Chapter 8

NATURAL FREQUENCIES OF STRUCTURES

8.1 Introduction-Oscillating Systems

A conservative system exhibits oscillation or vibration phenemona when disturbed from its equilibrium position and then released. The internal forces tend to restore the system to the equilibrium position and this is resisted by the inertia forces of the system masses. To illustrate the principle, the oscillation of a pendulum about its verictal equilibrium postion is analysed.

8.1.1 Oscillation of a Simpe Pendulum

Example I
As an example consider the simple pendulum shown in Figure ({fig81consisting of a mass M suspended by a long wire of length l of negligible mass displaced by a small angle β from its vertical equilibrium position.

From the figure the arc length is $l\beta$ so that the angular acceleration of the mass $l\ddot{\beta}$. Then the equation of motion is written takimg moments about the support pin,

$$ml\ddot{\beta} = -mg\sin\beta \approx -mg\beta \tag{8.1}$$

That is,

$$l\ddot{\beta} + g\beta = 0 \tag{8.2}$$

The solution of the differential equation is of the form,

$$\beta = A\sin kt \tag{8.3}$$

and is harmonic in nature. Substitution in the equation (8.2) gives,

$$(-\frac{l}{g}k^2 + 1)\beta = 0 \tag{8.4}$$

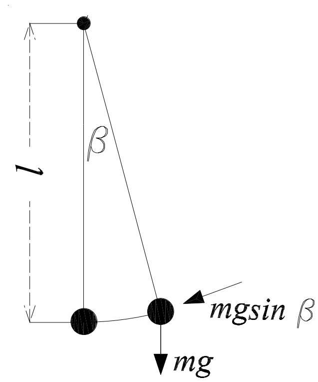

Figure 8.1: Simple pendulum forces in displaced position

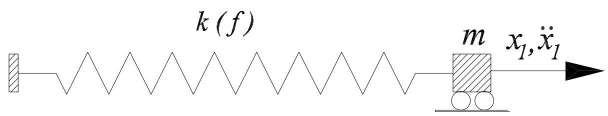

Figure 8.2: Single degree spring-mass system

Solution of this equation is possible only if the coefficient of β is equal to zero. That is if,

$$k^2 = \frac{g}{l} \qquad k = \sqrt{\frac{g}{l}} \tag{8.5}$$

The period of the vibration T is such that

$$k(t_0 + T) = k(t_0) + 2\pi, T = \frac{2\pi}{k} \quad T = 2\pi\sqrt{\frac{l}{g}} \tag{8.6}$$

It is seen that there is an interchange between Potential Energy at β_{max} and Kinetic Energy maximum at $\beta = 0$. In the case of the pendulum gravity provides the restoring force. The longer the wire the longer the period and the weaker the gravity field the longer the period. Elastic systems also exhibit the property of interchange of Potential and Kinetic Energies. In elastic systems Potential Energy is the elastic strain enegry stored internally in the material of the system.

8.1.2 Simple spring-lumped mass systems

Example II Single spring-mass system

Consider the mass m attached to a rigid support by the spring (considered to be weightless), of stiffness k or flexibility $f = \dfrac{1}{k}$ and supported on frictionless rollers. When the spring is neither extended nor compressed, the system is in equilibrium. However if the mass is displaced by an amount x it will oscillate about the equilibrium system coming to rest finally because of damping in the system. The elastic force acting on the mass at any displacement x will be $-kx$.

D'Alembert's inertia force $-m\ddot{x}$. so the the equation of dynamic equilibrium is written,

$$m\ddot{x} + kx = 0 \tag{8.7}$$

This is an equation for harmonic motion and the solution is,

$$x = A\sin\omega t \tag{8.8}$$

so that substitution in equation (8.7) gives,

$$(m\omega^2 + k)x = 0 \tag{8.9}$$

Solutions are possible if,

$$\omega = \sqrt{\frac{k}{m}} \tag{8.10}$$

The period of vibration is given,

$$T = 2\pi\sqrt{\frac{m}{k}} \;=\; 2\pi\sqrt{mf} \tag{8.11}$$

This problem can also be looked upon as the inertia force $-m\ddot{x}$ producing the displacement x. That is,

$$-m\ddot{x}f = x \tag{8.12}$$

Then,

$$mf\omega^2 = 1 \tag{8.13}$$

and as before T is given by equation (8.11). th of the examples thus far are single degree of freedom systems. That is a single displacement quantity may be used to describe the displacement of the system. It is seen that the solution is possible for free vibration is and only if the coefficient of the displacement function is zero. The theory must be extended for multi-degree freedom systems in which the coefficient matrix of the system is singular and it is necessary to solve the system equations to determine the independent vibration or eigenmodes. In order to introduce the general problem, a number of two degree of freedom systems are first analysed.

Example III Two degree of freedom system

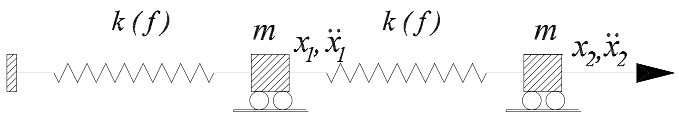

Figure 8.3: Two degree spring-mass system

The two spring-two mass system is shown in Figure 8.3. This is a statically determinate system so that the structure flexibility matrix $[F] = [b]^T[f][b]$ is easily calculated. The D'Alembert inertia fporces on the two masses 1 and 2 are $-m\ddot{x}_1$ and $-m\ddot{x}_2$ respectively. The forces in the springs are found to be,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \tag{8.14}$$

The spring flexibilities are each equal to f, so that the node(mass) deflections are given,

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = [b]^T[f][b] = f \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \tag{8.15}$$

The D'Alembert's forces are,

$$\begin{Bmatrix} -m\ddot{x}_1 \\ -m\ddot{x}_2 \end{Bmatrix}$$

The motion is periodic so that the displacements expressed as functions of time are,

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t$$

Substituting in equation (8.15), gives,

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = f \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} m & . \\ . & m \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} \tag{8.16}$$

Hence the undetermined multipliers, are given,

$$\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \omega^2 fm \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \tag{8.17}$$

Let $\lambda = \dfrac{1}{\omega^2 fm}$ and collecting terms on the left handside give the homogeneous equations,

$$\left[\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \lambda[I] \right] \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \tag{8.18}$$

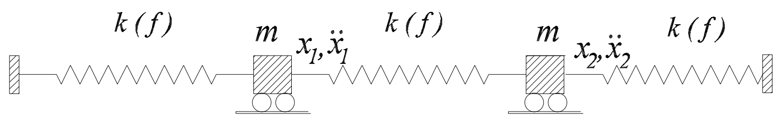

Figure 8.4: Spring-mass system, three springs, two masses

Solutions to these homogeneous equations is possible only if the The determinant of the coefficient matrix is equal to zero, that is,

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \tag{8.19}$$

Solving the resulting quadratic equation gives the values of λ,

$$\lambda = \frac{3 \pm \sqrt{5}}{2} \tag{8.20}$$

Substituting these values in the homogeneous equations, the modal shapes are obtained.

$$(1)\ \lambda_1 = \frac{3+\sqrt{5}}{2},\ \text{let } A_2 = 1,\ \text{then } A_1 = \frac{\sqrt{5}-1}{2}$$
$$(2)\ \lambda_2 = \frac{3-\sqrt{5}}{2},\ \text{let } A_2 = 1,\ \text{then } A_1 = \frac{-\sqrt{5}+1}{2}$$

The first of these modes is a sway motion with both masses having displacements of the same sign while the second is a breathing type of motion, the displacements of the masses having opposite sign. The periods of vibration are given,

$$T_{1,2} = \frac{2\pi}{\omega} = 2\pi\sqrt{\lambda_{1,2}}\sqrt{fm} \tag{8.21}$$

The more flexible the structure or the larger the masses the longer the period of vibration. Also the greater λ the longer T, so that the dominant mode is for $\lambda_1 = 3 + \sqrt{5}/2$. This occurs when the two masses are vibrating in harmony.

Example IV Statically indeterminate three spring-three mass system

In this example the system of three springs with two masses and two supports, shown in Figure 8.4, is statically indeterminate. In the text of Chapter 2, for example, this problem may have its flexibility matrix calculated via the stiffness method of analysis and this proceedure will be set out in the analysis below. Let the forces in the springs be (F_1, F_2, F_3) so that the equilibrium equations at the nodes 1 and 2 are,

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \qquad [A]\{S\} = \{R\} \tag{8.22}$$

This is clearly an indeterminate problem and the stiffness method is used, either to obtain the nodal stiffness matrix or the inverse, nodal flexibility matrix. Thus, first to obtain the stiffness matrix $[K]$,

$$[K] = [A][k][A]^T \tag{8.23}$$

In this equation,

$$[k] = k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Carrying out the matrix multiplications in equation (8.23),

$$[K] = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and the inverse flexibility,} \quad [F] = \frac{1}{3k} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{8.24}$$

Again let the motion be harmonic so that,

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin \omega t \tag{8.25}$$

and using D'Alembert's principle the deflections calculated from the inertia forces are given,

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{mf\omega^2}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} \tag{8.26}$$

Let $\lambda = 3/(mf\omega^2)$, so that for the solution of the homogeneous equations (6.66), the determinant of the coefficient matrix must vanish, that is,

$$\left| \begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix} - \lambda[I] \right| = 0 \tag{8.27}$$

That is, rearranging terms,

$$\left| \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right| = 0 \tag{8.28}$$

The roots of this determintal equation are,

$$\lambda_{1,2} = 1, 3 \tag{8.29}$$

The mode shapes are obtained by substituting these values in the homogeneous equations,

$$(1)\ \lambda = 3,\ \text{let}\ A_2 = 1,\ \text{then}\ A_1 = 1 \quad \text{antisymmetric mode}$$

$$(2)\ \lambda = 1,\ \text{let}\ A_2 = -1, \tag{8.30}$$

$$\text{then}\ A_1 = \quad \text{symmetric mode} \tag{8.31}$$

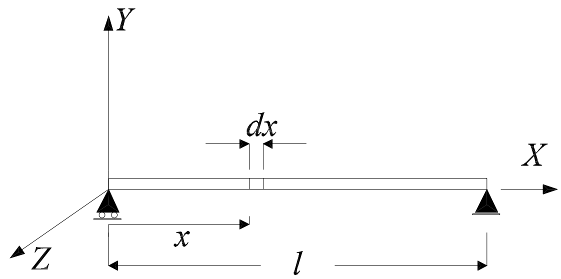

Figure 8.5: Simply supported beam with distributed mass

Substituting these values of $\lambda_{1,2}$ the periods for the two vibration modes are,

$$T_{1,2} = \frac{2\pi}{\omega} = 2\pi\sqrt{\lambda_{1,2}}\sqrt{\frac{mf}{3}} \tag{8.32}$$

The longest period is $T_1 = 2\pi\sqrt{3}\sqrt{mf/3} = 2\pi\sqrt{mf}$ and is associated with the antisymmetric mode of vibration. In this case there is zero force in the centre spring. Because there are two masses $(2m)$ and two active springs $(2k)$, this period is identical to that of the single spring-mass (k, m) system of *Exercise II*.

8.1.3 Beams with distributed mass

Exercise V Simply supported beam-uniformly distributed mass

Having examined the simple cases of one and two degree of freedom systems, the example of a simply supported beam is studied beceause with distributed mass this system has an infinite number of vibration modes. The beam is shown in Figure 8.5 and the distributed load for the mass density of m/unit length is, $w = m\ddot{x}$. The relationship between the load and the bending forces is given,

$$EI\frac{d^4y}{dx^4} = w = -ddotx \tag{8.33}$$

Choose a sine curve for the spatial function for the deflection y,

$$y = \sin\frac{n\pi x}{l} \quad n = 1, 2, 3, \text{ etc.} \tag{8.34}$$

Then $y = 0$ at $x = 0$ and $x = l$, so the boundary conditions are statisfied. In time the time domain, y is a harmonic function so that,

$$y_t = A\sin\frac{n\pi x}{l}\sin\omega t \tag{8.35}$$

The elastic force per unit length is then given,

$$EI\frac{d^4y}{dx^4} = EI(\frac{n\pi}{l})^4 A \sin\frac{n\pi x}{l}\sin\omega t \tag{8.36}$$

For the first harmonic, $n = 1$.

$$-m\ddot{y} = m\omega^2 A \sin\frac{n\pi x}{l}\sin\omega t \tag{8.37}$$

The equation of dynamic equilibrium using equation (8.33) is thus,

$$EI\frac{d^4y}{dx^4} + m\ddot{y} = 0 \tag{8.38}$$

Substituting for n=1,

$$EI(\frac{\pi}{l})^4 = m\omega^2 \tag{8.39}$$

The solution gives,

$$\omega = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{m}} \tag{8.40}$$

and the period T is given,

$$T = \frac{2\pi}{\omega} = \frac{2l^2}{\pi}\sqrt{\frac{m}{EI}} = 0.638l^2\sqrt{\frac{m}{EI}} \tag{8.41}$$

Example VI Simple span beam lumped mass
The beam is shown in Figure 8.6, and has one half of its mass lumped at the centre. The beam is now a single degree of freedom system with the beam providing the elastic spring support. The spring flexibility using elementary beam theory, is thus,

$$f = \frac{l^3}{48EI} \tag{8.42}$$

Then the period of vibration of the beam is approximated by the spring/mass period and is equal to,

$$\begin{aligned} T &= 2\pi\sqrt{mf} \\ &= 2\pi\sqrt{\frac{ml}{2}\frac{l^3}{48EI}} \\ &= 0.648\sqrt{\frac{m}{EI}} \end{aligned} \tag{8.43}$$

This value is in error from the distributed mass value, see equation (8.41). by only 1.5.

The mass of the individual members can thus be approximated by lumping the member masses at the nodes of the structure. The mass of a member, length l, mass per unit length m, will be replaced by $ml/2$ at the end nodes that the member connects. Because this

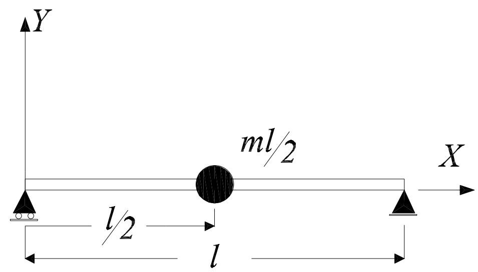

Figure 8.6: Simply supported beam central lumped mass

mass will be used as the dynamic force on the node, it will be applied in the coordinate directions of the problem being analysed as follows:

Truss	X,Y
Beam	Y
Frame	X,Y
Grid	Y

8.2 Calculation of natural frequenciesof structures

The elastic properties of the system are modelled in the usual way as a distributed system using the truss, beam, frame or grid elements. The masses are lumped at the nodes so that the D'Alembert equations of dynamic equilibrium are written for the undamped free vibration,

$$[K]\{r\} + [M]\{\ddot{r}\} = 0 \tag{8.44}$$

The motion is harmonic so write r and $\ddot{r}$ as functions of the time t,

$$\begin{aligned} [M]\{\ddot{r}\} &= -[M]\omega^2\{X\}\sin\omega t \\ \{r\} &= \{X\}\sin\omega t \end{aligned} \tag{8.45}$$

Substituting in equation (8.44) gives the set of linear homogeneous equations,

$$([K] - [M]\omega^2)\{X\} = 0 \tag{8.46}$$

These equations may be solved for eigenvalues and eigenvectors. Also premultiplying by the flexibility matrix $[F] = [K]^{-1}$, dividing by ω^2, and substituting $\lambda = 1/\omega^2$ gives the eigenvalue equation in terms of flexibility,

$$[\lambda[I] - [F][M]]\{X\} = 0 \tag{8.47}$$

The eigenvector with the largest period can be found by matrix iteration choosing a trial vector X_o that contains part of this mode. That is iterate by taking successive products,

$$\{X_1\} = [FM]\{X_o\}; \quad \{X_2\} = [FM]\{X_1\} \quad \text{etc.} \tag{8.48}$$

See the application of this technique in the calculation of the structure buckling loads in Chapter 7. Alternatively a symmetric eigensolver routine can be used by first symmetrizing equation (8.47) by making the substitution, $\{Y\} = [M^{1/2}]\{X\}$. With $[M] = [M^{1/2}][M^{1/2}]$. Rewrite equation (8.47),

$$(\lambda[M^{1/2}] - [M^{1/2}][F][M^{1/2}]\{Y\} = 0 \tag{8.49}$$

This is now written with $[M^*] = [M^{1/2}]$ and $[F^*] = [M^{1/2}][F][M^{1/2}]$,

$$[\lambda[M^*] - [F^*]]\{Y\} = 0 \tag{8.50}$$

This is a symmetric eigenvalue problem. Solution gives the eigenvalues λ and the eigenvectors Y. The X vectors are obtained,

$$\{X\} = [M^{1/2}]^{-1}\{Y\} \tag{8.51}$$

Finally the periods of vibration are obtained using λ_i,

$$T_i = \frac{2\pi}{\sqrt{\lambda_i}} \tag{8.52}$$

Example V11 Cantilever beam-uniformly distributed mass natural frequencies The cantilever beam of length l is shown in Figure 8.7. It has a mass of m units per unit length and a uniform bending stiffness EI. As shown in Figure 8.7, it has been subdivided into 10 equal sub-elements. See also the command sequence in B17 in the file DATN.DAT. There are 11 nodes and each node has mass of $ml/10$ except at the free end which has one half of this value. The flexibility matrix for this beam is easily calculated using the expression $[F] = [b]^T[f][b]$. The $[b]$ matrix can be written down by inspection for a cantilever beam subdivided into n sub-elements, as follows,

$$\underset{(2n \times n)}{[b]} = \frac{l}{n}\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 0 & 1 & 2 & \cdots & n-2 \\ 0 & 0 & 1 & 2 & \cdots & n-2 \\ & & & & \ddots & \vdots \\ & & & & \cdots & 1 \\ & & & & \cdots & 1 \\ & & & & \cdots & 0 \end{bmatrix} \tag{8.53}$$

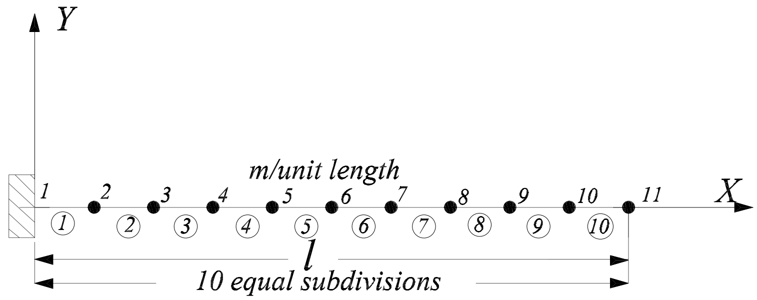

Figure 8.7: Cantilever beam uniformly distributed mass

The matrix $[f]$ of member flexibilities is of the form,

$$\underset{(2n \times 2n)}{[f]} = \frac{l}{n6EI} \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 2 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 2 & 1 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & \cdots & \cdots & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & 2 \end{bmatrix} \tag{8.54}$$

Similarly the mass matrix is given as the diagonal $(n \times n)$ matrix,

$$[m] = \frac{ml}{2n} \lceil 2m\ 2m\ 2m \cdots\ 2m\ m \rfloor \tag{8.55}$$

Using the above matrices to calculate $[F]$ amd $[M]$ in equation (6.9), the natural frequencies can be calculated in terms of $\sqrt{(ml^4/EI)}$. The value obtained for the fundamental period is,

$$T = \frac{2\pi}{3.50}\sqrt{\frac{ml^4}{EI}} \tag{8.56}$$

This value compares with (3.516) from the analytic solution with dortributed mass, see Timoskenko. The cantilever beam problem is reworked in Exercise VIII using the STATICS-2020 to calculate the beam flexibility matrix see B18 in the DATN.DAT file.

Example VIII Cantilever beam natural frequencies using command BEAMEX

The cantilever beam has distributed mass as in *Example VII* and the command sequence to carry out the analysis is available in DATN.DAT, B18. The command BEAMEX is used to generate A, B and C matrices for the cantilever beam. The command BEAMEQ sets up the equilibrium equations and inverts to produce the $[b]$ matrix. The cantilever beam is as shown in Figure 8.7 with 10 equal segments and 11 nodes including the support. Now when the equilbrium matrix is inverted and the beam flexibility matrix calculated it includes the rotational degrees of freedom and is a (22 × 22) matrix, including both support flexibilities. In terms of the transverse and rotational flexibilities the $[F]$ matrix is partitioned,

$$[F] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{8.57}$$

The matrices $[F_{11}]$, $[F_{21}]$ are extracted from $[F]$ with the command EXTRAC and diagonal terms corresponding to the support degrees given small flexibilities at least 1/100 of the next smallest flexibility. The mass matrix generated in the usual way is used with $[F_{11}]$ to calculate eigenvalues and mode shapes. No rotary inertia is included. That is, equation (8.47) is used to calculate λ and Y. The X mode shapes are obtained from equation (8.50). In order to calculate the nodal rotations in the mode shapes(for plotting), first $[F_{11}]$ is inverted and nodal forces R obtained for the X deflections,

$$\{R\} = [F_{11}]^{-1}\{X\} \tag{8.58}$$

Then the accompanying nodal rotations are calculated from,

$$\{\theta\} = [F_{12}]\{R\} \tag{8.59}$$

Combining X_i, θ_i for the ith mode, the shape may be plotted, interpolating deflections of elements between the node points. The results obtained for the period T,

$$T_i = \frac{2\pi}{\alpha_i}\sqrt{\frac{ml^4}{EI}} \tag{8.60}$$

The first 4 modes calculated in this way, are given in the Table 8.1 below.

Table 8.1

I	α_i	REF(1)	% difference
4	116.59	120.91	3.6
3	60.12	61.70	2.55
2	21.69	22.30	2.74
1	3.50	3.52	0.45

If the simply supported beam with distributed mass in Figure 8.5 is subdivided into 10 equal elements (11 nodes including support) and the above theory again applied to obtain the α_i values, the results and the comparison with the theory are given in Table 8.2. This problem can be run using SUBMIT B19 and the mode shapes drawn and compared with the corrsponding sine curves.

Table 8.2

I	α_i	REF(1)	% difference
4	157.52	157.91	2.45
3	88.77	88.83	0.07
2	39.47	39.47	–
1	9.87	9.87	–

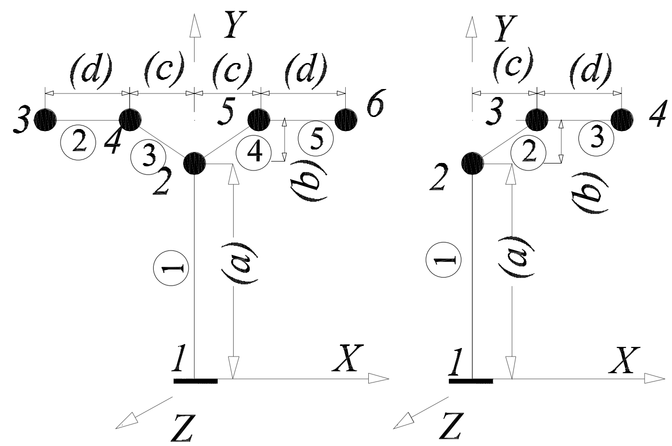

Figure 8.8: Frame structure with nodal masses

The error in the 1st and 2nd modes is less than 0.01%.

8.2.1 Statically indeterminate beams

The procedure to calculate eigenvalues and mode shapes for indeterminate beams is identical to that given in Example VII, once the beam flexibility matrix has been determined, in this case as the inverse of the stiffness matrix $[K]$. The two span beam of Figure (3.17) (7) is analysed for equal spans of length unity and with m=1 and EI=1. The period is calculated as,

$$T_i = \frac{2\pi}{\alpha_i}\sqrt{\frac{ml^4}{EI}} \tag{8.61}$$

the values of α_i so obtained will be the same as for the simple sapn beam of unit length so that again for 10 divisions per span the results for the eigenvalues will be as calculated in Table 8.2. the three cases, specially setup to illustrate the indeterminate beam modal analysis are in B27, B28 and B29. For these problems the command to generate the data is,

BEAMEX E=(27-28 or 29) L=?,? N=?

L(1) gives the overall length and L(2) the end span lengths, the beams being symmetric about their centre lines. N=? gives the number of beam elements in each span(the same in all spans). For example in B27, 10 subdivisions have been used.

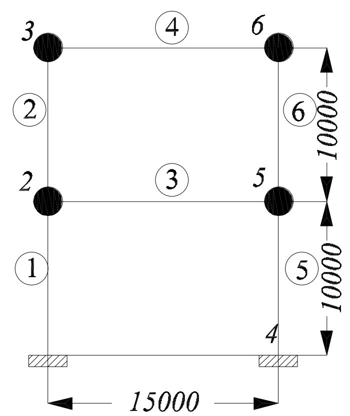

Figure 8.9: Two storey frame structure with nodal masses

8.3 Frame structures-natural frequencies and mode shapes

The theory is developed for any two dimensional plane frame vibrating in the plane of the frame. If vibrations occur out of the plain it may be possible that a grid analysis can be used. If however, both inplane and out of plane displacements are coupled then a three dimensional frame analysis that is beyond the scope of the present studies, should be used. The general principles still apply however in the eignvalue analysis. A statically determinate plane frame (the light post problem) is shown in Figure 8.8.The basic principles of plane frame analysis used here have been developed in Chapter 4. If the plane frame is determinate then the theory to set up the equilibrium equations is very simple, inversion gives the member force transformation matrix and hence via the member flexibility matrices to obtain the nodal flexibility matrix. See section(4.22). The frame will have a mass from its members that will be lumped at the nodal points. In Figure 8.8 there are 5 such nodes. It has been assumed that the post carries some extra masses on nodes 3 and 6 and these have been included. Because accelerations occur in both the X and Y directions, the same nodal mass will be added at a node for both of these directions. No rotary inertia terms will be used so that the eigenvalue analysis can be carried out on the $X-Y$ displacements only. The same process of extracting the relevant flexibility submatrix $[F_{11}]$ as for the beam structure is used. The light post shown in Figure 8.8 as modelled has 10 active degrees of freedom. This approximation will be adequate for the fundamental frequency. However nodes will be required for obtaining accurate values of higher frequencies. Of course the rotation degrees of freedom are active, they simply do not have any rotary inertia forces. The properties used for the members 1 to 5 are given in Table 8.3.

Table 8.3

Number	diameter(mm)	wall(t mm)	mass/metre(Kg)	area mm^2	I-mm^4
1,3,4	150	1	3.3	438	1.3×10^6
2,5	100	1	2.8	381	$.76 \times 10^6$

Masses on nodes 3 and 6 have been increased to 30Kg to simulate light fittings on the pole extremities. The data for the analysis of the light-pole is given in the DATN.DAT file under B19. The natural frequencies can be plotted with the command

PLTFRM A B C DEF N=6 M=?

The student should plot and print at least the first 4 modes and explain their shapes. For statically indeterminate frames the stiffness matrix has been calculated and inverted in the matrix K. With this modification the calculation of the natural frequencies follows the same sequence as the statically determinate case. The two storey frame whose geometry can be generated using FRAMEX E=3, is shown in Figure 8.9 dimensions are given in millimetres and the frames are at 15m spacing. The columns consist of ASC sections and the beams ASB. Assuming a concrete floor supported by member 3 and 1/2 of this mass acts on member 4 the nodal masses are:

Table 8.4 nodal masses

nodes	masses(Kg)
2,5	12630
3,6	6300

Table 8.5 member section properties

numbers	area	second mement	Young's modulus
	mm^2	mm^4	MPa
1,2,5,6	12.4E3	223E6	300E3
3,4	16.0E3	986E6	300E3

8.4 Natural frequency exercise module

A number of exercises, some of which have been discussed in Sections 8.3 and 8.4 are given on the DATN.DAT file, numbers 42.46 and 61-66. The list is given in the table below for reference.

(61) Cantilever beam	10 div/span frequencies (B17) (V,DEF)
(62) Cantilever beam	10 div/span frequencies: BEAMEX E=2(B18) (V,DEF)
(63) Simple span beam	10 div/span frequencies (B19) (V,DEF)
(64) Two span beam	10 div/span frequencies Figure 3.17(7) (B27)(V,DEF)
(65) Three span beam	10 div/span frequencies Figure 3.17 (8) (B28) (V,DEF)
(66) Four span beam	10 div/span frequencies Figure 3.17 (9) (B29) (V,DEF)
(42) Determinate frame	Figure 4.7 (9) frequencies (F10) (V,DEF)
(46) Two storey frame	Figure 4.6 (3) frequencies (F31) (V,DEF)

Some of the examples are of interest. For (63) the simple span beam, good results are obtained for the first mode, even with a coarse mesh subdivision of the beam. By way of contrast the cantilever beam (61) and (62) requires a relatively fine mesh subdivision (10 equal intervals) to obtain a similar degree of accuracy. The 2,3 and 4 span continuous beams (64)-(66) have their first modes as a higher mode obtained for the simple span beam for the cases where zero displacement corresponds to support points of the continuous beam. For the frame structures (42) and (46), the member masses are lumped at node points that define the member junctions or changes in geometry. With this coarse subdivision modes corresponding to inertia forces along members will not be captured. For better accuracy on the higher modes, additional nodes subdividing members are required. This is in contrast to static analysis in which the member elastic stiffness is adequately approximated. Note that for statically determinate structures the flexibility matrix is FL, where as for indeterminate structures it has been calculated in K, the inverted stiffness matrix. Any of the exercises may be used as analysis templates for other problems as given in the commands BEAMEX and FRAMEX. An annotated listing of this command sequence for problem (64)-(B27), the two span continuous beam, is given below.

```
C ...
C ... (64) continuous beam distributed load analysis and calculation of
C ... natural frequencies
C ...
C ... Two span continuous beam equal spans
B27
C ...
C ... choose two span continuous beam exercise E=27
C .... total length =2, 10 elements per span
BEAMEX E=27 L=2.0.1.0 N=10
C ... calculate the beam equilibrium matrix (EQ ) and make a copy
C ... of its transpose in EQTT. A=coordinate array, B=element numbers array,
C=boundary array
BEAMEQ A B C
C ... distributed load only
LOADR F R=1 C=20 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
C ... setup nodal load vector in LO
BEAMLD A B E F C=0 D=1
C ...
C ... read in Young's modulus and element second moment of area in IN
LOADR INT R=1 C=21
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
C ... Calculate member stiffnesses in MS as diagonal submatrices
BMMSTF B IN MS
C ... calculate gobal stiffness matrix K and invert
BMGSTF EQ MS K
```

```
C ...
C ... From K(FL) extract out F11(A1) the flexibility matrix for Y forces and
C ... F21(B1) cross flexibility from nodal forces to nodal rotations
EXTRAC K A1 B1 T=1
PRINT A1
C .. Duplicate A1 into A2 and invert A2 to obtain nodal stiffness matrix
ZERO A2 R=21 C=21
ADD A2 A1
INVERT A2 T=1
C ... load nodal masses as row matrix M1 and then duplicate into square
C ... matrix with values as diagonal terms copies an M and M2
LOADR M1 R=1 C=21
0.05 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.1 0.05
ZERO M R=21 C=21
ZERO M2 R=21 C=21
STODG M2 M1 N=1 L=1
STODG M M1 N=1 L=1
PRINT M
C .. calculate square root of diagonal terms
SQREL M
PRINT M
C .. calculate F1F*=M*A1M* generalized flexibility
MULT A1 M T
MULT M T F1
PRINT F1
C .. load V1 as row vector and transpose into column vector V (to suit EIGEN )
LOADR R V1 R=1 C=21
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
TRAN V1 V
C ... Calculate mode shapes N and eigenvectors V
EIGEN F1 N V T=16
C .. invert diagonal terms of V
INVEL V
PRINT V
C ... load scale factor for matrices SC here =1.0
LOADR SC R=1 C=1
1.0
SCALE V SC
C ... Now take square root of all diagonal terms (see theory)
SQREL V
PRINT V
INVERT M T=1
```

```
C .. Now transform eigenvectors N back to N1 in Y coordinate system
MULT M N N1
PRINT N1
C ... calculate nodal forces that produce the Y shape and then the
C ... accompanying nodal rotations in X2
MULT A2 N1 N3
MULT B1 N3 X2
PRINT X2
C ... Now put nodal deflections and nodal rotations into DEF
C ... (deflections in top half and rotations into bottom half)
ZERO DEF R=42 C=21
STOSM DEF N1 L=1,1
STOSM DEF X2 L=22,1
PRINT DEF
C ... calculated beam bending moments (MO), shears (VO) and reactions (SO)
C ... finally calculate beam deflections RV for the original loading
BEAMMO MO VO SO
MULT K LO RV
RETURN
```

Remember the command to plot mode shapes,

PLTBEM A B C DEF N=6=6 M=?

N=6 for mode shapes and the value of M gives the required mode to be plotted.