1^2\zeta_2) \quad \zeta_2 \quad -L(\zeta_1\zeta_2^2)] \begin{bmatrix} y_I \\ \phi_I \\ y_J \\ \phi_J \end{bmatrix} = [a]\{v\} \tag{3.42}$$

In this equation,

$$\zeta_2 = \frac{x}{L} \quad \text{and} \quad \zeta_1 = 1 - \zeta_2 \tag{3.43}$$

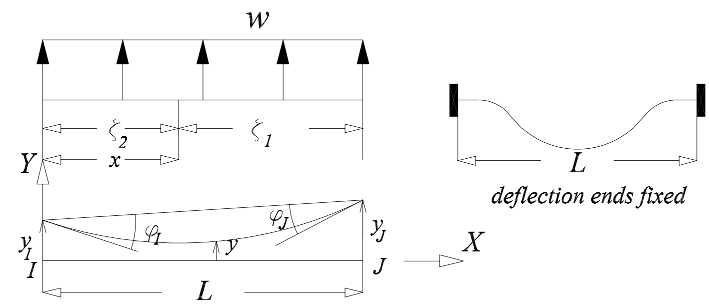


Figure 3.12: Element end displacements and deflections

Indeterminate beam deflections

When the [b] matrix hase been calculated for the indeterminate beam it is stored in the matrix KA, and the structure flexibility is kept in K. Then as the load vector is in the array, LO, the nodal deflections R from the joints fixed condition are calculated,

MULT K LO R

Because these deflections have been calculated from the position of all node displacements fixed, the within element deflections must be added for elements that have distributed loads. That is, if a uniformly distributed load w/unit length acts on the member

in the positive Y direction as shown in Figure 3.12, then for $\{v\} = 0$, deflection occurs from this fixed position and is given,

$$v_{xF} = \frac{wL^4}{24EI}(\zeta_1^2\zeta_2^2) \tag{3.44}$$

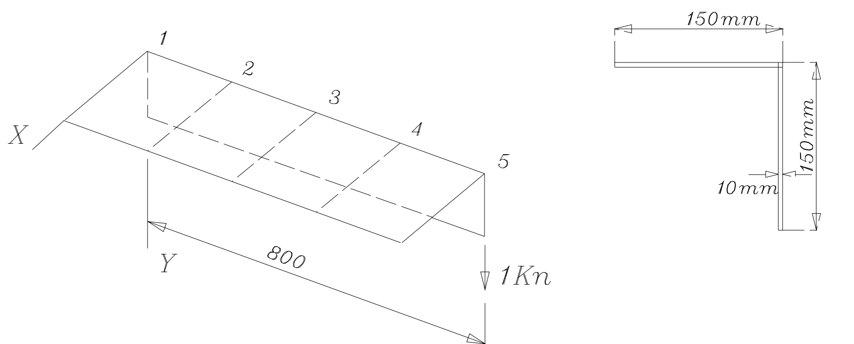
This is added automatically in the plotting process to y_x calculated in equation (3.42). Beam deflections are plotted with the PLTBEM command using the option N=4. If the nodal deflections have been calculated in the array $\{R\}$, the plot command is:

PLTBEM A B C R N=4

The matrices A, B and C have been defined previously.

3.4.3 Example: Calculation of beam deflections in non-principal axes

The cantilever beam shown in Figure 3.13(a) has a load of 1kN applied to node 5, the beam being of 8 metres long and having 4 equal members. The cross section is given in



(a) Beam dimensions (b) Beam cross section

Figure 3.13: Cantilever beam

Figure 3.13(b) and has the following properties:

$$I = 10^{-5} \begin{bmatrix} 0.703 & -0.424 \\ -0.423 & 0.703 \end{bmatrix} m^4 \tag{3.45}$$

From this,

$$\frac{I_x I_y - I_{xy}^2}{I_x} = 0.448 \ ; \qquad -\frac{I_{xy}}{I_y} = -0.5733 \tag{3.46}$$

This problem will be solved as an exercise number 2 of the beam sample examples. Thus the commands are:

```
BEAMEX E=2 L=8
BEAMEQ A B C
LOADR E R=1 C=3
5 1 0.001
BEAMLD A B C E F C=1 D=0
LOADR IN R=1 C=5
2 0.448 0.448 0.448 0.448
BEAMFL IN FL
BEAMDF FL R IN D=0
PRINT R
```

RESULTS:

$$\delta_{5y} = 190.48mm$$

$$\delta_{5x} = -0.5733 \times 190.48 = -109.2mm \quad (3.47)$$

3.5 Summary of beam projects

3.5.1 Beam bending moments, shear force and reactions

(I) Using examples 1 to 6 provided in the software. In these examples shown in Figure (3.15), the user has the ability to choose the beam span, so that these examples may be used in simple design calculations.

Command sequence

BEAMEX	E=? L=?	E=1 to 6, L=total length
		matrices A,B,C,[D] are generated
		A=coordinates
		B=member end node numbers
		C=support details
		[D]= hinge data for example 5 only
BEAMEQ	A B C [D]	set up and invert equilibium equations
LOADR	E R=? C=3	(concentrated loads)
LOADR	F R=1 C=?	distributed loads one per element
BEAMLD	A B E F C=? D=?	generates nodal loads from E
		of concentrated loads (C=1 present)
		or member U.D.L. in F (D=1 present)
BEAMMO	M V S	calculate moments M, shears V, reactions S

(II) Using input data of any determinate beam.

Command sequence

LOADR	A R=1 C=?	coordinate of beam nodes
LOADI	B R=? C=2	node numbers of beam elements, in order
		1 2
		2 3 etc.
LOADI	C R=? C=2	one row per reaction component
		node number, reaction type, 1=Y force, 2 = Z moment
LOADI	D R=? C=2	one value per each internal hinge, if present
BEAMEQ	A B C [D]	same as in (I) above

3.5.2 Indeterminate beam analysis, examples 7-10 Figure (3.16).

The theory for statically indeterminate beams is given in section 3.3.4. For the examples given in Figure (3.16) to data is generated with the command BEAMEX, and the whole sequence is given below.

Command sequence

BEAMEX	E=? L=?	E=7 to 10, L=total length
BEAMEQ	A B C	matrices A,B,C, are generated A=coordinates B=member node numbers C=support details
LOADR	E R=? C=3	(Concentrated loads)
LOADR	F R=1 C=?	distributed loads one per element
BEAMLD	A B E F C=? D=?	generates nodal loads from E of concentrated loads or member U.D.L. in F which must be pre-loaded C=1 concentrated loads present, D=1 distributed loads present,
LOADR	IN R=1 C=?	Young's modulus, member I values
BMMSTF	B IN MS	generate member stiffness matrices MS
BMGSTF	EQ MS K	force transformation matrix in KA
BEAMMO	M V S	bending moments, shears, beam reactions
PLTBEM	A B C (M or V) N=2-3	plot bending moment or shear force diagram

Note! At any stage, LIST gives matrices in the incore data base, and PRINT (array name) prints the array values. It can be useful to use the LIST command to check that calculations are proceeding as planned. If the program stops because of an attempt to access a matrix not defined, then the RESUME command will restore incore data base when STATIC is restarted. All matrices viewed on the screen with the PRINT command are also written to the file (name).OUT. The command sequence as entered by the user is written in the file (name).JNL. This journal file can be edited and used together with the SUBMIT command to shorten screen input.

3.6 Rigid body reaction module

The theory for the calculation of reactions for a planar rigid body is given in section 3.2. The basis of the method is that each force and each reaction must be identified by the following:

(1) Its type, that is, force or moment.

(2) The angle of its positive direction with the positive X direction in the case of a force only.

(3) The coordinates if a point on its line of action, again for a force only. Items (2) and (3) are simply left as zero for a moment.

The data is setup in the matrix A, which is read using the LOADR command of STATICS-2020. Reaction values are given unit magnitudes in this matrix. The command

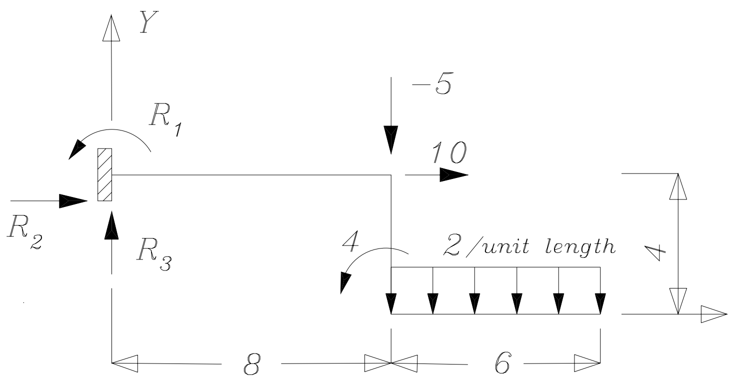


Figure 3.14: Cantilever beam reactions

REACT reduces all the forces, moments and reaction coefficients to the origin in two matrices B for applied forces and moments and C (3×3) for reaction coefficients. The reaction matrix C is invertible using the T=2 option. An example has been given in section 3.2.

3.6.1 Rigid body reaction exercises

(R1) The rigid body in Figure 1.7 has additional forces,

1. horizontal +15 at (4,2)
2. vertical +12 at (8,5)
3. inclined at 45^o of +8 at (8,5)

Calculate the reactions for all the forces on the rigid body including those in Figure 1.7.

(R2) A cantilever beam is shown in Figure 3.14. In the analysis of the beam to calculate the three reactions replace the distributed load in two ways,

1. A force of (0,−12) at coordinates (11,0).
2. A pair of forces of magnitude −6 at the coordinates (8,0) and (14,0).

Calculate reactions (R_1, R_2, R_3) for both substitutions showing that the reactions for the two cases are identical.

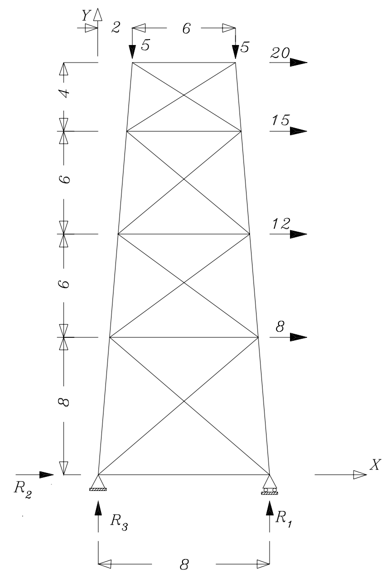


Figure 3.15: Tower structure rigid body reactions.

(R3) The tower shown in Figure 3.15 has a pin support at node 1 and a roller support at node 2. Use REACT to calculate the reactions for the loads shown on the tower. Use the PLTRBD command to show that the force polygon closes.

(R4) The truss (1) in Figure 2.8 has a vertical force of 10 units downwards on node 5 and a horizontal force of 10 on node 6 in the positive X direction. Calculate the reactions.(Note; coordinates may be obtained from the TRUSEX E=1 command.)

(R5) The truss(2) in Figure 2.8 has vertical forces of (-10) acting on nodes 2,4,6,10 and a horizontal force of $+10$ acting on node 7. Calculate reactions. (node coordinates can be obtained from the TRUSEX command.

(R6) The truss(11) in Figure 2.10 has horizontal forces each of magnitude $+10$ applied at nodes 3, 4 and 6. Calculate reactions at nodes 1 and 2 using the reaction command, REACT.

(R7) The beam (3) in Figure (3.15) forces of (-10) applied vertically downwards, on nodes 3, 5 and 9. Use REACT to calculate the beam reactions.

3.7 Statically determinate beam module

The theory for statically determinate beams is given in sections 3.2.1 to 3.2.3. The beam structure is characterized by having two equilibrium equations per node (F_y, M_z). It is

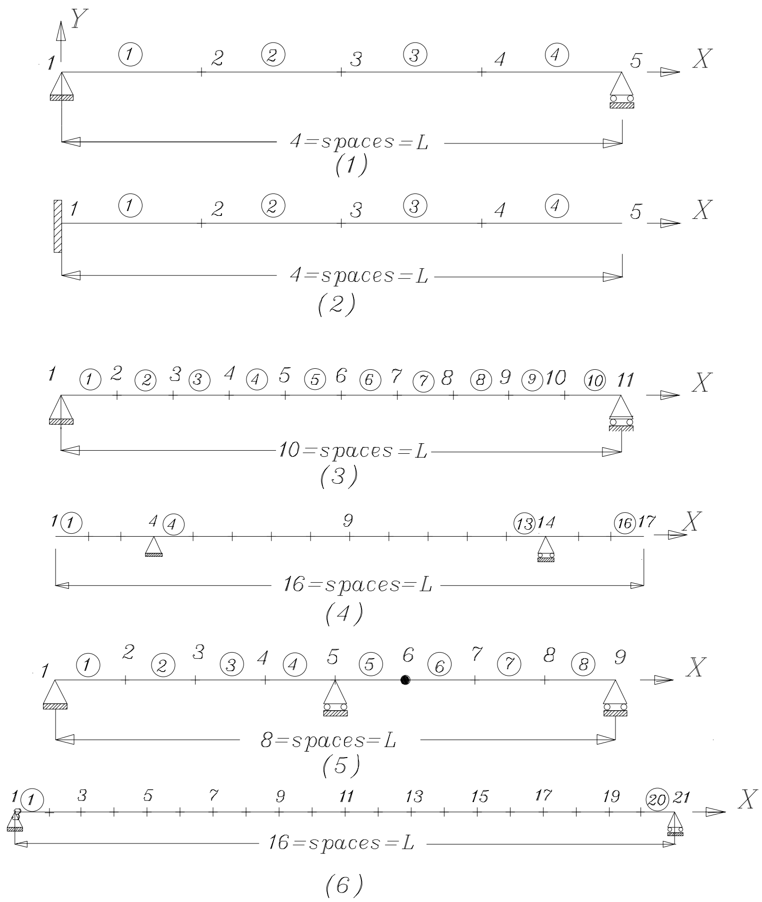


Figure 3.16: Determinate beam examples 1 to 6

further complicated by uniformly distributed loads on members and the possibility of internal hinges being present in which case the number of reaction components is greater that two. In such a case, the number of equilibrium equations will be greater than 2× the number of joints, an additional moment equilibrium equation being generated at each hinge. Because if this complication the STATICS-2020 software sets up and then inverts the equilibrium matrix which is defined (EQ). At the same time it maintains the information necessary to calculate member bending moments when distributed loads are present, (see Figure 3.10). The basic information for the beam will be stored in the arrays A,B,C and D.
A is a row vector giving nodal x coordinates.
B is a matrix with two columns and has as many rows as there are beam elements. It contains the $I - J$ member node numbers.
C contains the reaction data, one row per reaction component. See sections 3.2.1 and 3.2.2 for more details. The treatment of distributed loads is given in section 3.2.2. The basic command for setting up the equilibrium equations is

BEAMEQ A B C [D]

Because, for statically determinate beams, $[b]$ is obtained in this command in EQ$$, the user is not required to calculate the inverse. All that is now required is for the loads on the beam, concentrated forces, moments and uniformly distributed loads to be specified. The software provides six (6) beam examples shown in Figure 3.16, whose data may be generated with the command

BEAMEX E=? L=?

Where E= (1 to 6) and L gives the overall length of the beam. This command then automatically generates the A, B, C and D matrices wit D having been inverted. The load information is supplied by the user in the two arrays:

E = concentrated forces and moments
F = distributed load magnitudes

This information is then processed and inserted into the load vector R using the command

BEAMLD A B E F C=? D=?

See section 3.2.3 for more details. A number of beam exercises (**B1**) to (**B6**) are given. These are elementary problems and can be solved without the use of the STATICS-2020 software. Answers obtained by STATICS-2020 should always be checked by elementary statics. However if the further step of calculating beam deflections is undertaken then use of STATICS-2020 saves considerable amount of calculations.
Note: once the BEAMLD command has been used, the bending moments, shear forces and reactions are calculated with the BEAMMO command and plotted with PLTBEM.

3.7.1 Statically determinate beam exercises

(B1) Calculate the reactions and draw the bending moment and shear force diagrams for the following beams in Figure 3.16 with the loading values given in 1 to 16 below. All span lengths are to be 20 units.

1. Beam 1. Unit load in the Y direction at node 2.
2. Beam 1. Unit load in the Y direction at node 5.
3. Beam 3. Unit load in the Y direction at nodes 7,8 and 9.
4. Beam 3. Load of 10 at node 4, load of -10 at node 8.
5. Beam 4. Unit loads in the $-Y$ direction at nodes 1 and 17.
6. Beam 5. Load of 10 on node 3.
7. Beam 5. Load of 10 on node 6.
8. Beam 5. Loads of 10 on nodes 3 and 7.
9. Beam 1. Distributed load of -1 kN/m on portion 1 to 3.
10. Beam 2. Distributed load of +1kN/m from node 3 to 5.
11. Beam 3. Distributed load of -1kN/m from node 3 to 7.
12. Beam 4. Distributed load on beam from node 1 to 17.
13. Beam 5. Distributed load on beam from node 6 to 9.
14. Beam 6. Loads of +10 on nodes 3,7 and 9 and -10 on node 17.

(B2) Use STATIC-2020 to check results for the exercises (B1-11), (B1-15) and (B-16). USE the PLTBEM command to view the results.

(B3) For the example (B1-14) use STATIC-2020 to calculate the bending moments and shear forces if the length of the beam is 40 units. How have the bending moments and shear forces changed from those in (B1-14). Use the PLTBEM command to view the results.

(B4) Apply a unit moment of +1 at the nodes of the beams in Figure 3.16 as given below. The beam span length is 20m in all cases. Draw the bending moment and shear force diagrams.

1. Beam 1. Node 3
2. Beam 2. Node 5
3. Beam 3. Node 4
4. Beam 4. Node 1
5. Beam 5. Node 1
6. Beam 6. Node 7

(B5) Use the STATICS-2020 with the BEAMEX command to check the results in (B3-4).

(B6) The beam 5 in Figure 3.16 is 20m long and has a uniformly distributed load of $-2kN/m$ applied over its length. Use STATICS-2020 to draw the bending moment and shear force diagrams. Use the PLTBEM command to display the bending moment and shear force diagrams.

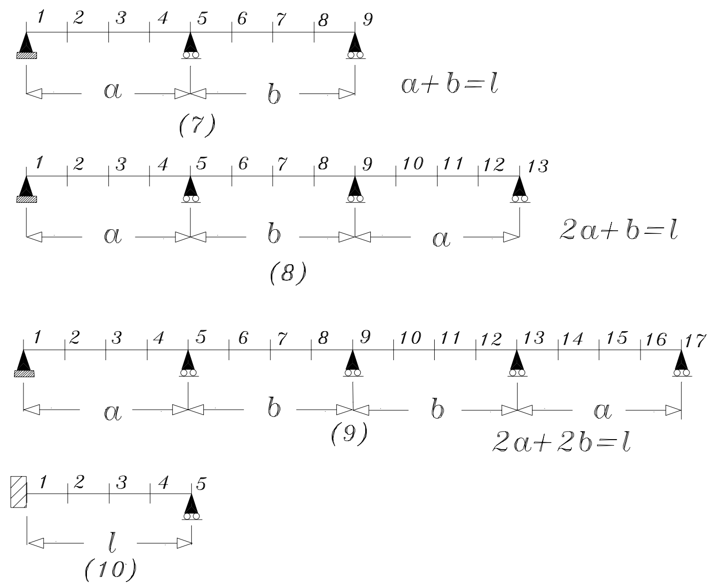


Figure 3.17: Indeterminate beam examples 7 to 10

3.8 Statically indeterminate beam module

The theory for the analysis of statically indeterminate beams is given in section 3.2.4. It is seen that the computational method is identical to that for the analysis of statically indeterminate trusses with the necessary modifications for beam elements. The STATICS-2020 software has four preprogrammed examples, shown in Figure 3.17 and these have their data generated using the BEAMEX command. When their data is generated, the message that the structure is indeterminate is displayed and that the stiffness method of analysis may be used. The two commands BMMSTF and BMGSTF are available to do this. The BMMSTF command generates the beam element stiffness matrices and BMGSTF generates the global stiffness matrtix, inverts and calculates the $[b]$ matrix in the programme defined matrix KA. Once the load vector has been generated with the BEAMLD command, the command BEAMMO generates member bending moments and shear forces and of course the reactions for the determinate beam case. The exercises are designed to give practice in indeterminate beam analysis and an appreciation of the bending moments in indeterminate beams. The span proportions of the overall length are given in Figure 3.17. In all examples STATICS-2020 should be used.

(B7) The beam (7) in Figure 3.17 is of uniform section throughout and is subjected to the load cases 1 to 7 below. Use STATIC-2020 to calculate bending moments and shear forces. Display results using the PLTBEM command and hence draw the bending moment and shear force diagrams. The overall length of the beam is 10m.

1. Unit moment of +1 on node 1.
2. Load of −10 on node 3.
3. Load of −10 on node 7.
4. Combine cases 2 and 3 and compare with the analysis of the load −10 on nodes 3 and 7. Hence show that superposition of load cases applies.
5. Uniformly distributed load of $-2kN/m$ on members 1 to 4.
6. Uniformly distributed load of $-2kN/m$ on the whole length.
7. Show how the results in 6 can be obtained from those in 5.

(B8) The beam (7) in Figure 3.17 has the following relative I values

1. Elements 1-3 and 6-8, relative $I = 1$.
2. Elements 4 and 5, $I = 2$.

Rework the exercises **B1** to **B7** and compare the results. How is the bending moment at node 5 (support) altered by the change in I values. Repeat the exercise with $I = 0.5$ for elements 4 and 5, and compare results.

(B9) The beam (8) in Figure 3.17 has uniform I throughout. The overall length is 15 m.

1. Calculate the bending moments and shear forces in the beam for a uniformly distributed load of $-2kN/m$ over the whole length.
2. Compare the moments at supports (nodes 5 and 9) with that at node 5 in exercise **B6**-6.
3. Use PLTBEM to view the results for both bending moment and shear force diagrams.

(B10) The beam (9) in Figure 3.17 has an overall length of 20m. Carry out the following exercises:

1. Calculate bending moments and shear forces for a uniformly distributed load of $-2kN/m$.
2. Compare results with **B7**-6 and **B8**-1.
3. Use PLTBEM to view bending moment and shear force diagrams.
4. Analyse the beam for a load of −10 acting on node 7.

(B11) For each of the beams in **B7**, **B9** and **B10**, apply a moment of +10 kNm at node 1. Calculate bending moments and shear forces in the each beam. Use PLTBEM to view results and plot the bending and shear force diagrams.

(B12) The propped cantilever beam (10) in Figure (3.16) has a span of 10 m.

1. Calculate the bending moment at node 1 if a moment of +10 kNm is applied to node 5. What is the ratio of these two bending moments?
2. Analyse the beam for bending moment and shear forces for a uniformly distributed load of $-2kN/m$ over the whole span.
3. View the results with the PLTBEM command.
4. Calculate vertical reaction at node 5 in terms of w and L.
5. Compare the bending moment in (2) with those in the end span of beam (**B9**)-1 scaled to a span of 10 m.

(B13) For each of the examples, (**B7**)-6, (**B9**)-1 and (**B10**)-1, calculate the interior moments over the supports in terms of wl^2, where L = one span. That is $l = L/3$ for (**B7**)-6, etc..

(B14) In exercise (**B10**), the sections 4 to 6, 9 to 10 and 12 to 14 have $I = 2 \times I$ and I for the remaining sections. Calculate the bending moments for the uniformly distributed load $-2kN/m$ over the whole length. How have the bending moments changed?

1. Over the supports (nodes 5, 9 and 13)?
2. At the mid-span (nodes 3, 11 and 15)?

3.9 Determinate beam deflections

The theory for the calculation of beam deflections is given in section 3.4. The deformations of a beam element are composed of two parts,

(a) That part due to terminal moments M_i, M_j.

(b) That part due to any distributed load on a beam element which produces a quadratic variation from linearity (see Figure 3.10).

The calculation is accomplished using the two commands, BEAMFL and BEAMDF. When distributed loads are present the BEAMDF command automatically includes the effect (b) of the quadratic variation from linearity of the bending moment in an element.
The command
BEAMFL IN FL
calculates the beam flexibility matrix using the $[b]$ matrix residing in KA and the Young's modulus and member I values stored in the user defined array IN (see section 3.4 for more details).

(B15) The beam (1) in Figure 3.16 has a span of 20 units, and is subjected to a unit load (-1) on node 3. If $E = 100, I = 1$, calculate the deflection of the beam node points and show that the vertical deflection of node 3 $= -10/6$.

(B16) The beam (1) in Figure 3.16 with the properties given in (**B15**), has a distributed load (-1) acting on the portion of the span 1-2-3. Calculate the deflection of the node points of the beam. If the same load is now applied to portion 3-4-5, prove that the central deflection $= -(5 \times 10^3)/384$.

3.10 Multiple Load Cases-Section Selection

In this section the concept of design is introduced using a combination of load cases and selecting sections from the standard universal beam sections. In the data file DATN.DAT supplied with the software examples are given of the analysis of beams for multiple load cases. Values of member bending moments, shears and nodal deflections are stored for each load case as columns of a matrix previously defined as zero matrix. There will be one such matrix for each particular group (moment, shear, deflection). Using the matrix multiplication command the columns can be combined with any desired load factors. This is an elementary simulation of the design process in which loads might include (self weight, wind and super imposed live load). The combined load cases can then be retrieved in the same form as the produced in the initial analysis and the results viewed through the PLTBEM command. The group of commands can be setup in the DATN.DAT file with an appropriate marker and accessed from the screen using the SUBMIT command. This is done in problems (44) and (45) with markers B15 and B16 respectively. The two commands for storing and retrieving the data are STOVEC and GETVEC. For moment values there are 3 moments (M_i, M_c, M_j) so that the matrix defined for storage will have dimensions,

number of rows $= 3 \times$ number of members
number of columns = number of load cases

For example from DATN.DAT under B15,

ZERO	CMB	R=24	C=3	(inilialize CMB)
STOVEC	M	CMB	N=1	(load case 1)
STOVEC	M	CMB	N=2	(load case 2)
STOVEC	M	CMB	N=3	(load case 3)
LOAD	FAC	R=1	C=3	(load factors)
1.1	1.2	2.0		(factors)
MULT	CMB	FAC	TOT	(combined factored moment)
GETVEC	M	TOT	N=1	(reverse process to obtain combinations for each member)

In the above table the intermediate commands have been omitted before each STOVEC command. See the DATN.DAT file for details of the complete analysis sequence. The RETURN command in the DATN.DAT file returns the program to the user screen and the combined bending moments viewed with the PLTBEM command in the usual way. From the dialog window the user can select suitable member sizes, Univeral Beams only for Australian Standard sections based on a simple allowable stress criterion, of $F_{all} = 0.6F_{yp}$ using the command

BEMDES M G=?

Tihs command will give sections for both positive and negative bending moments that satisfy the sttress criterion. If no section can be found a message is printed to this effect. The values of G are 300 $\Rightarrow$ 440.0mPa and 350 $\Rightarrow$ 480.0mPa yield point values. The probems in DATN.DAT have been designed to give section sizes in the range available in the Australian Standard Sections. These section dimensions are given in millimeter units and the bending moments should be calculated in kNM units and the stresses are in mPa.