Chapter 7

AXIAL FORCE EFFECTS BUCKLING TENSION

7.1 Introduction

A fundamental assumption of the elementary theory of structures, as presented in Chapters 2 to 5 is that deflections produced by the deformations of the members of a structure are small when compared to the structure dimensions and do not have an effect on the equations of equilibrium. However, if the equations of equilibrium are written in the displaced position obtained by small displacements from the original position, it is found that linear terms appear that are dependent upon the axial forces in the members. For example, the cable of a suspension bridge carries a significant tensile force that has a stiffening effect on the bridge, whereas the compressive force in an arch bridge will reduce the arch stiffness and have a destabilizing effect on the arch. It is this destablizing effect that will be studied in this chapter and it leads to the concept of structure buckling. STATICS-2020 has a number of commands that may be used for the calculation of the approximate buckling loads of the several structural types (beams, frames and grids). The phenomena of buckling occurs because of the decrease of structure stiffness with increase in axial compressive forces eventually leads to a state of zero stiffness, (infinite flexibility), for which large and damaging deflections will occur. Theoretically, for a structure with distributed structural properties (any beam, frame or grid) there is an infinite number of these zero stiffness states for which one particular stiffness parameter becomes equal to zero. In the study of structure stability it is usually only the lowest value that is of concern to the engineer designing a structure. It is the objective of the STATICS-2020 commands to determine this lowest buckling or critical load. To introduce the study of the subject of buckling two examples are studied. The first is a simple column spring system with a single degree of freedom, see Figure 7.1 and the second a beam with distributed properties subjected to an axial compressive force, Figure 7.2.

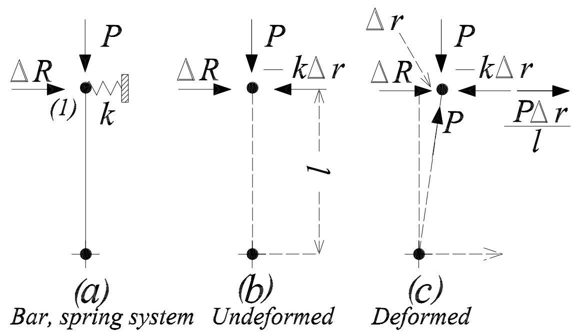

Figure 7.1: Hinged column supported by a spring at free end

7.1.1 Single bar-spring system with axial compressive load

In Figure 7.1 a single bar member of length l, hinged at its base, is supported horizontally by a spring of stiffness k at its free end and carries an axial compressive force P. A small increment of horizontal force ΔR is applied at node (1) at the free end. In the undeformed position in Figure 7.1(b), the equation of equilibrium of horizontal forces is,

$$\Delta R - k\Delta r = 0 \tag{7.1}$$

The horizontal displacement of node (1) has no effect on this equation. However examine the situation in Figure 7.1(c) which is drawn in the deflected position. The displacement, Δr, is considered to be small, so that to the first order the axial force in the member is still equal to P. The equilibrium equation of node (1) must now take into consideration the inclined force in the member and becomes,

$$\Delta R - k\Delta r + P\Delta\theta = 0 \tag{7.2}$$

P is considered to be positive when compressive and $\Delta\theta = \Delta r/l$. Rearranging equation 7.2,

$$\Delta R = (k - \frac{P}{l})\Delta r \tag{7.3}$$

The additional term is the geometric effect and represents the change in the horizontal,X, component of the member force at node (1). For the stiffness at node (1) to be equal to zero,

$$P_{cr} = kl \tag{7.4}$$

This value is the buckling load for the simple spring-column system. Of course, for structures with many degrees of freedom it is not such a simple matter to calculate P_{cr}, as is

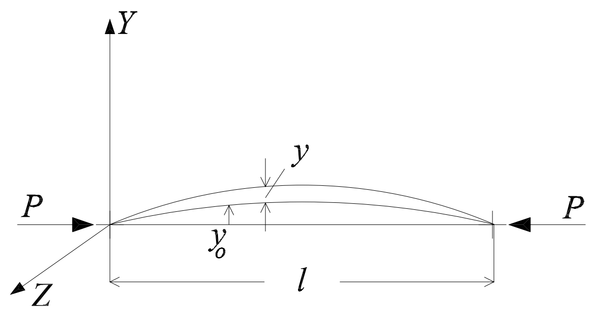

Figure 7.2: Column with imperfection subjected to axial force P

shown in section 7.1.2, of a simply supported column subjected to an axial compressive force and bending in one of the principal planes of the member cross section.

7.1.2 Column buckling-Euler load

Consider the simply supported column in Figure 7.2 bending about the Z axis in the $X-Y$ plane. The column is of uniform cross section and the $X-Y$ plane is a plane of symmetry so that when a bending moment M_Z is applied, the deformations of the column all occur in the $X-Y$ plane. The beam has an initial imperfection designated y_0 in the $X-Y$ plane and an additional deflection increment y is produced by the axial force P as shown in Figure 7.2. From the Figure 7.2, the bending moment at any section (taking into consideration the deflection y), is written,

$$M = -P(y_0 + y) \tag{7.5}$$

From the theory of beam bending, for M positive, the differential equation to the deflected curve is written,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{-P(y_0 + y)}{EI} \tag{7.6}$$

let, $P/EI = k^2$, so that k is real for P compressive, and rewrite equation 7.6,

$$\frac{d^2y}{dx^2} + k^2 y = -k^2 y_0 \tag{7.7}$$

Now consider the situation where $y_0 \to 0$ then,

$$\frac{d^2y}{dx^2} + k^2 y \to 0 \tag{7.8}$$

The solution to this equation, in the limit, is of the form,

$$y = A \sin kx \tag{7.9}$$

The boundary conditions are for the displacements to be zero, at $x = 0, l$

$$x = 0, \quad y = 0 \tag{7.10}$$
$$x = l, \quad y = 0 \tag{7.11}$$

From the second of these conditions,

$$0 = A \sin kl \tag{7.12}$$

This is satisfied for, $kl = n\pi$, or,

$$k^2 = \frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3, \ \dots \text{etc.} \tag{7.13}$$

The value of P_{cr} is given,

$$P_{cr} = \frac{n^2\pi^2 EI}{l^2}, \quad n = 1, 2, 3 \ \dots \text{etc.} \tag{7.14}$$

The lowest value for $n = 1$ is the Euler load for the column,

$$P_{Euler} = \frac{\pi^2 EI}{l^2} \tag{7.15}$$

This result proves to be useful for determining the accuracy of numerical methods used for determining approximate values of P_{cr} for complex structures.

7.1.3 Deflection of beams

In order to introduce the approximate solutions to the beam-column problem, deflection theory of beams is first studied. A simply supported beam, span l, is subjected to a load W at the point that divides the beam in the ratio (ζ_2, ζ_1), where $\zeta_2 = x/l$ and $\zeta_1 = 1 - \zeta_2$ as shown in Figure 7.3. The end rotations (ϕ_i, ϕ_j) are calculated for the load W by using the two elements (1) and (2) shown in the Figure 7.3. The expression for these end rotations is given using $[\bar{b}]^T[f][b]$ where

$$[\bar{b}]^T[f][b] = \begin{bmatrix} 1 & \zeta_1 & \zeta_1 & 0 \\ 0 & \zeta_2 & \zeta_2 & 1 \end{bmatrix} \frac{l^2\zeta_1\zeta_2}{6EI} \begin{bmatrix} \zeta_2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \\ \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} & \zeta_1 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{Bmatrix} \tag{7.16}$$

Multiplying out the righthand side of this equation and simplifying, the end rotations are calculated,

$$\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{-l^2\zeta_1\zeta_2}{6EI} \begin{bmatrix} (1+\zeta_1) \\ (1+\zeta_2) \end{bmatrix} W \tag{7.17}$$

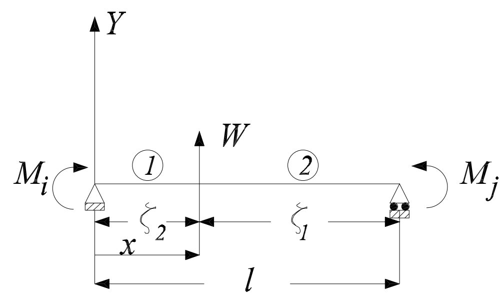

Figure 7.3: Beam with concentrated load W

This expression is used to calculate the end rotations of a variety of load patterns on the beam.

Example I

Uniformly distributed load of w/unit length over the whole span. Then, on an infinitesimal length $d\zeta_2$, the load is equal to,

$$W = w\, d\zeta_2\, l \tag{7.18}$$

To obtain the end rotations due to w on the whole span integrate $\zeta_2 = 0$ to 1. Then

$$\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{-l^3}{6EI} \int_0^1 \zeta_1 \zeta_2 \begin{bmatrix} 1+\zeta_1 \\ 1+\zeta_2 \end{bmatrix} d\zeta_2\, w = \frac{-wl^3}{24EI} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{7.19}$$

Example II

Linearly distributed load, w/ unit length at L.H.S and 0 at R.H.S.

The load on intensity at (ζ_1, ζ_2) is equal to,

$$W_\zeta = w\zeta_1\, d\zeta_2\, l \tag{7.20}$$

Then the end rotations are calculated,

$$\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix}_w = \frac{-l^3}{6EI} \int_0^1 \zeta_1^2 \zeta_2 \begin{bmatrix} 1+\zeta_1 \\ 1+\zeta_2 \end{bmatrix} d\zeta_2\, w = \frac{-wl^3}{360EI} \begin{bmatrix} 8 \\ 7 \end{bmatrix} \tag{7.21}$$

7.1.4 Calculation of beam deflection due to end moments M_i, M_j

The beam is shown in Figure 7.3 with end moments positive, M_i, M_j and a unit load W at the point (ζ_1, ζ_2) on the beam. For the end moments the member bending moments for the two segments of the beam are,

$$\begin{Bmatrix} M_{1i} \\ M_{1j} \\ M_{2i} \\ M_{2j} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \zeta_1 & \zeta_2 \\ \zeta_1 & \zeta_2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \tag{7.22}$$

and for the unit load, $W = 1$,

$$[\bar{b}] = l\zeta_1\zeta_2 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \tag{7.23}$$

The matrix $[f]$ of member flexibilities is the same as that given in equation (7.16), and note the interchange of the role of $[\bar{b}]$ and $[b]$. Hence the deflection at the load point is,

$$v_x \approx v_\zeta = [\bar{b}]^T[f][b] \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \tag{7.24}$$

$$v_x = \frac{-l^2\zeta_1\zeta_2}{6EI} \left[(1+\zeta_1) \quad (1+\zeta_2) \right] \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} \tag{7.25}$$

Again compare this expression with equation (7.17) for the end rotations. It will be necessary in the development of the geometric stiffness matrix to express v_x in terms of the end rotations (ϕ_i, ϕ_j) of the beam. Now from the expression for the stiffness matrix of the beam,

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \frac{2EI}{l} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.26}$$

Substitution in equation (7.25) then gives v_x in terms of (ϕ_i, ϕ_j).

$$v_x = -l\zeta_1\zeta_2[\zeta_1 \ \zeta_2] \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.27}$$

Note from the sign convention, positive end rotations produce negative deflections.

7.1.5 Effect of axial force on end rotations

It is now possible to calculate the first order (linear component) effect of the axial force P (compression positive) on the end rotations. The compressive axial force produces a bending moment M_P given by using equation (7.27). The expression for M_P is,

$$M_P = -Pv_x = Pl[\zeta_1^2\zeta_2 \quad \zeta_1\zeta_2^2] \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.28}$$

To calculate the increment in the end rotations produced by this bending moment, use the equation,

$$\{\Delta\phi_P\} = \int_o^l \frac{[\bar{b}]^T[b]}{EI} dx \tag{7.29}$$

where $[b]$ is given in equation (7.22) and $\bar{b}$ is,

$$[\bar{b}]^T = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}; \quad dx = l\, d\zeta_2 \tag{7.30}$$

Substituting and integrating 0 to 1 on ζ_2 gives the rotation increments $\Delta\phi_P$,

$$\{\Delta\phi_P\} = \frac{Pl^2}{60EI}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.31}$$

That is, if $(M_i \ M_j)$ are applied, the linear correction to the end rotations will be given by equation (7.31), so that,

$$\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} = \frac{l}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} + \frac{Pl^2}{60EI}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.32}$$

Imagine now the beam to be in an undeformed position. The second term on the right handside will be produced by fictitious moments $(M_i \ M_j)$, whose magnitudes are calculated,

$$\{M_F\} = \begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \frac{2EI}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\frac{Pl^2}{60EI}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.33}$$

Performing the multiplication,

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix}_F = \frac{Pl}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.34}$$

That is, if an axial force P (compressive positive) acts on the beam and the end rotations are $(\phi_i \ , \phi_j)$ then the applied moments are,

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \begin{Bmatrix} M_i \\ M_j \end{Bmatrix}_E - \begin{Bmatrix} M_i \\ M_j \end{Bmatrix}_F \tag{7.35}$$

That is,

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \left(\frac{2EI}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{Pl}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix} \tag{7.36}$$

Let $P = \lambda$ and write equation (6.65),

$$\{R\} = ([K_E] + \lambda[K_G])\{r\} \tag{7.37}$$

where,

$$[K_G] = \frac{-l}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \tag{7.38}$$

is the geometric stiffness for unit axial force. It is seen that for λ positive, (P compression) the geometric stiffness reduces the overall stiffness of the beam. An approximation to the

Euler load for the beam is obtained from equation (7.37) by examining the case $R = 0$. Then,

$$\frac{2EI}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{\lambda l}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = 0 \tag{7.39}$$

Solutions to this equation are possible for,

$$\lambda = P_{cr} = \frac{12EI}{l^2} \tag{7.40}$$

This compares with the Euler load and it is seen that the approximation is not sufficiently accurate. It may be improved by adding more nodes (degrees of freedom) to the beam.

7.1.6 Rotation of a member, change of nodal force components

The effect of the rotation of a member as a rigid body through a small angle $\Delta\theta$ as shown in Figure 7.4 is the alter the contribution to the transverse components of the forces at the nodes. From Figure 7.4, the rotation of the member, $\Delta\theta$, produces increments in the transverse forces ΔP_y shown. The member force is to the first order of approximation still equal to P. Then,

$$\Delta P_y = -P\Delta\theta = -\frac{P}{l}\Delta y \tag{7.41}$$

Combining this with the effect of the rotations relative to the chord, the member incremental geometric stiffness relationship is,

$$\begin{Bmatrix} \Delta M_i \\ \Delta M_j \\ \Delta P_y \end{Bmatrix} = -P\begin{bmatrix} \frac{l}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} & [0] \\ [0] & \frac{1}{l} \end{bmatrix}\begin{Bmatrix} \Delta\phi_i \\ \Delta\phi_j \\ \Delta y \end{Bmatrix} \tag{7.42}$$

This equation is written,

$$\{\Delta S\} = P[k_G]\{\Delta v\} \tag{7.43}$$

Note that there is a negative sign attached to $[k_G]$ which means that with P positive (compressive), the geometric stiffness tends to increase displacements and of course the reverse is true if the member force is tensile. The *member forces* for the member geometric stiffness matrix are three in number, as for the elastic stiffness of the frame member. There are the moment increments ΔM_i ΔM_j, however instead of the axial force for elastic stiffness, the geometric force, corresponding to the relative member rotation is the transverse force increment ΔP_y. It is now possible to derive the geometric stiffness matrix for the nodal components (ΔF_I ΔM_I ΔF_J ΔM_J)in the local global coordinate axes. The nodal

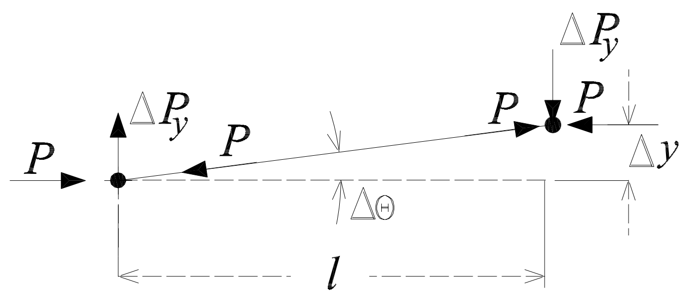

Figure 7.4: Rotation of member-transverse force components

force components are expressed in terms of $(\Delta M_i \; \Delta M_j \; \Delta P_y)$ by the transformation,

$$\{\Delta R_i\}_i = \begin{Bmatrix} \Delta F_I \\ \Delta M_I \\ \Delta F_J \\ \Delta M_J \end{Bmatrix} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} & 1 \\ -1 & 0 & 0 \\ \frac{1}{l} & -\frac{1}{l} & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \Delta M_i \\ \Delta M_j \\ \Delta P_y \end{Bmatrix} = [A_G]_i \{\Delta F_P\}_i \tag{7.44}$$

The notation $[A_G]$ is used to indicate that it is a geometric rather that an elastic effect. Contragredience gives the corresponding transformation between the nodal displacements and the member deformations to be,

$$\begin{Bmatrix} \Delta\phi_i \\ \Delta\phi_j \\ \Delta y \end{Bmatrix}_i = \begin{bmatrix} -\frac{1}{l} & -1 & \frac{1}{l} & 0 \\ \frac{1}{l} & 0 & -\frac{1}{l} & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \Delta y_I \\ \Delta\theta_I \\ \Delta y_J \\ \Delta\theta_J \end{Bmatrix} \tag{7.45}$$

Combining these transformations the nodal geometric stiffness for a single member is,

$$\{\Delta R_G\}_i = P[A_G]_i[k_G]_i[A_G]_i^T\{\Delta r\}_i \tag{7.46}$$

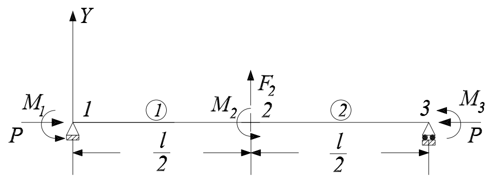

Figure 7.5: Simply supported beam-column, two segments.

Performing the matrix multiplications the expression is obtained for the single member,

$$\begin{Bmatrix} \Delta F_I \\ \Delta M_I \\ \Delta F_J \\ \Delta M_J \end{Bmatrix} = \frac{-Pl}{30} \begin{bmatrix} \frac{36}{l^2} & \frac{3}{l} & -\frac{36}{l^2} & \frac{3}{l} \\ \frac{3}{l} & 4 & -\frac{3}{l} & -1 \\ -\frac{36}{l^2} & -\frac{3}{l} & \frac{36}{l^2} & -\frac{3}{l} \\ \frac{3}{l} & -1 & -\frac{3}{l} & 4 \end{bmatrix} \begin{Bmatrix} \Delta y_I \\ \Delta \theta_I \\ \Delta y_J \\ \Delta \theta_J \end{Bmatrix} \tag{7.47}$$

Alternatively, the geometric stiffness matrix for the whole structure may be obtained in the same way as the elastic stiffness, combining all components of $[A_G]_i$ and member geometric stiffnesses $[k_G]_i$. This is a suitable approach for small structures and is used in STATICS-2020.

Example III
The solution for the simple span beam in Figure 7.3, for the buckling load considering only the end rotations gave 12 as the approximation to the "exact" value of π^2. This approximation can be significantly improved by subdividing the beam into two equal segments each of length $l/2$ as shown in Figure 7.5. The matrices used in the analysis are as follows in equations (7.48 to 7.51).

$$\begin{Bmatrix} \Delta M_1 \\ \Delta F_2 \\ \Delta M_2 \\ \Delta M_3 \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{l} & -\frac{2}{l} & 1 & -\frac{2}{l} & \frac{2}{l} & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \Delta M_{1i} \\ \Delta M_{1j} \\ \Delta P_{Y1} \\ \Delta M_{2i} \\ \Delta M_{2j} \\ \Delta P_{Y2} \end{Bmatrix} \tag{7.48}$$

The coefficient matrix on the right hand side of equation (7.48) is the $[A_G]$ matrix. With the individual member lengths equal to $l/2$, the matrix of member geometric stiffnesses is,

$$[k_G] = -\frac{Pl}{60}\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{120}{l^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{120}{l^2} \end{bmatrix} \tag{7.49}$$

Thence,

$$[K_G] = [A_G][k_G][A_G]^T = -\frac{Pl}{60}\begin{bmatrix} 4 & -\frac{6}{l} & -1 & 0 \\ -\frac{6}{l} & \frac{288}{l^2} & 0 & \frac{6}{l} \\ -1 & 0 & 8 & -1 \\ 0 & \frac{6}{l} & -1 & 4 \end{bmatrix} \tag{7.50}$$

The structure flexibility matrix $[F]$ is given,

$$[F] = \frac{l}{192EI}\begin{bmatrix} 64 & 12l & -8 & -32 \\ 12l & 4l^2 & 0 & -12l \\ -8 & 0 & 16 & -8 \\ -32 & -12l & -8 & 64 \end{bmatrix} \tag{7.51}$$

Using the notation of equation (7.37),

$$\{\Delta R\} = ([K_E] + \lambda[K_G])\{\Delta r\} \tag{7.52}$$

In which $\lambda = P$ and $[K_G]$ has been calculated for $P = 1$. Solutions can be obtained for $\{\Delta R\} = 0$ by solving the eigen value problem,

$$([K_E] + \lambda[K_G])\{\Delta r\} = 0 \tag{7.53}$$

With $[K_E]^{-1} = [F]$, the flexibility matrix, This equation can be rearranged,

$$(\frac{1}{\lambda}[I] + [F][K_G])\{\Delta r\} = 0 \tag{7.54}$$

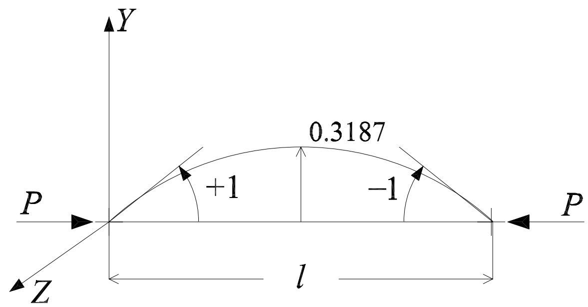

Figure 7.6: Mode shape for 2 segment column

Letting $\{\Delta\} = \{X\}$ and $[A] = -[F][K_G]$ finally the eigen value equation is written,

$$[A]\{X\} = \frac{1}{\lambda}\{X\} \tag{7.55}$$

The matrix $[A]$ is in general unsymmetric, and equation (7.55) can be easily solved by matrix iteration for the maximum value of $1/\lambda$, that is, for the smallest $\lambda = P_{cr}$. Now in the above example substitute $E = 100$, $l = 10$ and $I = 1$. From the equation (7.55), solving the eigenvalue equation,

$$[F][K_G]\{X\} = \frac{1}{\lambda}\{X\} \tag{7.56}$$

gives the critical load approximation to $\pi^2 \approx 9.87$,

$$P_{cr} = 9.9445 \tag{7.57}$$

an error of approximately 0.7%. The mode shape is shown, normalized with a maximum value equal to 1, in Figure 7.6. When the same problem is solved with four segments in the length the value obtained to two significant figures is 9.87. Both this and the worked example above are solved in B11 and B12 of the data file DATN.DAT. Details of the matrix commands and the steps necessary to carry out the solution are given in Section 7.3.

7.2 Matrix Iteration, calculation of buckling load

The iteration proceedure is set up to calculate the dominant root of the eigenvalue equations,

$$[A]\{X\} = \lambda\{X\} \tag{7.58}$$

in which, $[A] = [F][K_G]$ is an unsymmetric matrix and $\lambda = 1/P$, where P is a scalar multiplier of the axial forces in the members of the structure. The approximation to the dominant value of λ is calculated as,

$$\lambda = (\frac{r_n, r'_n}{r_n, r'_{n-1}}) \equiv (\frac{r_n^T r'_n}{r_n^T r'_{n-1}}) \tag{7.59}$$

where r_n and r'_n are the iteration vectors using $[A]$ and $[A]^T$ as outlined below. A starting vector Y_0 is chosen. This vector must contain part of the dominant buckling shape. Some experience may be required in choosing a suitable Y_0. Two sequences of iteration vectors are formed, as follows:

$$Y_0, \quad Y_1 = AY_0, \quad Y_2 = AY_1 = A^2 Y_0, \quad \text{etc.} \tag{7.60}$$

$$Y_0, \quad Y'_1 = A^T Y_0, \quad Y'_2 = A^T Y'_1 = [A^T]^2 Y_0, \quad \text{etc.} \tag{7.61}$$

Then the approximation to λ_1 is given by the ratio of the inner products,

$$\lambda \approx (\frac{Y_k^T Y'_k}{Y_k^T Y'_{k-1}}) \tag{7.62}$$

First it is shown that eigen vectors of A and A^T are orthogonal. It must be recognized that A and A^T have the same eigenvalues because the value of the determinant is not changed if rows and columns of a square matrix are interchanged. Let X_r be an eigenvector of A and X'_s one of the transpose, A^T. Then calculate the inner products,

$$\{(AX_r)^T X'_s\} = \lambda_r X_r^T X'_s \tag{7.63}$$

Now the lefthand side of this equation can be rewritten,

$$(AX_r)^T X'_s = X_r^T A^T X'_s = \lambda'_s X_r^T X'_s \tag{7.64}$$

Subtract these two equations,

$$0 = (\lambda_r - \lambda'_s)(X_r^T X'_s) \tag{7.65}$$

If the eigenvalues are real and distinct, $(\lambda_r \neq \lambda'_s)$ unless $r = s$. Therefore it follows that,

$$X_r^T X'_s = 0 \tag{7.66}$$

and the vectors can be normalized so that,

$$X_r^T X'_r = 1 \tag{7.67}$$

Now express Y_0 in both the X_i and the X'_i generalized coordinate systems.

$$Y_0 = a_1 X_1 + a_2 X_2 + ... + a_n X_n \tag{7.68}$$

$$Y_0' = Y_0 = b_1X_1' + b_2X_2' + ... + b_nX_n' \tag{7.69}$$

Then the inner product between Y_k' and Y_k is written,

$$(Y_k')^TY_k = ([A^T]^kY_0)^TA^kY_0 = ((Y_0)^TA^{2k}Y_0) \tag{7.70}$$

Substitute for Y_0, Y_0^T from equations(7.68) and (7.69), and because of the orthogonality properties of the eigenvectors, it follows,

$$\begin{aligned}(Y_k')^TY_k &= a_1b_1\lambda_1^{2k} + a_2b_2\lambda_2^{2k} + ... \\ &= \lambda_1^{2k}a_1b_1\{1 + \frac{a_2b_2}{a_1b_1}(\frac{\lambda_2}{\lambda_1})^{2k} + ...\}\end{aligned} \tag{7.71}$$

This series converges to:

$$(Y_k')^TY_k \Rightarrow \lambda_1^{2k}a_1b_1 \tag{7.72}$$

Similarly,

$$\begin{aligned}(Y_{k-1}')^TY_k &= a_1b_1\lambda_1^{2k-1} + a_2b_2\lambda_2^{2k-1} + ... \\ &= \lambda_1^{2k-1}a_1b_1\{1 + \frac{a_2b_2}{a_1b_1}(\frac{\lambda_2}{\lambda_1})^{2k-1} + ...\}\end{aligned} \tag{7.73}$$

and this series converges to:

$$(Y_{k-1}')^TY_k \Rightarrow \lambda_1^{2k-1}a_1b_1 \tag{7.74}$$

It follows that the approximation to λ_1 is,

$$\lambda_1 \approx \frac{(Y_k')^TY_k}{(Y_{k-1}')^TY_k} \tag{7.75}$$

This iteration process converges to the $2k$ power. Then Y_k gives the eigen vector mode approximation. The important point is that in the choice of Y_0, a_a and b_1 must not be zero, otherwise λ will converge to a root different from λ_1. A possible choice for Y_0 should contain a displacement term degree of freedom that corresponds to the largest diagonal term of the flexibility matrix, $[F]$.

7.3 Commands to calculate beam critical load

The geometric stiffness force transformation matrix $[A_G]$ is calculated in the command BEAMEQ that sets up the nodal equations of equilibrium (matrix EQ), and has been stored in an internally defined matrix GEQ. Then calls to BMMSTF (beam member stiffness) and BMGSTF (beam global stiffness) are used with the appropriate input to calculate KG. Thus :

LOADR B1 R=? C=?
(data input for member forces in B1)
BMMSTF B B1 MG (member geometric stiffness MG)
BMGSTF GEQ MG KG (global geometric stiffness KG)

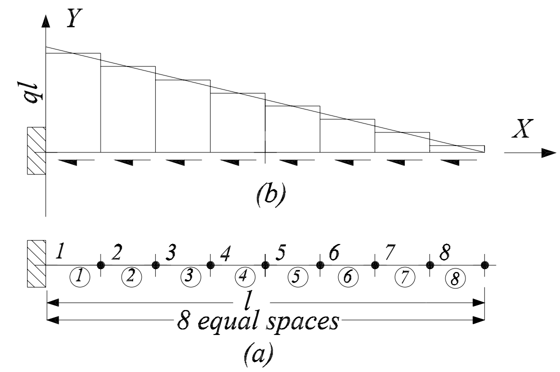

Figure 7.7: Cantilever column-uniformly applied axial load

The command CRITLD is then used to calculate the buckle mode shape and the critical load value. If the structure is determinate the flexibility matrix F must have been calculated as in the DATN.DAT files, problems B1 to B5 (numbers 17-21) and for indeterminate structures the flexibility matrix is calculated as the inverse of the elastic stiffness matrix and is stored in the matrix K. See problems B6 to B9 (numbers 22-26) for indeterminate beam analyzes. Three example command listings are given on DATN.DAT in 27 to 29 for the calculation of critical beam loads and corresponding modes shapes. The first two of these are determinate beams and the third a two span indeterminate beam. The first example 27 is for a simply supported beam with one central internal node. This example is useful in that all the matrices have sufficiently small dimensions to be printed out and their terms studied. Example 28 is simply 17 with the commands for the critical load calculations added. Similarly 29 is problem 22 with data arranged so that span lengths, Young's modulus and second moment of area are the same as for 27 and 28 and so results can be compared. The commands with trial vectors for problems 27 and 29 on the file DATN.DAT are given below.

Determinate structures

LOADR R0 R=1 C=6
0 0 1.0 0 0 0
CRITLD F KG R0 LA
The cantilever beam shown in Figure 7.7 has the critical load,

$$(ql)_{crit} = \frac{\alpha EI}{l^2}$$

With $\alpha = 7.83$. The analysis of Figure 7.7 gives the value of 7.79, an error of 0.51.

Indeterminate structures

```
LOADR R0 R=1 C=18
0 0 0 0 1.0 0 0 0 0 0 0 0 0 0 -1.0 0 0 0
CRITLD K KG R0 LA
```

This is an antisymmetric mode shape and is required for the 2 span beam.

7.4 Frame buckling

The theory for the generation of the nodal equilibrium equations for frame structures has been detailed in Chapter 4, section 4.2. For the frame member, the basic member forces are the axial force and end moments, $(F,\ M_i,\ M_j\,)$. In section 4.2, see equation (4.1) and Figure 4.2. It was shown how the member end forces expressed in terms of the local member axes components are derived in terms of the three values (F, M_i, M_j). For the effect of changes in geometry on these force components, the theory of section 7.1.6 applies and is here easily adapted for frames. It is convenient now to use the order $(\Delta P_y, \Delta M_i, \Delta M_j)$ so that the force and moment terms are arranged in the same order as in Chapter 4, equation (4.1). Thence the transformation to local coordinate components is expressed, by,

$$\Delta \left\{ \begin{array}{c} F'_{ix} \\ F'_{iy} \\ M_i \\ F'_{jx} \\ F'_{jx} \\ M_j \end{array} \right\} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{l} & \frac{1}{l} \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{l} & -\frac{1}{l} \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} \Delta P \\ \Delta M_i \\ \Delta M_j \end{array} \right\} \tag{7.76}$$

or symbolically,

$$\{\Delta F'_G\} = [A'_G]\{\Delta S_G\} \tag{7.77}$$

The transformation to global components is then given by the identical transformation to that is equation (4.2). That is,

$$\{\Delta F_G\} = [L_D]\{\Delta F'_G\} = [L_D][A'_G]\{\Delta S_G\} = [A_G]\{S_G\} \tag{7.78}$$

where,

$$[A_G] = [L_D][A'_G] \tag{7.79}$$

With the obvious change in the first column of the coefficient matrix on the right hand side of equation (7.76), the same procedure can be used to set up the matrix GEQ as for

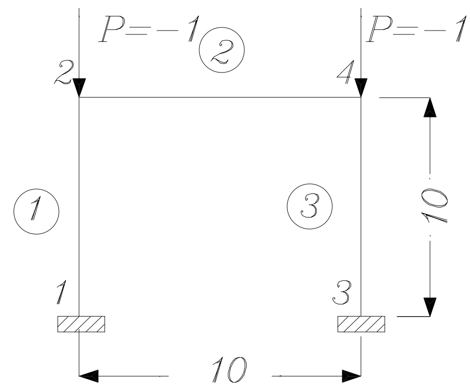

Figure 7.8: Portal Frame axial compressive forces

EQ. The member geometric stiffness matrix is written in the reordered sequence,

$$\begin{Bmatrix} \Delta P \\ \Delta M_i \\ \Delta M_j \end{Bmatrix} = -P \begin{bmatrix} \frac{1}{l} & [0] \\ [0] & \frac{l}{30}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \Delta y' \\ \Delta \phi_i' \\ \Delta \phi_j' \end{Bmatrix} \tag{7.80}$$

For then whole structure, the stiffness matrix can be calculated by setting up $[A_G]$ for all the member forces and $[k_G]$ becomes a matrix with $k_{Gi}]$ as diagonal blocks. Thus $[K_G]$ may be formed in an identical manner to $[K_E]$ once $[A_G]$ and $[k_G]$ have been assembled. the commands will be to form $[k_G]$

FRMSTF B B1 MG

and to form $[K_G]$

FRGSTF GEQ MG KG

The frame shown in Figure 7.8 has axial compressive loads of unit values in members 1 and 3. The values of A and I and Young's modulus are (1.0,1.0,10000.0) respectively. The frame buckles with a sidesway displacement and this must be included in the starting iteration vector. The commands and data are given in data set F1 on the DATN.DAT file. The value is obtained in the variable labeled LA,

$$P_{critical} = 740.0$$

The value obtained from Timoskenko is 740.2. If the bases are pinned rather than fixed then

$$P_{critical} = 181.4$$

This compares with the value from Timoskenko,

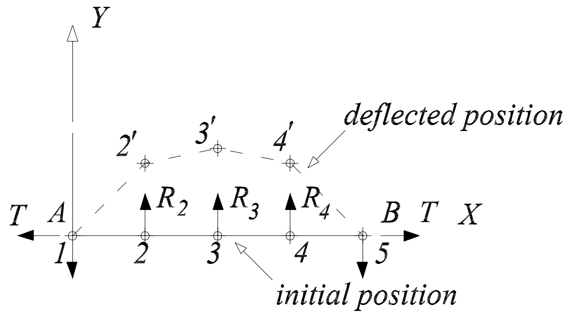

Figure 7.9: Initial and final positions of cable

$$P_{critical} = \frac{1.82EI}{l^2} = 182.0$$

The difference is 0.2%.

7.5 Tension structures

7.5.1 Introduction

Thus far in Chapter 7, the stability phenomena related to compressive forces in structures and structural members has been studied. A knowledge of buckling is necessary for the design of compression members, for example the top chord of a simply supported truss, the columns of a high rise building and the rib of a steel arch bridge. In contrast to these, modern building makes good use of light weight cables and fabric materials in which the positive stiffness of applications members can be utilized. Suspension and cable stayed bridges are quite old examples of this construction technique. Tension cable structures with light weight fabric membrane shells to provide roof covering are more recent additions to building forms, although here too the fabric tent is of very ancient origin. In the remainder of Chapter 7, the concept of geometric stiffness is studied in order to show how tension structures provide stiffness and load carrying capacity making them suitable for long spans and provision of large open space areas. Because of the very nature of cable net construction it will be necessary to develop the theory for two and three dimensional systems. The two dimensional case is introduced first and then the three dimensional analysis leads to the highly commercialized topic of shape finding of membrane structures. As a simple example of a planar cable structure, consider the cable AB, nodes 1-5 shown in Figure7.9. The cable is stretched between rigid supports A and B and has an initial tension T that may be induced by either jacking against one end or by tightening a turn buckle in the cable. When the loads $(R_1 - R_4)$ are applied at nodes (2-4) of the undeflected cable AB, the cable has zero elastic stiffness in the transverse Y direction. However, because of the initial tension it possesses geometric stiffness. If the

tension T is large in comparison to the load increments $(\delta R_1 - \delta R_4)$ the initial deflections will be small and a linear analysis may give a close approximation to the deflected shape. Finally however the cable may take up the position $(1-2'-4'-5)$ shown by the broken line in Figure 7.9, and deflections are no longer small. To trace the load path from the initial to the final position will require a non-linear, iterative analysis. If small increments of the loads and the deflections are considered the analysis is not too difficult to accomplish. After the initial step, the following situation arises:
(1) elastic stiffness is present.
(2) cable tension has changed.
(3) initial geometry must be updated.
(4) in the updated position iteration is necessary to re-establish equilibrium.
(5) slack members are not allowed and if they occur initial tensions in the cables must be increase.

7.5.2 Analysis of a plane cable

In this first example a simple case is considered of a cable hanging under its own weight and supported at its ends. The cable can then be deflected by additional nodal forces and a suspension bridge type of cable profile generated. A simple planar cable is shown in Figure 7.10 (a) under the influence of the vertical forces R_1 and R_2. The weight of the cable will be lumped at these nodes and for each segment the weight is replaced by equal vertical nodal forces at its ends. If the unstrained lengths l_{10}, l_{20}, etc. are known, and the end forces H and V are assumed, the position at end I may be calculated. For uniform load, V will be assumed to be one half the total applied load and $H = WL/8S$; where W is the total vertical load, l the span between supports and S the centre sag. If both supports are at the same level then V must be in both static and moment equilibrium with the applied loads. The deflection of I is calculated and if it is different from the support location B, H and V must be modified in such a way as to bring I into coincidence with B. This is achieved in a manner which calculates the flexibility coefficients at I including the effects of the axial forces in the cable members. The first step is to calculate the tangent flexibility matrix of the cable member with axial force P, components (X, Y) for small variations $(\Delta X, \Delta Y)$ of P. The member is shown in Figure 7.10(b) with a rotation increment of $\delta\alpha$. That is, it is required to find $[F_T]$, the tangent flexibility matrix, such that for the member in Figure 7.10(b),

$$\left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\} = [F_T] \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \tag{7.81}$$

From Figure 7.10(b),

$$\delta y' = l\delta\alpha \ ; \left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\} = \left\{ \begin{array}{c} -\sin\alpha \\ \cos\alpha \end{array} \right\} \delta y' = l \left\{ \begin{array}{c} -\sin\alpha \\ \cos\alpha \end{array} \right\} \delta\alpha \tag{7.82}$$

Also

$$\Delta P = P\delta\alpha, \qquad \text{and by projection} \quad \Delta P = [\,-\sin\alpha \quad \cos\alpha\,] \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \tag{7.83}$$

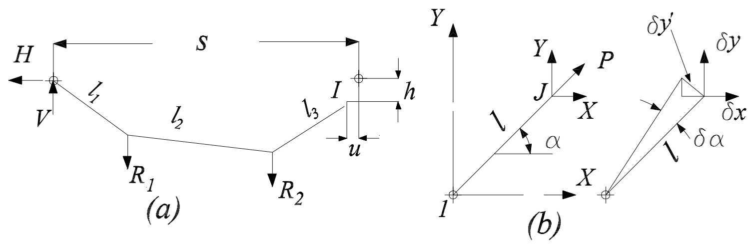

Figure 7.10: Cable analysis-member rotation

Thence,

$$\delta\alpha = \frac{\Delta P}{P} = \frac{1}{P}\left[-\sin\alpha \quad \cos\alpha \right] \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \tag{7.84}$$

Combining equations (7.82) and (7.84) the effect of member rotation is given,

$$\begin{aligned} \left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\} &= \frac{l}{P} \left\{ \begin{array}{r} -\sin\alpha \\ \cos\alpha \end{array} \right\} \left[-\sin\alpha \quad \cos\alpha \right] \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \\ &= \frac{l}{P} \begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix} \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \end{aligned} \tag{7.85}$$

That is, if the cable rotates to cause $(\Delta X, \Delta Y)$ variation in the components of P, equation (7.85) gives the $(\delta x, \delta y)$ changes in the member projections.
The elastic flexibility matrix is calculated for member force and deformation,

$$\Delta v = \frac{l}{EA}\Delta P = f\Delta P \tag{7.86}$$

and so the X, Y components of deflection $(\delta x, \delta y)$ are given,

$$\left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\} = \left\{ \begin{array}{c} \cos\alpha \\ \sin\alpha \end{array} \right\} \Delta v; \text{ that is,} \quad \Delta r = [a]\Delta v \tag{7.87}$$

Contragredience gives,

$$\Delta P = [a]^T \Delta R \tag{7.88}$$

So that for the elastic deformation,

$$\begin{aligned} \left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\}_E &= [a] f [a]^T \Delta R \\ &= \frac{l}{EA} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} \Delta X \\ \Delta Y \end{array} \right\} \end{aligned} \tag{7.89}$$

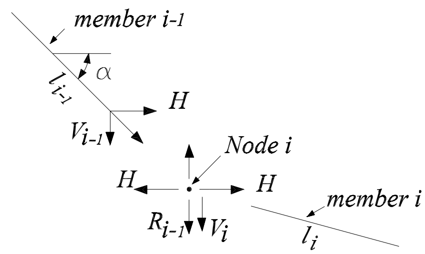

Figure 7.11: Forces at Cable Node

The total flexibility is the sum of elastic and geometric effects so that

$$[F_T] = [F_E] + [F_G] = \frac{l}{P}\begin{bmatrix} \sin^2\alpha + \frac{P}{EA} & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha + \frac{P}{EA} \end{bmatrix} \tag{7.90}$$

If then H and V are varied and produce variations of components $(\Delta X, \Delta Y)_i$ in the cable member i, the displacements at I will the sum over the i members,

$$\left\{ \begin{array}{c} \delta u \\ \delta v \end{array} \right\} = \sum_{i=1}^{N} \left\{ \begin{array}{c} \delta x \\ \delta y \end{array} \right\} \tag{7.91}$$

7.5.2.1 Cable force

If the cable values of H and V are assumed a recurrence relationship can be established to determine the corrected values of cable tensions P_i, and thus finally H and V From Figure 7.11,

$$V_1 = V \ ; \ V_i = V_{i-1} - R_i \text{ for } i > 1 \tag{7.92}$$

Cable tension,

$$P_i = (H_i + V_i^2)^{\frac{1}{2}} \tag{7.93}$$

Member position,

$$\left\{ \begin{array}{c} \sin\alpha_i \\ \cos\alpha_i \end{array} \right\} = \frac{1}{P_i} \left\{ \begin{array}{c} V_i \\ H \end{array} \right\} \tag{7.94}$$

Strained lengths.

$$l_i' = l_i(1 + \frac{P_i}{EA}) \tag{7.95}$$

Projections on the coordinate axes,

$$\left\{ \begin{array}{c} x_i \\ y_i \end{array} \right\} = l_i' \left\{ \begin{array}{c} \cos\alpha_i \\ \sin\alpha_i \end{array} \right\} \tag{7.96}$$

and $s = \sum x_i$, $h = \sum y_i$. The errors in the approximation are given by the differences between these values,

$$\Delta u = s - \sum x_i \quad \Delta v = h - \sum y_i \tag{7.97}$$

Then,

$$\begin{aligned} \begin{Bmatrix} \Delta u \\ \Delta v \end{Bmatrix} &= [F_T] \begin{Bmatrix} \Delta H \\ \Delta V \end{Bmatrix} \\ \begin{Bmatrix} \Delta H \\ \Delta V \end{Bmatrix} &= [F_T]^{-1} \begin{Bmatrix} \Delta u \\ \Delta v \end{Bmatrix} \end{aligned} \tag{7.98}$$

The new approximations to H and V are,
$H = H + \Delta H$; $V = V + \Delta V$. The process is repeated until the convergence criterion is reached,

$$\frac{|\Delta H|}{H} < \text{ tolerance} \tag{7.99}$$

In the examples studied it have been found possible to use a tolerance of 10^{-4} or even 10^{-5} depending on the finally accuracy required. If a cable is assumed to be parabolic under uniform self weight. Its shape is approximated knowing the span and the centre sag. The cable length is calculated,

$$l_0 = \int_0^s \sqrt{1 + \frac{dy}{dx}^{\frac{1}{2}}} dx \tag{7.100}$$

This is easily integrated numerically using either trapezoidal or Simpson's rules to obtain the initial length and is incorporated in the initial shape finding command. The command,

CABEXE A D=??? N=?

The three parameters in D are 1) the clear span, 2) the assumed centre sag, and 3) the span divided by the number of segments to be used. If N=0, the cable length is subdivided into equal lengths whereas for N=1 the X segments are equal and the cable segment lengths will vary. Thus the command CABEXE simply generates the initial parabolic shape and divides it into the desired number of segments and stores them in A. The command

CABSHP A S B C E=?,?,?

produces a cable shape under a given load condition. The load values at all internal nodes are contained in the two columns of the matrix B and in the present case these are generated as input. The first column gives the self weight nodal loads and the second column a limited number of symmetrically applied point loads of opposite direction to the self weight forces. In this way a cable profile similar to that used in suspension bridge construction can be obtained. E=?,?,? gives the three parameters, 1) Young's modulus of elasticity of the cable, 2) the cable area and 3) the number of load case increments for the second case that are to be added to the self weight. If the number of load cases is zero only the dead weight shape is produced. The X, Y coordinates of the final cable profile are given in C. A= initial segment geometry(segment lengths from CABSHP), S contains

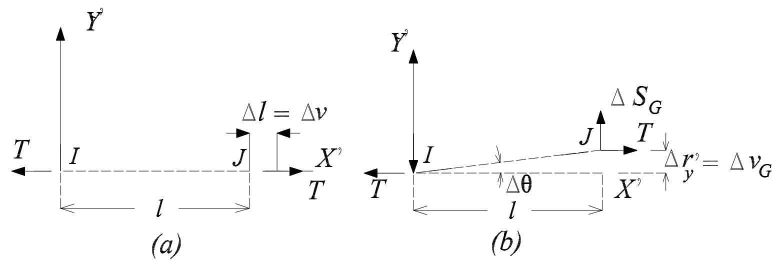

Figure 7.12: Elastic deformation and rigid body rotation

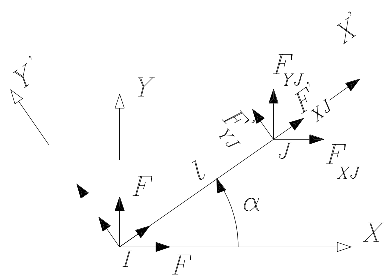

Figure 7.13: Local and global force components

1) span, 2) 0.0, 3)assumed initial horizontal tension, 4) calculated initial vertical support at the left hand support, 5) assumed central sag. A plot of the final profiles is obtained using,

PLTCAB A B

An example is given in DATN.DAT (C2), of a cable of 1500feet over all span and an initial sag of 300 feet. The cable has an area of 0.307 sq. inches, Young's modulus of 30.0 $\times 10^3$ k/sq. inch. (1k=1000lbs). The interesting result of the analysis is that the distance between tower supports will vary according to the loading and if a fixed distance is specified an iteration is required on the cable segment lengths.

7.5.3 Elastic and geometric stiffness

The theory is developed here for general two dimensional systems. The difference between elastic and geometric stiffness of a single member is illustrated in Figure 7.11. The elastic stiffness is associated with a small axial deformation of the member $\Delta r_x'$ whereas the geometric stiffness is associated with a small rigid body rotation, $\Delta\theta$. For the cable member the elastic distortion and the geometric displacement are shown in Figure 7.13(a) and (b) respectively. In Chapter 2 only elastic forces are considered in calculation of truss deflections because deformations are small. Now this is no longer the case. For the elastic force increment ΔS_E the member forces at the ends I and J of the member in its local X', Y' coordinate axes are,

$$\begin{Bmatrix} F'_{xi} \\ F'_{xj} \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \Delta S_E \tag{7.101}$$

and for all four X', Y' components at ends I and J,

$$\begin{Bmatrix} F'_{xi} \\ F'_{yi} \\ F'_{xj} \\ F'_{yj} \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \Delta S_E \tag{7.102}$$

For the geometric force increment, ΔS_G, for the member with axial force T, the member end forces, from 7.12(b),

$$\begin{Bmatrix} F'_{yi} \\ F'_{yj} \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \Delta S_G \tag{7.103}$$

and for all four X', Y' components,

$$\begin{Bmatrix} F'_{xi} \\ F'_{yi} \\ F'_{xj} \\ F'_{yj} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{Bmatrix} \Delta S_G \tag{7.104}$$

The force components in the global X, Y axes, see Figure 7.13,are given using the transformation of vector components, equation (7.104),

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} F'_x \\ F'_y \end{Bmatrix} \tag{7.105}$$

Thence from equations (7.102) and (7.105) for the ΔS_E force component applying equation (7.105), to both ends of the member,

$$\Delta \begin{Bmatrix} F_{xi} \\ F_{yi} \\ F_{xj} \\ F_{yj} \end{Bmatrix} = \begin{Bmatrix} -\cos\alpha \\ -\sin\alpha \\ \cos\alpha \\ \sin\alpha \end{Bmatrix} \Delta S_E \tag{7.106}$$

And for the geometric stiffness, global components, using equation(7.104) and (7.105),

$$\Delta \begin{Bmatrix} F_{xi} \\ F_{yi} \\ F_{xj} \\ F_{yj} \end{Bmatrix} = \begin{Bmatrix} \sin\alpha \\ -\cos\alpha \\ -\sin\alpha \\ \cos\alpha \end{Bmatrix} \Delta S_G \tag{7.107}$$

the equations (7.106) and(7.107) are then used to set up the global force equilibrium equations involving all nodal forces and reactions just as for the truss structure, see section 2.2.

$$[A_E]\{\Delta S_E\} = \{\Delta R\} \tag{7.108}$$

and in the same way the equation (7.107) is for the geometric force components,

$$[A_G]\{\Delta S_G\} = \{\Delta R\} \tag{7.109}$$

To analyze any system, the stiffness method is used. Both loads and support displacements must be considered. Contragredience is applied to both equations (7.108) and (7.109) to obtain the corresponding relationships between global nodal displacements and member deformations, that is

$$\{\Delta v_E\} = [A_E]^T\{\Delta r\} \tag{7.110}$$

$$\{\Delta v_G\} = [A_G]^T\{\Delta r\} \tag{7.111}$$

Again it is seen that once the equilibrium equations have been determined so also have the displacement transformations. The elastic stiffness is obtained from the properties of the member, its length l, area of cross section A and Young's modulus of elasticity E see equation (2.31),

$$k_{Ei} = \frac{EA_i}{l_i}; \quad \Delta S_{Ei} = \frac{EA_i}{l_i}\Delta v_{Ei} \tag{7.112}$$

For all members of the cable system, define the $(n \times n)$ diagonal matrix, for the n members

$$[k_E] = [k_{Ei}]; \quad k_{Ei} = \frac{EA_i}{l_i} \tag{7.113}$$

For the geometric stiffness is given from its length l and the tension T in the member,

$$k_{Gi} = \frac{T_i}{l_i}; \quad \Delta S_{Gi} = \frac{T_i}{l_i}\Delta v_{Gi} \tag{7.114}$$

and the geometric stiffness matrix for all members is the diagonal matrix,

$$[k_G] = [k_{Gi}]; \quad k_{Gi} = \frac{T_i}{l_i} \tag{7.115}$$

The elastic tangent stiffness matrix is obtained by combining equations, (7.108),(7.110), (7.113),

$$[A_E]\{\Delta S_E\} = [A_E][k_E][A_E]^T\{\Delta r\} = [K_E]\{\Delta r\} = \{\Delta R_E\} \tag{7.116}$$

and similarly the tangent geometric stiffness matrix is obtained by combining equations, (7.109),(7.111),(7.115),

$$[A_G]\{\Delta S_G\} = [A_G][k_G][A_G]^T\{\Delta r\} = [K_G]\{\Delta r\} = \{\Delta R_G\} \tag{7.117}$$

Combining equations(7.116) and (7.117),

$$\{\Delta R_G\} = ([K_E] + [K_G])\{\Delta r\} = [K_T]\{\Delta r\} \tag{7.118}$$

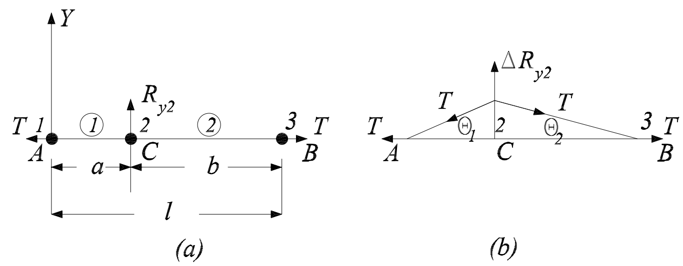

Figure 7.14: Straight cable-transverse load

The equations(7.108), (7.109),(7.110), (7.111),(7.118), may be used in the non-linear analysis of a tension cable structure, given here in two dimensions. It can easily be extended to three dimensional cable structures.

7.5.3.1 Example of geometric stiffness

A simple example is given to illustrate the meaning and use of geometric stiffness of a cable structure. Consider the cable, shown in Figure 7.14, length l with a tension T and supported rigidly at its ends, A and B (nodes 1 and 3). A load P is applied in the Y direction at C (node 2) that divides the length l into segments of lengths $(a, b), a + b = l$ Assuming the displacements are small T large, and l relatively small, calculate the vertical deflection of node 2. Find the relationship between T and P if the deflection of node 2 is limited to $l/400$. First, the complete $[A_G]$ matrix is written for the tension force in members (1) and (2), and then the equations that are not required are discarded. Note that for these members in the intimal state, $\cos\alpha = 1,\ \sin\alpha = 0.0$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta S_{y1} \\ \Delta S_{y2} \\ \Delta R_{cx1} \\ \Delta R_{cy1} \\ \Delta R_{cx3} \\ \Delta R_{cy3} \end{Bmatrix} = \begin{Bmatrix} \Delta R_{x1} \\ \Delta R_{y1} \\ \Delta R_{x2} \\ \Delta R_{y2} \\ \Delta R_{x3} \\ \Delta R_{y3} \end{Bmatrix} \tag{7.119}$$

The first, third and last equations can be deleted, as can the reactions, $\Delta R_{cx1}, \Delta R_{cx3}$ as equations in the X direction are not considered.

$$[A_G]\Delta S_G = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta S_{y1} \\ \Delta S_{y2} \\ \Delta R_{cy1} \\ \Delta R_{cy3} \end{Bmatrix} = \begin{Bmatrix} \Delta R_{y1} \\ \Delta R_{y2} \\ \Delta R_{y3} \end{Bmatrix} \tag{7.120}$$

Assume that the reaction stiffnesses are infinite and with constant tension T in the cable,

$$[k_G] = \left\{ \begin{array}{cccc} \frac{T}{a} & 0 & 0 & 0 \\ 0 & \frac{T}{b} & 0 & 0 \\ 0 & 0 & \infty & 0 \\ 0 & 0 & 0 & \infty \end{array} \right\} \tag{7.121}$$

Then forming $[K_G]$,

$$[K_G] = [A_G][k_G][A_G]^T \tag{7.122}$$

$$\begin{bmatrix} \infty & -\frac{T}{a} & 0 \\ -\frac{T}{a} & \frac{T}{a}+\frac{T}{b} & -\frac{T}{b} \\ 0 & -\frac{T}{b} & \infty \end{bmatrix} \left\{ \begin{array}{c} \Delta r_{y1} \\ \Delta r_{y2} \\ \Delta r_{y3} \end{array} \right\} = \left\{ \begin{array}{c} \Delta R_{y1} \\ \Delta R_{y2} \\ \Delta R_{y3} \end{array} \right\} \tag{7.123}$$

The solution of these three equations give, $\Delta r_{y1} = \Delta r_{y3} = 0$ and,

$$\left\{ \frac{T}{a} + \frac{T}{b} \right\} \Delta r_{y2} = \Delta R_{y2} \tag{7.124}$$

that is,

$$\Delta r_{y2} = \left(\frac{ab}{Tl} \right) \Delta R_{y2} \tag{7.125}$$

This same result may be obtained from Figure 7.14(b) by resolving forces at C in the Y direction. It is seen that with the geometric stiffness matrix included the equations are written in the undeformed position and the displacements then obtained from that position correspond to calculating the equilibrium in a displaced position close to the undeformed position. Finally, Δr_{y2} is limited to $1/400$ of the span, so that with $a = b = l/2$ in equation (7.125),

$$\frac{l}{400} = \frac{l}{4T} \Delta R_{y2} \tag{7.126}$$

gives, the value of the required tension,

$$T = 100 \Delta R_{y2} \tag{7.127}$$

7.5.4 Three dimensional cables

The derivation of the *geometric stiffness* for a cable in three dimensional space presents additional problems because the direction of the transverse displacements cannot be determined à-priori as in the two dimensional case. A cable member $I - J$ in three dimensional space is shown in Figure 7.15. The axes Y', Z' in a plane perpendicular to $I - J$ are chosen and the components of displacements at each end are (v'_{YI}, v'_{ZI}) and (v'_{YJ}, v'_{ZI}). These

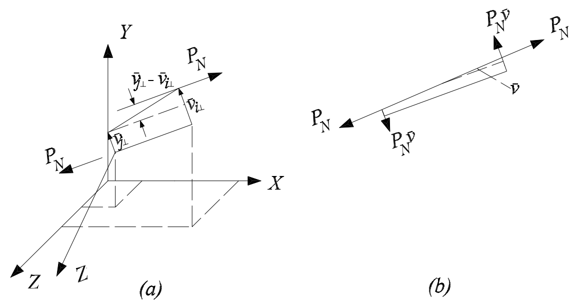

Figure 7.15: Cable members in three dimensions

may be combined into a single vectors v'_I, v'_J at the ends I and J and then the relative displacement in the $Y'Z'$ plane J to I is $v'_j - v'_I$. However its direction depends on the displacement components. To overcome this problem the relative components in the $X'Y'$ and $X'Z'$ planes can be used or alternatively it is possible to work directly with XY and XZ components and derive a geometric stiffness matrix for the member, directly in the XYZ axes. This means that the $[A_G]$ matrix will not be formed explicitly except in the global axes.

7.5.4.1 Three dimensional cable net stiffnesses

The member $I - J$. length l, has direction cosines,

$$\{c\} = \left\{ \begin{array}{c} c_x \\ c_y \\ c_z \end{array} \right\} \tag{7.128}$$

Then the global components of the member force P are given,

$$\{P_I\} = \left\{ \begin{array}{c} P_x \\ P_y \\ P_z \end{array} \right\} = \left\{ \begin{array}{c} c_x \\ c_y \\ c_z \end{array} \right\} P = \{C\}S \tag{7.129}$$

and for both the ends I and J, for a force increment ΔP

$$\left\{ \begin{array}{c} \Delta P_I \\ \Delta P_J \end{array} \right\} = \left\{ \begin{array}{r} -C \\ C \end{array} \right\} \Delta S \tag{7.130}$$

Contragredience applied to equation (7.130) gives the change of the length of the member Δl for the increments in the nodal displacements,

$$\Delta l = \Delta v_E = [-C \ C] \left\{ \begin{array}{c} \Delta v_I \\ \Delta v_J \end{array} \right\} \tag{7.131}$$

And the member stiffness k_E is the inverse of the flexibility,

$$\Delta S_E = k_E \Delta v_E \tag{7.132}$$

Combining equations (7.130)- (7.132) the stiffness relationship for the elastic forces and nodal displacements is given,

$$\begin{aligned}
\frac{EA}{l}\begin{Bmatrix}\Delta P_I\\ \Delta P_J\end{Bmatrix} &= \begin{Bmatrix}-C\\ C\end{Bmatrix}[-C\ C]\begin{Bmatrix}\Delta v_I\\ \Delta v_J\end{Bmatrix}\\
&= \frac{EA}{l}\begin{bmatrix}CC^T & -CC^T\\ -CC^T & CC^T\end{bmatrix}\begin{Bmatrix}\Delta v_I\\ \Delta v_J\end{Bmatrix}
\end{aligned} \tag{7.133}$$

this is the usual expression for the elastic stiffness of a single truss or bar type of element. Combining equation (7.133) for all members,

$$[A_G]\{S\} = \{R\} \tag{7.134}$$

Where $[k_E]$ is a diagonal matrix with terms k_{Ei} as diagonal terms, an ∞ or a large number for the support stiffnesses. Then for all the nodal forces,

$$\Delta\{R\} = [A_E][k_E][A_E]^T\Delta\{v\} = [K_E]\Delta\{v\} \tag{7.135}$$

gives the elastic stiffness relationship for the whole cable net.

7.5.4.2 Geometric stiffness

The displacements (v_x, v_y) of an end I or J may be resolved into vector components parallel $v_{\|}$ and and perpendicular $v_{\perp}$ to the member. That is, in vector form,

$$\{v\} = \{v_{\|}\} + \{v_{\perp}\} \tag{7.136}$$

The magnitude of $v_{\|}$ is given by the projection of X, Y, Z components on the direction $I - J$.

$$|v_{\|}| = \{C\}^T\{v\} \tag{7.137}$$

and the vector along $I - J$ is given by the resolution of $|v_{\|}|$ into its components.

$$\{v_{\alpha\|}\} = \{C\}\{C\}^T\{v_\alpha\} \tag{7.138}$$

The expression for the component perpendicular to the member is thus given by,

$$\{v_{\alpha\perp}\} = \{v_\alpha\} - \{C\}\{C\}^T\{v_\alpha\} \tag{7.139}$$

The vectors $\{v_{I\perp}\}$ and $\{v_{J\perp}\}$ at the two ends of the member are shown in Figure 7.15, $\alpha =$ or J. It is seen that the measure of the rotation of the member is given by $\{\bar{v}\}$, such that,

$$\{\bar{v}\} = \frac{1}{l}[\{v_{J\perp}\} - \{v_{I\perp}\}] \tag{7.140}$$

Substituting from equation (7.139) gives,

$$\{\bar{v}\} = \frac{1}{l}\left\{-[I_3 - \{C\}\{C\}^T]\ [I_3 - \{C\}\{C\}^T]\right\}\begin{bmatrix} v_I \\ v_J \end{bmatrix} \tag{7.141}$$

Using equations (7.140), (7.141)

$$\Delta\begin{bmatrix} P_I \\ P_J \end{bmatrix}_G = P_N\begin{bmatrix} -\{\bar{v}\} \\ \{\bar{v}\} \end{bmatrix} = P_N\begin{bmatrix} -I_3 \\ I_3 \end{bmatrix}\{\bar{v}\} \tag{7.142}$$

The vectors $\{\Delta P_{IG}\}$ and $\{\Delta P_{JG}\}$ are the changes in the global components due to the rotation $\theta = \{\bar{v}\}$, see Figure 7.15. Then,

$$\Delta\begin{bmatrix} P_I \\ P_J \end{bmatrix}_G = \frac{P_N}{l}\begin{bmatrix} [I_3 - \{C\}\{C\}^T] & -[I_3 - \{C\}\{C\}^T] \\ -[I_3 - \{C\}\{C\}^T] & [I_3 - \{C\}\{C\}^T] \end{bmatrix}_G \tag{7.143}$$

This is the equation for the member geometric stiffness

$$\{\Delta P_{Gi}\} = [k_{Gi}]\{v_i\} \tag{7.144}$$

Combining equations (7.133) and (7.144)

$$\{\Delta P_i\} = \{\Delta P_{Ei}\} + \{\Delta P_{Gi}\}[k_{Gi} = [k_E + k_G]\{v_i\} = [k_T]\{v_i\} \tag{7.145}$$

Finally, $[k_T]$ is written as a 6×6 partitioned matrix:

$$[k_T] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \tag{7.146}$$

where,

$$[k_1] = \left[\frac{EA}{l_0} - \frac{P_N}{l}\right]\{C\}\{C\}^T + \frac{P_N}{l}[I_3] \tag{7.147}$$

7.5.5 Iterative calculation

Simple Newton-Raphson type iteration can be carried out as follows. The total member force P_N is given by

$$\begin{aligned} \{R_i\} &= 0 \quad \text{zero initial load vector} \\ \{x_i\} &= 0 \quad \text{set initial position to initial coordinates} \\ \{R_i\} &= 0 \quad \text{input load increment} \end{aligned} \tag{7.148}$$

$$P_N = \frac{EA}{l_0}(l - l_0) = \frac{EA}{l_0}\Delta v_{N0} \tag{7.149}$$

in equation (7.149), l_0 are unstrained lengths and Δv_{N0} are thus the initial member strains.

$$\begin{aligned} (1)\ \{R_{i+1}\} &= \{R_i\} + \{\Delta R_i\} \text{ increment load vector} \\ (2)\ \{\Delta v_{Ni}\} &= [a_{Ni}]\{\Delta r_{Ni}\} \text{ calculate } \{\Delta v_{Ni}\} \\ \{\Delta R_{i+1}\} &= \{R_i\} - [a_{Ni}]\{P_N\} \end{aligned} \tag{7.150}$$

Check convergence of iteration,

$$\frac{\sum \Delta R_{i+1}}{\sum \Delta R_i} \leq \text{ toler go to 1 next load loop} \tag{7.151}$$

(Next iteration step)

$$[K_{Ti}] = [K_{Ei}] + [K_{Gi}] \quad \text{tangent stiffness} \tag{7.152}$$

$$\Delta r_{i+1} = [K_{Ti}]^{-1} \quad \text{displacement increment} \tag{7.153}$$

$$\{P_{N(i+1)}\} = \{P_{Ni}\} + [k_i][a_{Ni}]\Delta r_{i+1} \quad \text{cable force increments} \tag{7.154}$$

$$\{x_{i+1}\} = \{x_i\} + [k_i][a_{Ni}]\Delta r_{i+1} \quad \text{update coordinates} \tag{7.155}$$

Go to (2) start of iteration