Chapter 4

PLANE FRAME ANALYSIS

4.1 Introduction

A plane frame combines the axial force effects of truss members with bending moments and transverse shears of beam elements. Consider the structure shown in Figure 4.1. The joint equilibrium equations must now involve forces F_x, F_y in the X, Y directions and the moment M_z about the Z axis out of the plane of X and Y. To develop the joint equilibrium equations in the global coordinate axes, member forces are firstly set up for a single member in the local, member coordinate, axes and then transformed to global axes components. For the whole structure, the three conditions still apply for the solution of the equilibrium equations with NR rows and NC columns. That is,

$$\begin{array}{lcl} \text{equations(NR)} & > & \text{(member forces + reactions)(NC) (unstable)} \\ \text{equations(NR)} & = & \text{(member forces + reactions)(NC) (determinate)} \\ \text{equations(NR)} & < & \text{(member forces + reactions)(NC) (indeterminate)} \end{array}$$

The software tests for these three conditions and the determination of the member forces is possible for conditions (2) and (3). It should be mentioned that the conditions are necessary but not sufficient. The exception of unstable indeterminate structures does not usually occur and is beyond the scope of this text.

4.2 Determinate frame structures

4.2.1 Introduction

In this section the basic theory of Chapters 2 and 3 are extended to include plane frames that have three equilibrium equations per node. Because member forces now include both axial and shear forces these interact at the nodes. For example the simple structure shown in Figure 4.1 cannot be solved directly as a beam or as a truss. In this chapter the necessary relationships are developed so that the nodal equilibrium equations can be set up in terms of the basic member forces. The basic member forces for the frame member

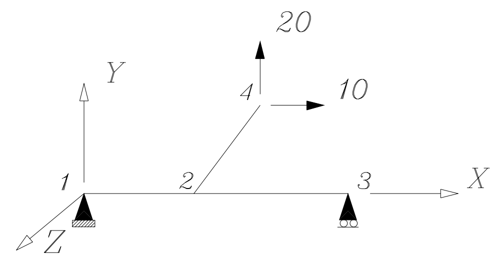

Figure 4.1: Simple planar structure

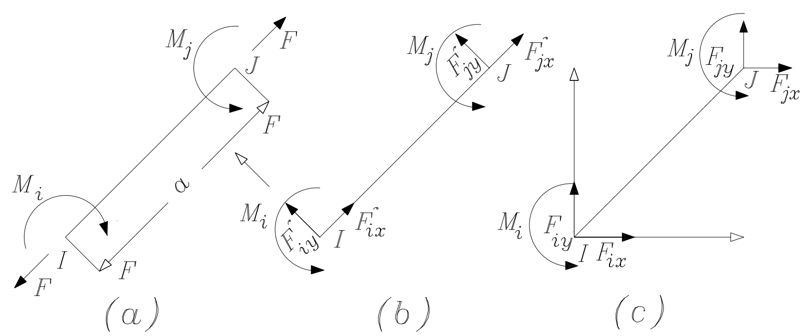

Figure 4.2: Member forces, local and global axes

are (F, M_i, M_j). That is, axial force and end moments. Their positive sense and local coordinate axes are shown in Figure 4.2(a).

4.2.2 Member forces – nodal equations of equilibrium

The nodal components in the X', Y' axes Figure 4.2(b), of the member are given by the transformation,

$$\left\{\begin{array}{c} F'_{ix} \\ F'_{iy} \\ M_i \\ F'_{jx} \\ F'_{jx} \\ M_j \end{array}\right\} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{a} & \frac{1}{a} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{a} & -\frac{1}{a} \\ 0 & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} F \\ M_i \\ M_j \end{array}\right\} \tag{4.1}$$

These six force components are then transformed to global components using, see Figure 4.2(c), the transformation for the components at each end of the member,

$$\left\{\begin{array}{c} F_x \\ F_y \\ M_z \end{array}\right\} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} F'_x \\ F'_y \\ M_z \end{array}\right\} \quad \text{or} \quad \{F\} = [L]^T\{F'\} \tag{4.2}$$

Thus if equation (4.1) is written,

$$\{F'\} = [A']\{S\} \tag{4.3}$$

then using equation (4.2) the global components are given by,

$$\{F\} = [L_D]^T\{F'\} = [L_D]^T[A']\{S\} = [A]\{S\} \tag{4.4}$$

The matrix $[A]$ is thus defined by,

$$[A] = [L_D]^T[A'] = \begin{bmatrix} L^T & 0 \\ 0 & L^T \end{bmatrix} [A'] \tag{4.5}$$

The joint equilibrium equations can be written as before,

$$\text{applied force} + \text{reaction} + \text{member } \textit{node} \text{ force} = 0$$

so that,

$$\text{member force} - \text{reaction} = \text{applied force}$$

Using the equations for all nodes and members of the frame, together with the reaction contributions enables the setting up all the joint equilibrium equations for the planar structure. For example, in Figure 4.1,

number of member forces $= (3 \times 3)$	$=$	9
number of reactions	$=$	3
number of joint equations	$=$	12

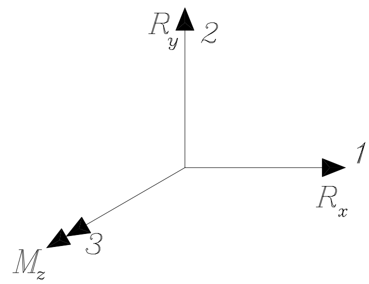

Figure 4.3: Reaction numbering sequence at a joint

This structure is stable and determinate. Reactions are restricted to global components only and are given in the sequence 1, 2, 3 as shown in the Figure 4.3. For each reaction a (-1) is added to the corresponding (row, column) location in the $[A]$ matrix. When all the equilibrium equations are assembled they are written in symbolic form,

$$[A_{SM}|A_{SR}]\left\{\begin{array}{c} S_M \\ S_R \end{array}\right\} = \{R\} \tag{4.6}$$

or simply,

$$[A]\{S\} = \{R\} \tag{4.7}$$

The equilibrium matrix is generated in STATICS-2020 by using the command

FRAMEQ A B C

The matrix A contains the X, Y coordinates of the frame nodes in the natural order of node numbering. The topology matrix B has three columns and is an integer matrix. The first column being the member number, the second and third columns being the member end node numbers (I, J). The matrix C is an integer matrix containing the reaction data. If for example node I is fully fixed, the reaction array will contain three rows with the following information:

I 1
I 2
I 3

Then a (-1) is then placed in the appropriate (row, column) location of the $[A]$ matrix. The matrix $[A]$ in equation (4.7) is the program defined matrix EQ. If $[A]$ is a square nonsingular matrix, then within the command FRAMEQ, the matrix EQ is inverted so that it now contains the $[b]$ matrix. In the 9 examples provided in STATICS-2020, see Figure 4.6, the command,

FRAMEX E=? D=?,?,?,?

generates the data matrices A, B and C for the appropriate frame in the Figure 4.6 or 4.7. If E is negative the frame (Exercise 1 to 6) will have pinned bases rather than fixed

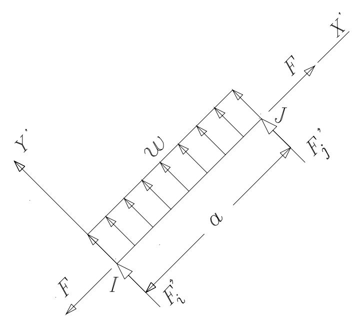

Figure 4.4: Uniformly distributed load in member coordinates

as shown in the Figure. The dimensions (a, b, c, d) given in Figure 4.6 are supplied in D, in that sequence.

4.3 Node and member loads

Loads are restricted to concentrated loads and moments applied to the nodes and uniformly distributed loads applied over the whole of a member length. For the nodal forces the command FRAMLD may be used. The command is,

FRAMLD B E F C=? D=?

The options are:

C = 0, 1; concentrated loads 0 = none, 1 = present

D = 0, 1; distributed loads 0 = none, (+, −)1 = present

The load vector is generated from E and F in the program defined array LO. The matrix E for concentrated loads will have one row per load value with the following information:

[node number] [force or moment direction (1-F_x, 2-F_y, 3-M_z)] [magnitude]

The (3) component is a moment about the Z axis, counter clockwise positive. For distributed load w per unit length, with $D = +1$ the load will be applied over the whole member length and will be in the direction of the positive Y' coordinate so that the transformation to global components is required. The load w per unit length is shown in the positive Y' sense in Figure 4.4. If $D = -1$ the distributed loads are in the global Y coordinate system and are per unit length of the inclined member.

For this load the end forces will be, for the statically determinate frame,

$$\{F'\} = \frac{wa}{2} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad (4.8)$$

These forces are transformed by pre-multiplication by $[L_D]^T$ to obtain the global $X - Y$ components to be added to the nodal load vector LO. Once the global load vector has been formed then member forces and reactions are obtained using the command, FRMFRC as follows,

FRMFRC M V S X

In which, M are the member bending moments, V the end shears, S the structure reactions and X the member axial forces.

4.3.1 Statically determinate structure deflections

The theory follows that given for truss structures in section 2.2.5 and for beams in section 3.4.1. Thus from equation (4.7) for the determinate frame, the transformation for member forces is obtained,

$$\{S\} = [A]^{-1} = [b]\{R\} \quad (4.9)$$

The ith submatrix, $[f_i]$, of the matrix of member flexiblities $[f]$ is obtained by combining the truss member flexibility, section 2.2.7, equation(2.21) and the beam member flexibility, section3.4.1, equation(3.33) as,

$$[f_i] = \frac{l_i}{6EI_i} \begin{bmatrix} \frac{I_i}{l_i} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad (4.10)$$

For distributed loads on the member in the positive Y'(local member) coordinate axes, the end rotations v_i^* are given,(see also equation(3.35)),

$$\{v_i^*\} = \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{-l_i^3}{24EI_i} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} w_i \quad (4.11)$$

Then the nodal deflections are obtained using the contragredient law,

$$\{r\} = [b]^T\{v_S + v^*\} \quad (4.12)$$

$\{v_i^*\}$ terms, including zero values for members without distributed load. Thence,

$$\begin{aligned} \{r\} &= [b]^T([f][b]\{R\} + v^*) \\ &= [b]^T[f][b]\{R\} + [b]^T\{v^*\} \end{aligned} \quad (4.13)$$

The elastic flexibility matrix for the frame is,

$$[F] = [b]^T[f][b] \tag{4.14}$$

This equation eqtn(uationq486) is valid even for statically indeterminate frames.

4.3.2 Statically indeterminate frames

In the case for which NR<NC, the analysis can still proceed and the $[b]$ matrix in,

$$\{S\} = [b]\{R\} \tag{4.15}$$

calculated using compatibility conditions of the deformations in an identical manner to that for statically indeterminate trusses and beams. Starting from,

$$[A]\{S\} = \{R\} \tag{4.16}$$

the contragredient principle shows that the corresponding displacement transformation is given,

$$\{v\} = [A]^T\{r\} \tag{4.17}$$

in which $\{v\}$ contains the member changes in length and the end rotations relative to the chord. The relationship between $\{S\}$ and $\{v\}$ is easily established for a prismatic member (n), length l_n, area of cross section, A_n and second moment of area, I_n

$$\left\{\begin{array}{c} F \\ M_i \\ M_j \end{array}\right\}_n = \{S_n\} = \frac{EI_n}{l_n}\begin{bmatrix} A_n/I_n & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}_n \left\{\begin{array}{c} \Delta l \\ \phi_i \\ \phi_j \end{array}\right\}_n = [k_n]\{v_n\} \tag{4.18}$$

It is seen that the expression for frame member stiffness simply combines axial and bending stiffnesses of the member. For all member forces (including reactions), these equations are combined as,

$$\{S\} = [k]\{v\} \tag{4.19}$$

The stiffness corresponding to a reaction is set equal to a large number e.g. (10^{20}). Combining equations (4.16), (4.17) and (4.18),

$$[A][k][A]^T\{r\} = [K]\{r\} = \{R\} \tag{4.20}$$

The structure stiffness matrix $[K]$ is now nonsingular and symmetric and the node deflections $\{r\}$ are obtained by solving equation (4.20),

$$\{r\} = [K]^{-1}\{R\} \tag{4.21}$$

The force transformation matrix from nodal to member forces is given, in the equation

$$\{S\} = [k][A]^T[K]^{-1}\{R\} = [b]\{R\} \tag{4.22}$$

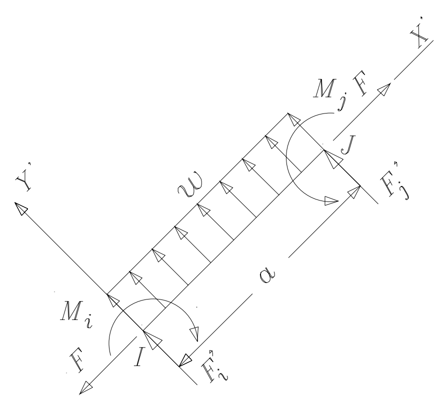

Figure 4.5: Member forces with U.D.L., fixed end moments

That is,

$$[b] = [k][A]^T[K]^{-1} \tag{4.23}$$

Notice that this is an identical matrix transformation to those for indeterminate trusses and beams so that the matrix theory presents a uniform approach to structural analysis. If the frame is statically indeterminate the condition is indicated and the user may continue the analysis as outlined above to obtain the [b] matrix. The steps are thus the same as for indeterminate trusses and beams. The user must firstly supply the area, second moment of area and Young's modulus of elasticity of each member. That is, a matrix with three columns is to be read in with the following values for each member,

[area], [second moment of area] [Young's modulus]

That is, there is one row for each member with 3 values. The program assumes that all members are of the same material and only the (1, 3) row value needs to be loaded with the Young's modulus. Of course, the area and second moment of area must be in units consistent with those used to describe the frame nodal geometry. The command,

LOADR IN R=? C=3

is used to load these member properties into the matrix (IN). The concentrated node loads and member distributed loads are processed via the command,

FRAMLD B E F C=? D=?

B=member nodal numbers.

If concentrated loads are present in E then C=1, otherwise C=0.

If distributed loads are present in F then D=(+, −)1, otherwise D=0.

The matrix E has one row per node load with the information:

node number, force component identification (1, 2 or 3), magnitude

The $(3 - Z)$ component being a moment about the Z axis, counter clockwise positive, see Figure 4.2(c). The matrix F is a row matrix giving uniformly distributed loads for all

members. A zero is required for a member without loads.
The command,

FRMSTF B IN MS

then calculates the matrix $[k]$ of member and reaction stiffnesses in MS. The equilibrium matrix has already been generated and stored in the program defined matrix EQ, so that the global stiffness command that generates $[b]$ and stores it in the program defined array KA, is

FRGSTF EQ MS K

Note that the same routines are used for all equations such as equation (4.22). Finally member forces and reactions are obtained with the command,

FRMFRC M V S X

In which, M are the member bending moments, V the member end shears, S the structure reactions and X the member axial forces.

4.3.3 Nodal forces, statically indeterminate frames

Forces applied directly to the nodes are treated in the identical way as for statically determinate frames. However for distributed loads on members, in the kinematically determinate state, $\{r\} = 0$, fixed end moments are induced by the zero rotation conditions and these with signs reversed must be applied to the nodes at the ends of the member. Thus from Figure 4.5, with w/unit length as shown in the member coordinate system the nodal release forces induced in the local coordinate system are,

$$\{F'\} = \frac{wa}{2}\left\{\begin{array}{c} 0 \\ 1 \\ a/6 \\ 0 \\ 1 \\ -a/6 \end{array}\right\} \tag{4.24}$$

When member forces are calculated from the nodal loads the fixed end moments must be included in the member forces. This is done automatically in FRMFRC.

4.3.4 Frame analysis

The examples 1 to 7 available in the software are shown in Figure 4.6 and they represent typical simple frames that are met in practical analysis. Because frames tend to be indeterminate only the commands for this case are given in this section. The frame dimensions are shown in Figure 4.6 and are entered in the sequence D=a,b,c,d.

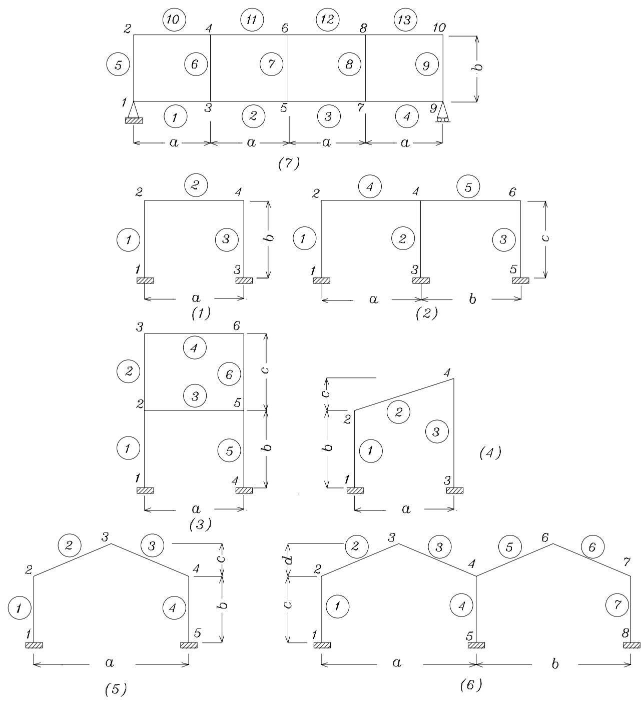

Figure 4.6: Frame examples 1 to 9

Command sequence

FRAMEX	E=? D=?,?,?,?	E=1 to 7, D gives frame dimensions see Fig. 4.6
PLTFRM	A B C	plot frame, nodes, members, reactions
FRAMEQ	A B C	set up joint equilibrium equations in EQ
LOAD	E R=? C=2	read in node concentrated loads, if present
LOAD	F R=? C=?	read in member distributed forces, if present
FRAMLD	B E F C=? D=?	generate node forces
LOAD	IN R=? C=3	read in area,second monent of area, Young's modulus each member
FRMSTF	B IN MS	generate member stiffness matrix in MS
FRGSTF	EQ MS K	generate global stiffness K, b matrix in KA
FRMFRC	M V S X	calculate member forces (M,V,X), reactions S
PRINT	M	print member moments
PRINT	V	print member end shears
PRINT	X	print member axial forces
PRINT	S	print structure reactions

4.3.5 Notes

Note! At any stage, LIST gives matrices in the incore data base, and PRINT (array name) prints the array values. If the program stops because of an attempt to access a matrix not defined, then the RESUME command will restore incore data base when STATIC is restarted

4.4 Frame analysis module

The theory for the analysis of plane frames is given in Sections 4.2 to 4.3. See these sections for the basic theory for member forces, node equilibrium and the setting up of the node equilibrium equations. the basic equation to do this is:

FRAMEQ A B C

in which A stores the node coordinates, B the member node connectivity matrix and C the support boundary conditions. Node forces can be generated using the command,

FRAMLD A B E F C=? D=?

The matrices A and B are as given in the command FRAMEQ, and E and F are matrices

containing data for concentrated node forces and uniformly distributed loads on members respectively. See Section 4.3 for the node forces for statically determinate and indeterminate frames that are different because the first solves by statics whereas the second uses the stiffness method (see the above sections for the details). The command FRAMEQ generates the equilibrium equations and for the determinate frame inverts the matrix and stores this in the programme defined array, EQ. This is the $[b]$ matrix. Member forces are obtained from the node loads by the command, (See Section 4.3).

FRMFRC M V S X

For statically indeterminate frames, the same strategy as for the analysis of statically determinate trusses and beams is used to generate $[b]$. (See Section 1.7.4, equation(1.72)). Then the commands used are,

FRMSTF B I MS
FRGSTF EQ MS K

The matrix K is the generated stiffness matrix and has been inverted in the process of calculating KA. This latter command generates the $[b]$ matrix stored in the program defined array, KA. When FRAMLD is used for statically indeterminate frames, the node forces include the release fixed end moments (See Section 4.3.2). The exercises given in Figure 4.6 are all for statically indeterminate frames. The set of commands to analyse these exercises are given in Section 4.3.1. The exercises for frame analysis are given in assignments, **(F1)** to **(F12)**.

4.4.1 FRAME ANALYSIS TUTORIAL EXERCISES

In all these exercises use a Young's modulus of elasticity of 200×10^3Mpa if using metric units and 30×10^6lb/sq inch if using Imperial units.

(F1) The frame (1) in Figure 4.6, has dimensions a=b=3m. All members are composed of 100mm o.d. tube with a wall thickness of 5mm. (Note if Imperial measurements are being used substitute 25mm = 1 inch).

1. Analyse the frame for a horizontal load of $2kN$ in the positive X direction on node 4. Draw the bending moment dirgram for the members. What are the magnitudes of axial forces in members (1) and (2)?
2. (2) The truss (8) in Figure 2.11 has the same dimenions as in (F1)-1, and all members have the same area of $500mm^2$. Use the TRUSS commands to analyse this truss for the same horizontal force and compare member forces with those from (F1)-1.

(F2) The frame (2),Figure 4.6, has dimensions a=b=c=3m and members are the same size as those in (F1). A horizontal load of $2kN$ acts horizontally on node 6. Calculate the bending moments and shears in the members.

1. Draw the bending moment diagram
2. How is the horizontal shear shared between members (1), (2) and 93).

(F3) The frame (1) in Figure 4.6 has a U.D.L of $-2kN/m$ acting on member (2). Analyse the frame and draw the bending moment diagram.

(F4) The frame (2) in Figure 4.6 has a U.D.L. of $-2kN/m$ acting on members (4) and (5). Analyse the frame and draw the bending moments so obtained with those for the frame in (F3).

(F5) The frame (3) in Figure 4.6 has dimensions a=b=c=3m. A U.D.L. of $-2kN/m$ acts on members (3) and (4). Analyse the frame and draw the bending moment diagram.

(F6) The frame given in exercise (F5) is subjected to the following loads:

1. Horizontal load in the positive X direction of $2kN$ on node (5).
2. Horizontal load in the positive X direction of $2kN$ on node (6).

For the two cases analyse the frame and compare bending moments in the members (1), (2), (5) and (6). Compare your results with those in (F1).

(F7) The frame (2) in Figure 4.6 has members (4) and (5) composed] of tube 200mm diameter with wall thickness 5mm, with column members as in (F2). Analyse the frame for the U.D.L. given in (F3). How do bending moments in the frame differ from those in (F3)?

(F8) The frame (4) in Figure 4.6 has the folowing dimensions, a=b=3m, c=1.5m. Analyse the frame for a horizontal force of $2kN$ on node (4). Compare the results with those from (F1). It is subjected to a U.D.L. of $-2kN/m$ on the members 2 to 4. Analyse the frame and draw the bending moment diargam. Compare the answers with those in (F3).

(F10) The gable frame (3) in Figure 4.6 has the dimensions, a=6m, b=3m c=1.5m. The frame members are of tubular section, O.D. = 150mm, wall thickness 4mm. Analyse the frame for the following loads in each case draw the bending moment diagram.

1. Horizontal load of $2kN$ on node 3.
2. Vertical load of $-2kN$ on node 3.
3. Horizontal load of $2kN$ on node 4.

(F11) the gable frame (6) in Figure 4.6 has member sizes the same as those in (F10). Analyse the frame for the following load cases.

1. Horizontal load of kN on node 2.
2. Horizontal load of $2kN$ on node 3.
3. Vertical load of $-2kN$ on node 3.
4. Horizontal load of $2kN$ on node 4.
5. Vertical load of $-2kN$ on each of the nodes 3 and 6.

(a) For each of the horizontal load cases, how is the shear force distributed between the vertical members 1,4 and 7?
(b)Compare the bending moments in the frame for load cases (3) and (5).

(F12) the Vierendeel girder (7) in Figure 4.6 has dimensions a=b=3m and all members are composed of tube, outside diameter 250mm with wall thickness, 5mm. Analyse the frame for loads of $-1kN$ on each of the nodes 4, 6 and 8.

1. Draw the bending moment diagram.
2. How are the transverse shears carried by the frame members?
3. Do you consider the Vierendeel girder an efficient way of carrying transverse loads?

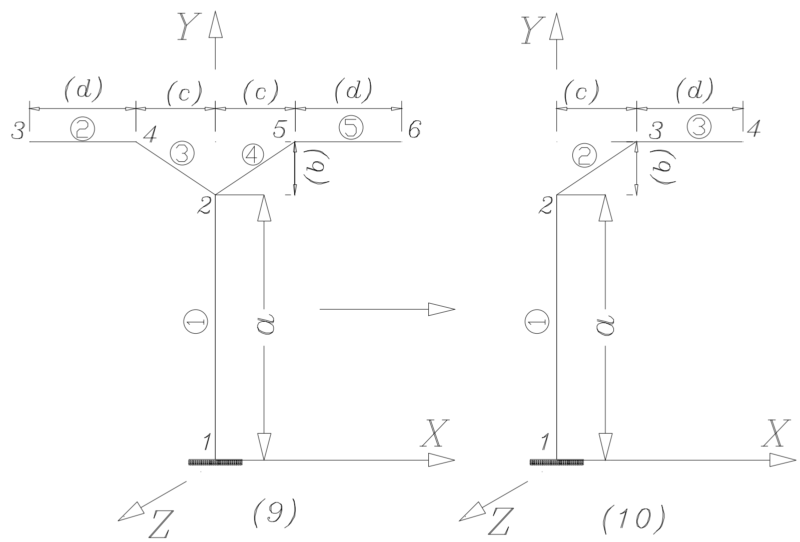

Figure 4.7: Determinate frames

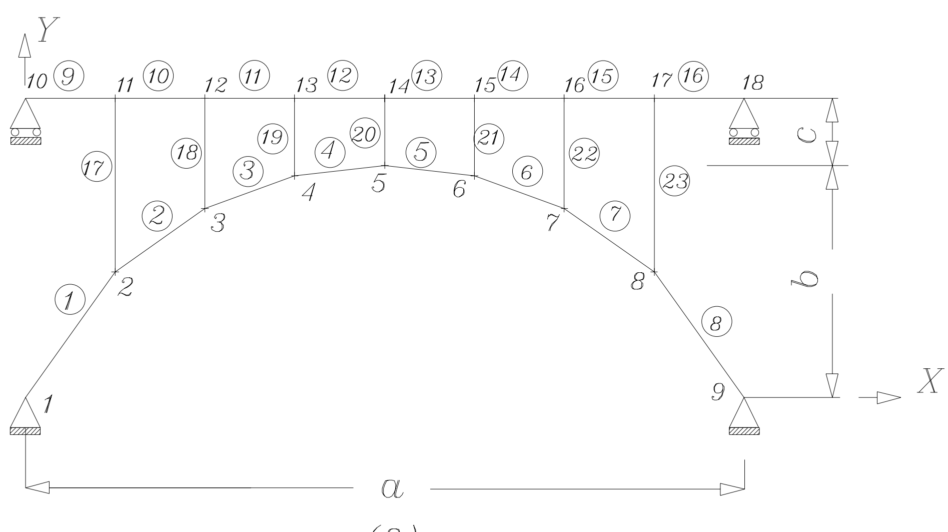

Figure 4.8: Two pinned arch

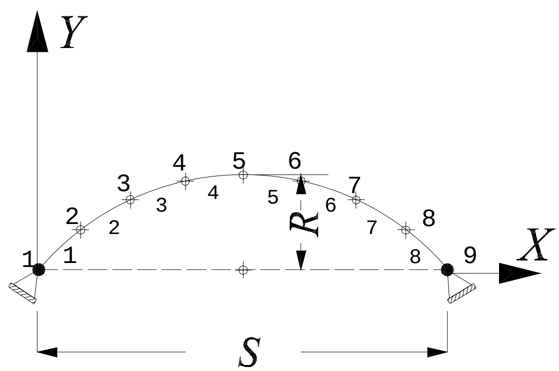

Figure 4.9: Two pinned arch circular arch