Welcome to Vibrationdata

Acoustics • Shock • Vibration • Signal Processing **November 2006 Newsletter**

Happy Thanksgiving!

Music brings joy into our lives. Soon after creating the Earth and man, God gave us the gift of music.

Jubal was "the father of all such as handle the harp and organ," according to Genesis 4:21.

Pythagoras of Samos was a Greek philosopher who sought a mathematical basis for harmony, which is a combination of notes considered to be pleasing. He devised a musical scale based on a perfect fifth, which is pair of notes with a frequency ratio of 3:2.

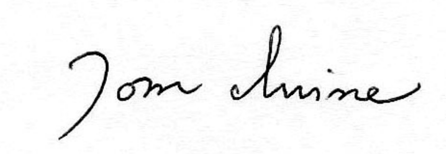
The first article presents the musical notes of the piano keyboard. The second article builds upon the first by considering harmonic ratios.

Sample sound files of individual notes and harmonic pairs are given at:

http://www.vibrationdata.com/newsletters.htm

I hope that you enjoy these articles.

Sincerely,



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Feature Articles

Piano Acoustics by Tom Irvine

Introduction

The piano is a musical instrument that is classified as either a string or percussion instrument depending on the classification system used.

Forerunners of the Piano

The modern piano or pianoforte had its origins in the ancient harp.

The plucked strings of the harp produced musical notes resulting from the vibration of the strings.

The fundamental frequency f of a given string varies according to portion equation

$$f \propto \frac{1}{L}\sqrt{\frac{T}{\rho}} \quad (1)$$

where

L = length

T = tension

ρ = mass per length

A variety of innovations were made to the harp over the years resulting in numerous other instruments.

The psaltery was common in the fourteenth and fifteenth centuries. It is a shallow closed box where stretched strings are sounded by plucking with the fingers.

The hammered dulcimer is similar in appearance to the psaltery except that the dulcimer uses wooden hammers to strike each string. Neither the psaltery nor hammered dulcimer has a keyboard.

The harpsichord family is thought to have originated when a keyboard was attached to the end of a psaltery. The keyboard provided a mechanical means to pluck the strings.

Figure 1. Clavichord

The clavichord is a stringed keyboard instrument that produces sound by striking brass or iron strings with small metal blades called tangents.

Invention of the Piano

Bartolomeo Cristofori (1655-1731) of Padua, Italy is credited with inventing the piano around the year 1700. Cristofori was an expert harpsichord maker. He invented a method by which hammers would string the string but not remain in contact with the string, as a tangent remains in contact with a clavichord string. Otherwise, the hammer would damp the sound if it remained in contact with the string. Furthermore, his method allowed the hammers to return to their rest position smoothly. His design also allowed the rapid repetition of a note.

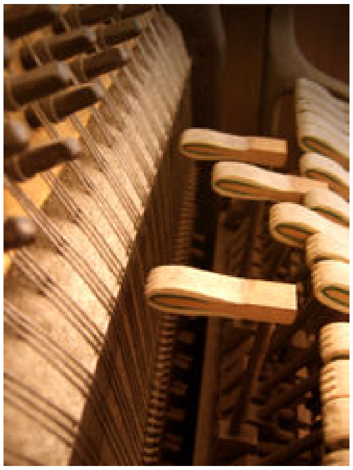


Figure 2. Interior of an Upright Piano

The felt-covered hammers are shown. Each note has three strings in the treble range.

Other inventors added further enhancements to Cristofori's piano design. Gottfried Silbermann (1683-1753) was an organ builder who invented the predecessor of the modern damper pedal, which lifts all the dampers from the strings at once.

A number of years passed before the piano became a popular instrument. Wolfgang Amadeus Mozart (1756-1791) composed concertos and sonatas for the Viennese-style piano, which had a softer, clearer tone, with less sustaining power than today's pianos.

The design of the piano improved. The hammers in early pianos consisted of wooden heads covered with leather, but felt hammers have been standard since about 1830.

Music wire evolved from handmade ductile iron to continuously drawn carbon steel. In addition, the use of precision casting for the production of iron frames made the piano more powerful.

Another important innovation was the use of three strings rather than two for all but the lower notes.

Piano bass strings are made by winding copper or iron wire over a core of plain steel music wire to increase the mass density so that the length of the string can be kept within the space available. The core itself is stiff and causes the higher harmonics to become slightly sharp. This effect is called inharmonicity. This effect may occur for treble strings as well.

Modern Piano

Furthermore, the number of octaves was increased from the five in Mozart's piano to the 7 1/4 octaves in today's piano.

The term *fortepiano* is often used to describe modern pianos in contrast to the eighteenth century pianos.

The modern piano has 88 keys, arranged in one-twelfth octave steps. A one-octave separation occurs when the higher frequency is twice the lower frequency.

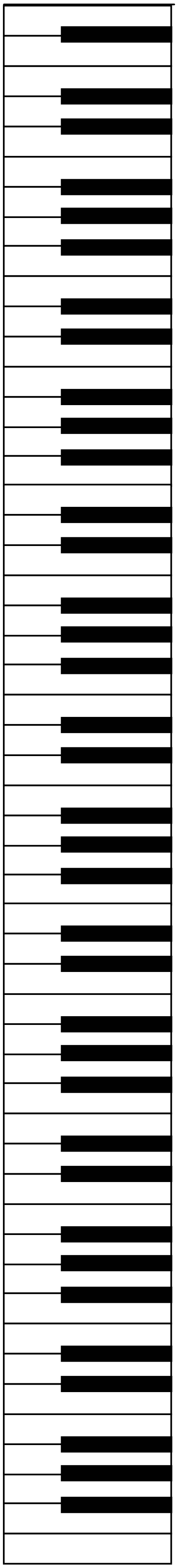
The lowest note on a modern piano is an A with a frequency of 27.5 Hz. The highest is a C with a frequency of 4186 Hz. The octave span is calculated as

$$\ln\left[\frac{4186\text{ Hz}}{27.5\text{ Hz}}\right] / \ln[2] = 7.25\text{ octaves}$$

Some sources represent the octave span as 7 1/3 because there are four keys beyond the 84 required for 7 octaves. The difference is simply a matter of the counting convention.

The fundamental frequency of each piano key is shown in Figure 4.

Middle C



A0 27.5
A0# 29.135
B0 30.868
C1 32.703
C1# 34.648
D1 36.708
D1# 38.891
E1 41.203
F1 43.654
F1# 46.249
G1 48.999
G1# 51.913
A1 55.000
A1# 58.270
B1 61.735
C2 65.406
C2# 69.296
D2 73.416
D2# 77.782
E2 82.407
F2 87.307
F2# 92.499
G2 97.999
G2# 103.83
A2 110.00
A2# 116.54
B2 123.47
C3 130.81
C3# 138.59
D3 146.83
D3# 155.56
E3 164.81
F3 174.61
F3# 185.00
G3 196.00
G3# 297.65
A3 220.00
A3# 233.08
B3 246.94
C4 261.63
C4# 277.18
D4 293.66
D4# 311.13
E4 329.63
F4 349.23
F4# 369.99
G4 392.00
G4# 415.30
A4 440.00
A4# 466.16
B4 493.88
C5 523.25
C5# 554.37
D5 587.33
D5# 622.25
E5 659.25
F5 698.46
F5# 739.99
G5 783.99
G5# 830.61
A5 880.00
A5# 932.33
B5 987.77
C6 1046.5
C6# 1108.7
D6 1174.7
D6# 1244.5
E6 1318.5
F6 1396.9
F6# 1480.0
G6 1568.0
G6# 1661.2
A6 1760.0
A6# 1864.7
B6 1979.5
C7 2093.0
C7# 2217.5
D7 2349.3
D7# 2489.0
E7 2637.0
F7 2793.8
F7# 2960.0
G7 3136.0
G7# 3322.4
A7 3520.0
A7# 3729.3
B7 3951.1
C8 4186.0

<u>Piano Keyboard</u>

The number beside each key is the fundamental frequency in units of cycles per seconds, or Hertz.

<u>Octaves</u>

For example, the A4 key has a frequency of 440 Hz. Note that A5 has a frequency of 880 Hz. The A5 key is thus one octave higher than A4 since it has twice the frequency.

<u>Overtones</u>

An overtone is a higher natural frequency for a given string. The overtones are "harmonic" if each occurs at an integer multiple of the fundamental frequency.

Figure 4.

Measured Piano Notes

Figure 3a. Kawai Upright Piano

Figure 3b. Petrov Grand Piano

The pianos in Figures 3a and 3b were used for the data shown in Figures 5 through 8. The sound from individual keys was recorded directly by a notebook PC with an uncalibrated microphone.

Middle C

The sound pressure time histories from Middle C are shown for the Kawai and Petrov pianos in Figures 5a and 5b, respectively. The Petrov time history shows more sustained reverberation. The envelope of each signal depends in part on beat frequency effects.

The nominal fundamental frequency for Middle C is 262 Hz. The corresponding spectral plots in Figures 6a and 6b, however, show that the harmonic at 523 Hz has the highest amplitude of all the spectral peaks for each given piano. This harmonic is one octave higher than the fundamental frequency. Each of these frequencies is rounded to the nearest whole number.

Some scientific manufacturers once adopted a standard of 256 Hz for middle C, but musicians ignored it, according to Culver, C. A. *Musical Acoustics.*

D above Middle C

The spectral functions for D above Middle C for the Kawai and Petrov pianos are shown in Figures 6a and 6b, respectively. Again, the first overtone has a higher amplitude than the fundamental, although the difference is small for the Petrov piano.

A above Middle C

The fundamental frequency has the highest spectral peak for A above Middle C for each piano in Figures 8a and 8b. The spectral function shows a fairly balanced blend of harmonics in terms of the respective amplitudes for the Kawai piano.

This A note is also known as “concert pitch.”

Inharmonicity

The spectral functions in Figures 6 through 8 are calculated via a Fourier transform.

Note that the higher harmonics tend to become slightly sharp, as shown in Figures 6 through 8. This deviation results in inharmonicity.

The deviation is due to the bending stiffness of the wires. The physical model of a “string” lacks bending stiffness. It is rather a function of tension, density, and length as shown in equation (1).

Data Conclusion

The graphs in Figures 6 through 8 provide a useful demonstration of the harmonics for each given note for each piano.

Each represents a single keystroke and is thus a “snapshot.” Each response depends on the applied pressure of the finger against the key, the piano tuning and other variables.

There are a great number of nuances in the perceived quality of a piano’s sound. Fully characterizing each of these in terms of engineering graphs might well require a Master’s thesis level of effort, but graphs may be insufficient to account for individual preference.

The saying that “Beauty lies in the eyes of the beholder” applies to music as much as it does to visual art.

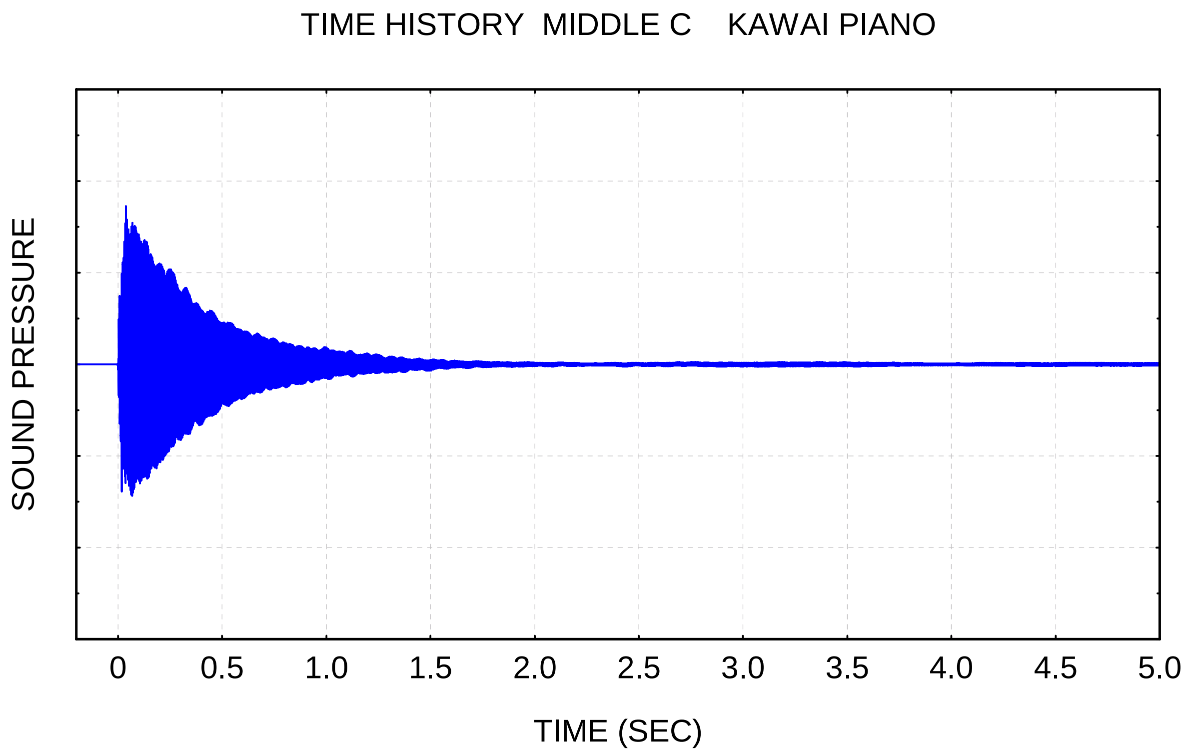


Figure 5a.

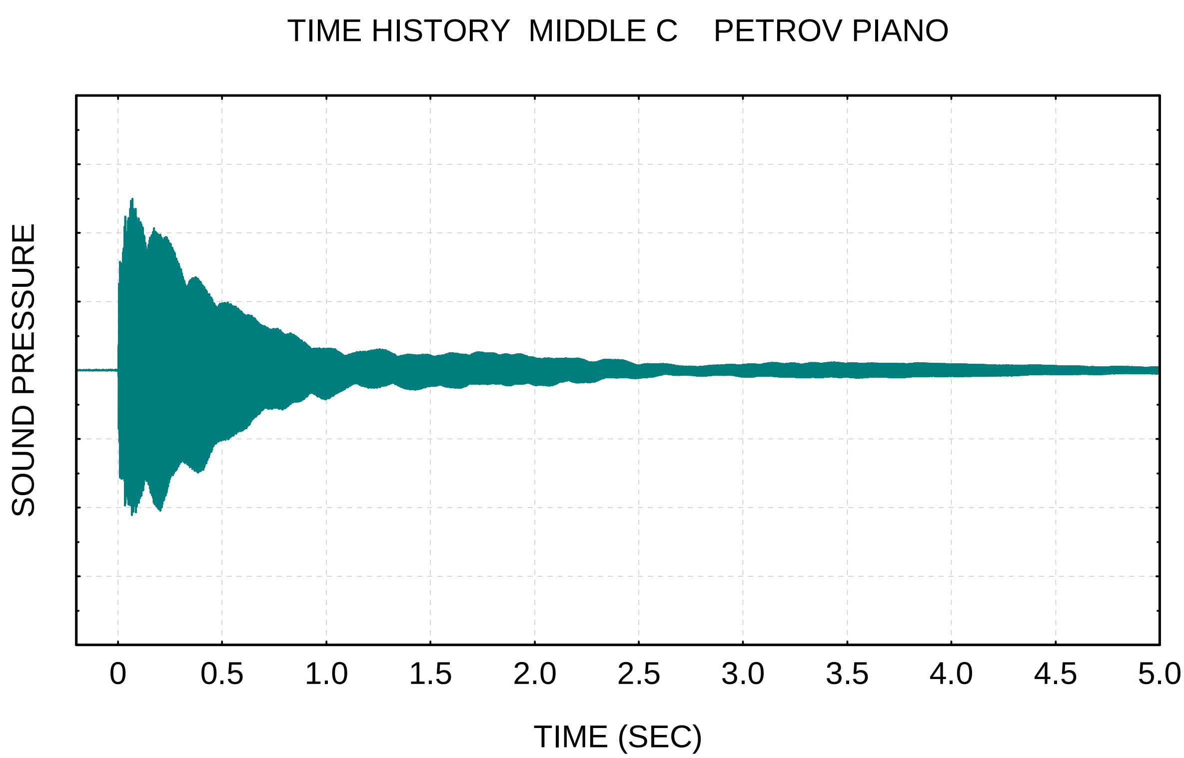


Figure 5b.

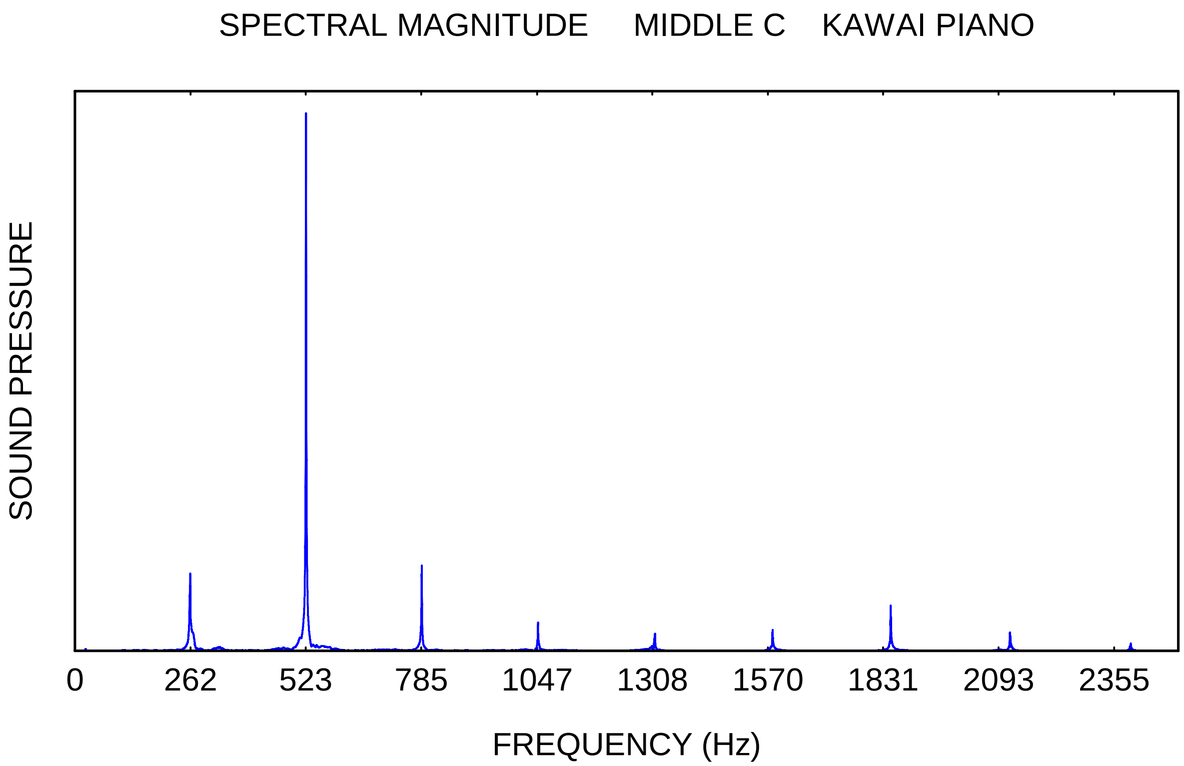


Figure 6a.

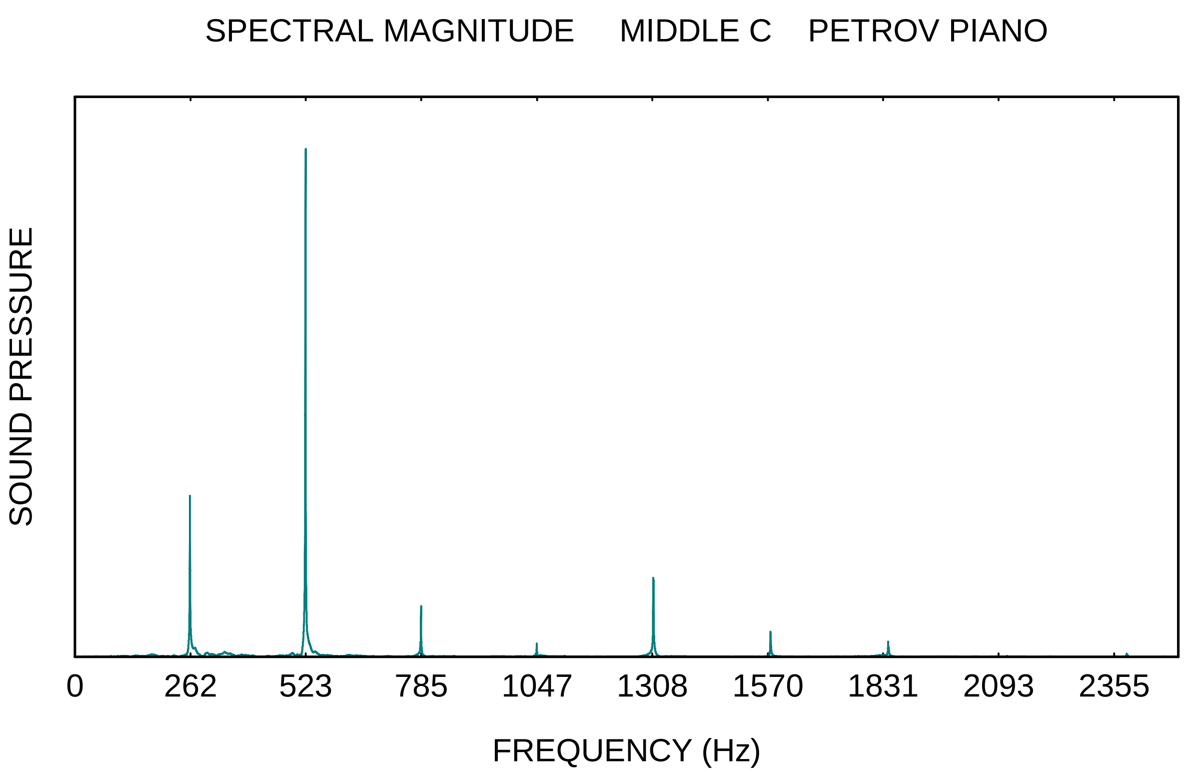


Figure 6b.

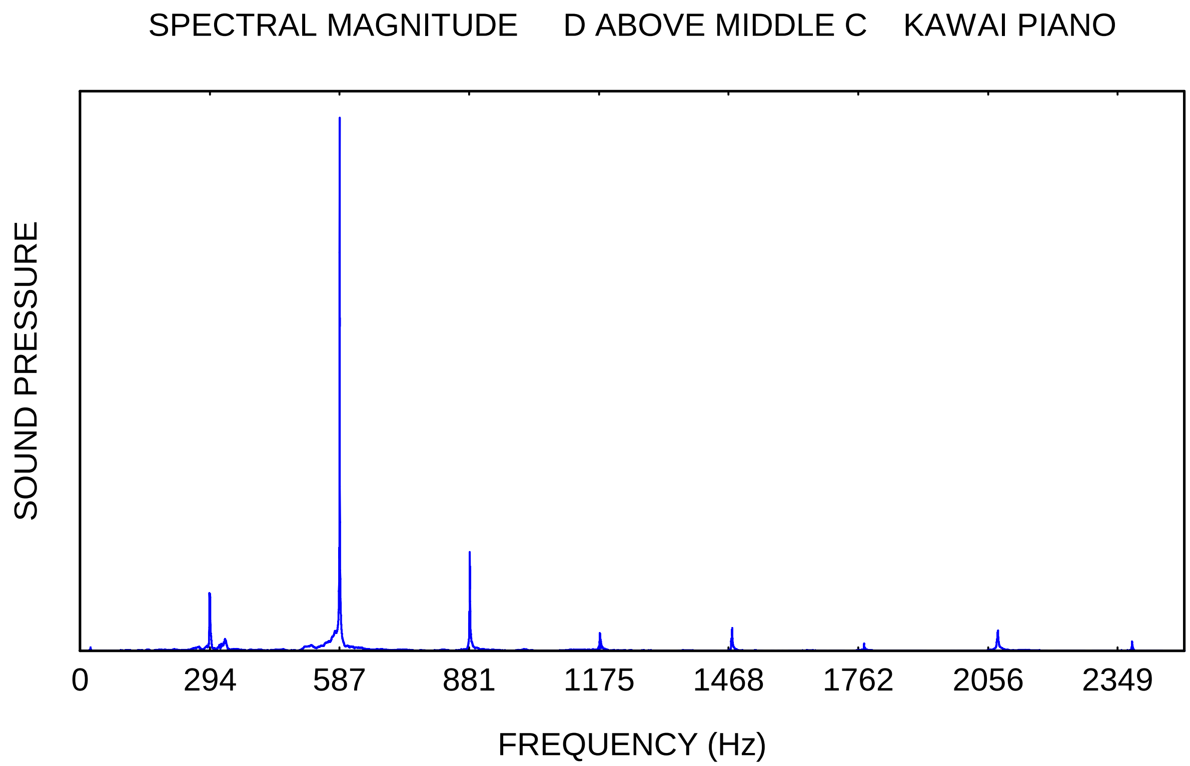


Figure 7a.

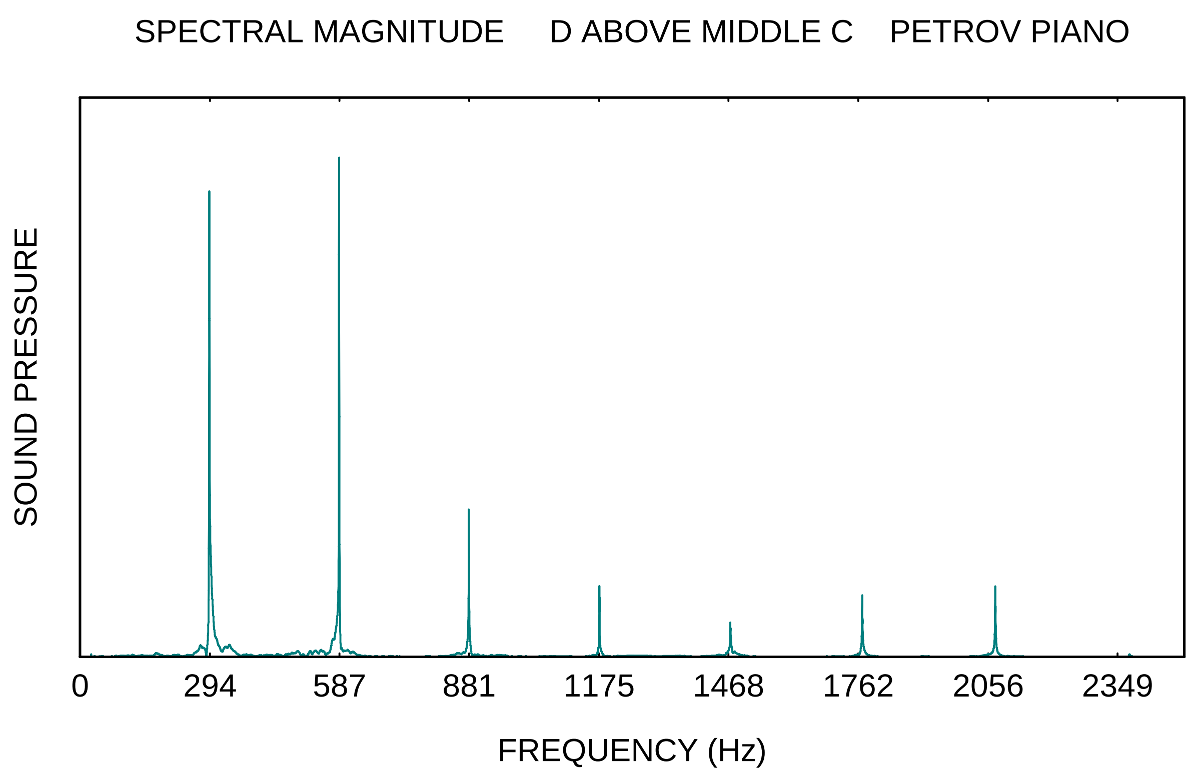


Figure 7b.

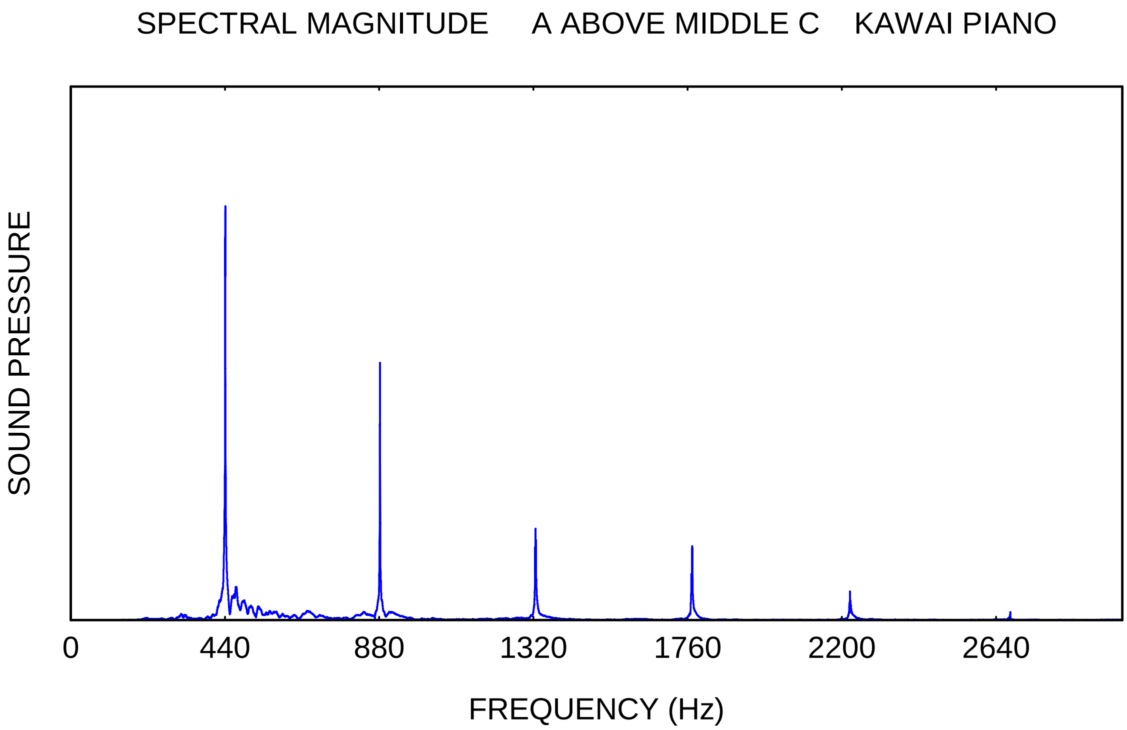


Figure 8a.

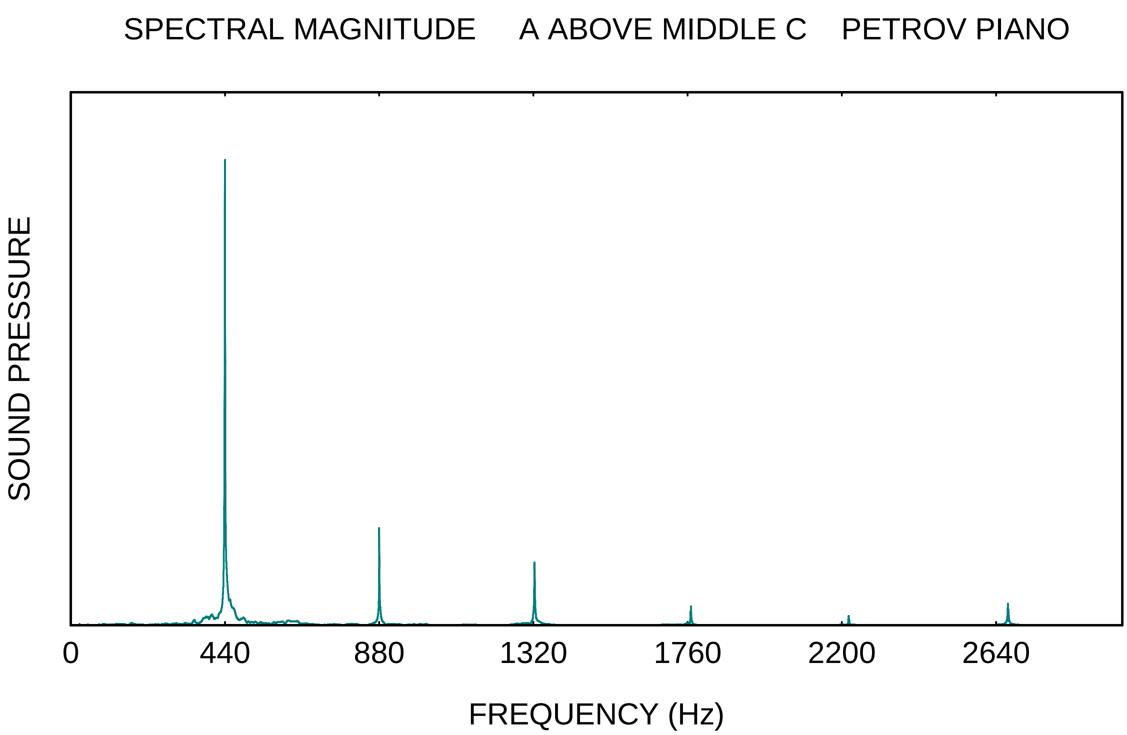


Figure 8b.

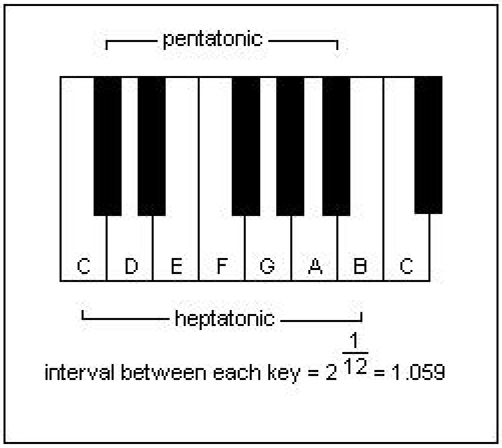


Figure 1.
Keyboard with Equal Temperament

Musical Scales by Tom Irvine

History

The modern musical scale can be traced to Pythagoras of Samos.

Pythagoras was a Greek philosopher and mathematician, who lived from approximately 560 to 480 BC. Pythagoras and his followers believed that all relations could be reduced to numerical relations. This conclusion stemmed from observations in music, mathematics, and astronomy.

Pythagoras studied the sound produced by vibrating strings. He subjected two strings to equal tension. He then divided one string exactly in half. When he plucked each string, he discovered that the shorter string produced a pitch which was one octave higher than the longer string.

A pleasing, harmonious sound is produced when two notes separated by one octave are played simultaneously on a piano or any other musical instrument.

Ptolemy (c. 90 – c. 168 AD) was a Greek mathematician and astronomer who used “Just Intonation.” This is a system of labeling intervals according to the ratio of frequencies of the two pitches. Important intervals are shown in Table 1.

Hermann von Helmholtz (1821-1894) was a German physicist who further studied the mathematics of musical notes. He published his theories in a book called “The Sensations of Tone as a Physiological Basis for the Theory of Music.”

Ratios

The harmonic theories of Pythagoras, Ptolemy and Helmholtz depend on the frequency ratios shown in Table 1.

These ratios apply both to a fundamental frequency and its overtones, as well as to relationship between separate keys. The ratios may also be expressed in reverse order.

Table 1. Standard Frequency Ratios

Ratio	Name	Example
1:1	Unison	-
1:2	Octave	A4 & A5
1:3	Twelfth	A4 & E6
2:3	Fifth	A4 & E5
3:4	Fourth	A4 & D5
4:5	Major Third	A4 & C5#
3:5	Major Sixth	A4 & F5#

The Example column shows notes in terms of their respective fundamental frequencies.

Consonance

Now consider two strings which are plucked simultaneously. The degree of harmony depends on how the respective fundamental frequencies and overtones blend together.

Music notes which blend together in a pleasing manner are called consonances. Notes with a displeasing blend are dissonances.

Helmholtz gave a more mathematical definition of these terms:

> When two musical tones are sounded at the same time, their united sound is generally disturbed by the beats of the upper partials, so that a greater or less part of the whole mass of sound is broken up into pulses of tone, and the joint effect is rough. This relation is called Dissonance. But there are certain determinant ratios between pitch numbers, for which this rule suffers an exception, and either no beats at all are formed, or at least only such as have so little intensity that they produce no unpleasant disturbances of the united sound. These exceptional cases are called Consonances.

Helmholtz has defined degrees of consonance as shown in Table 2.

Octave

Again, a one-octave separation occurs when the higher frequency is twice the lower frequency. The octave ratio is thus 2:1.

A note's first overtone is one octave higher than its fundamental frequency.

Table 2. Consonances

Degree	Interval
Absolute	Octave, Twelfth, Double Octave
Perfect	Fifth, Fourth
Medial	Major Sixth, Major Third
Imperfect	Minor Sixth, Minor Third

Consider a modern piano keyboard. The beginning key on the left end is an A0 note with a fundamental frequency of 27.5 Hz. A piano key has harmonic overtones at integer multiples of its fundamental frequency. Thus, the A0 key also produces a tone at 55.0 Hz, which is one octave higher than the fundamental frequency. The second overtone is at 82.5 Hz.

The twelfth key to the right of A0 is A1, counting both the black and white keys. The A1 note has a fundamental frequency of 55.0 Hz. The A1 note is thus one octave higher than the A0 note, in terms of their respective fundamental frequencies. In fact, there is a one-octave separation between any two piano keys which are twelve keys apart.

A pleasing, harmonious sound is produced when two notes separated by one octave are played simultaneously on a piano or other musical instrument. Helmholtz calls such a pair an absolute consonance. Thus, the A0 and A1 keys are an absolute consonance. This effect is shown for the A0 note and the A1 note in Table 3.

Table 3. Comparison of Two Notes and Their Respective Overtones up to 165 Hz	
A0 Frequency (Hz)	A1 Frequency (Hz)
27.5	
55.0	55.0
82.5	
110.0	110.0
137.5	
165.0	165.0

The overtones of the A1 note thus coincide with the evenly numbered overtones of the A0 note. Again, these two notes are separated by one octave.

Hermann Helmholtz wrote:

> A note accompanied by its Octave consequently becomes brighter in quality, because the higher upper partial tones on which brightness of quality depends, are partially reinforced by the additional Octave.

Twelfth

A twelfth is two notes which form a frequency ratio of 1:3.

A note's second overtone is a twelfth higher than its fundamental frequency. Recall the A0 note with its fundamental frequency of 27.5 Hz. Its second overtone is 82.5 Hz, which is three times higher than its fundamental frequency.

Table 4. Comparison of Two Notes and Their Respective Overtones up to 250 Hz	
A0 Frequency (Hz)	E2 Frequency (Hz)
27.5	
55.0	
82.5	82.407
110.0	
137.5	
165.0	164.814
192.5	
220.0	
247.5	247.221

Ideally, there would be a key with a fundamental frequency of 82.5 Hz. The nearest is the E2 key which has a fundamental frequency of 82.407 Hz. This frequency approximately meets the goal. Thus, the E2 key is considered as a twelfth higher than A0. A comparison is shown in Table 4.

The comparison shows that A0 and E2 have three tones very nearly in common in the frequency domain up to 250 Hz.

Fifth

A fifth is two notes which form a frequency ratio of 2:3.

A note's second overtone is a fifth higher than its first overtone.

Recall the A0 note with its fundamental frequency of 27.5 Hz. A fifth higher would be 41.25 Hz. Such a note does not exist in an exact sense. On the other hand, the E1 note has a frequency of 41.203 Hz, which is approximately equal to the exact fifth. Thus, E1 is considered as a fifth higher than A0. A comparison is shown in Table 5.

Thus, A0 and E1 have two overtones very nearly in common in the frequency domain up to 165 Hz.

Table 5.
Comparison of Two Notes and Their Respective Overtones up to 165 Hz

A0 Frequency (Hz)	E1 Frequency (Hz)
27.5	
	41.203
55.0	
82.5	82.406
110.0	
	123.609
137.5	
165.0	164.812

Measured Piano Notes

Two notes were played simultaneously on the Kawai piano.

Examples for a fifth, fourth and twelfth are shown in Figures 2, 3 and 4, respectively.

The ratios in Table 1 also apply to the fundamental frequency and its harmonics for an individual key. An example is shown in Figure 5 from the D note of the Petrov piano.

References

1. Hermann Helmholtz, On the Sensations of Tone, Dover, 1954.

2. American Heritage Dictionary, Houghton Mifflin Company, Boston, 1982.

APPENDIX A

Glossary

Beat - a resultant modulation which occurs when two notes, which are slightly different in pitch, are played simultaneously. The modulation can be perceived as a slow periodic change in volume or timbre.

Cents - a logarithmic unit of measure used for musical intervals. Typically cents are used to measure extremely small intervals, or to compare the sizes of comparable intervals in different tuning systems. The interval of one cent is much too small to be heard between successive notes. 1200 cents are equal to one octave, which is a frequency ratio of 2:1.

The cent interval between two frequencies f_2 and f_1 is:

$$\text{cents} = 1200 \ln\left[\frac{f_2}{f_1}\right] / \ln[2]$$

Chord – a combination of usually three or more notes sounded simultaneously.

Chromatic Scale - the scale that contains all twelve pitches of the Western tempered scale.

Consonance - a simultaneous combination of sounds conventionally regarded as pleasing.

Diatonic Scale - is a seven-note musical scale comprising five whole-tone and two half-tone steps, in which the half tones are maximally separated.

Dissonance - a simultaneous combination of sounds conventionally regarded as lacking harmony.

Equal Tempered Scale – a scale, in which every half step is exactly the same size in terms of proportion. An octave must have the ratio 2:1. There are 12 half-steps in an octave. Each half-step must therefore have a ratio of $2^{1/12}$.

Even Tempered Scale – See Equal Tempered Scale.

Half Step – see Semitone.

Harmony - a combination of musical notes considered to be pleasing.

Harmonic - a tone in the harmonic series of overtones produced by a fundamental tone

Harmonic Series - a series of tones consisting of a fundamental tone and the overtones produced by it, whose frequencies are at integral multiples of the fundamental frequency.

Inharmonicity – an effect whereby the higher harmonics become slightly sharp due to the bending stiffness of the piano wires. This effect tends to be more significant in the bass wires.

Interval - the difference in pitch between two musical tones.

Just Intonation – a system of labeling intervals according to the ratio of frequencies of the two pitches. Important intervals are those using the lowest integers, such as 1:1, 2:1, 3:2, etc.

Octave - the interval of eight diatonic degrees between two tones, one of which has twice the frequency as the other.

Overtone - a harmonic.

Partial - a harmonic.

Pitch - the frequency of a tone.

Semitone – is a musical interval. It is also called a half-step. It is a pitch interval halfway between two whole tones. It is the smallest interval commonly used in western music, and is considered the most dissonant.

Triad - a chord of three tones, especially one consisting of a given tone with its major or minor third and its perfect, augmented, or diminished fifth.

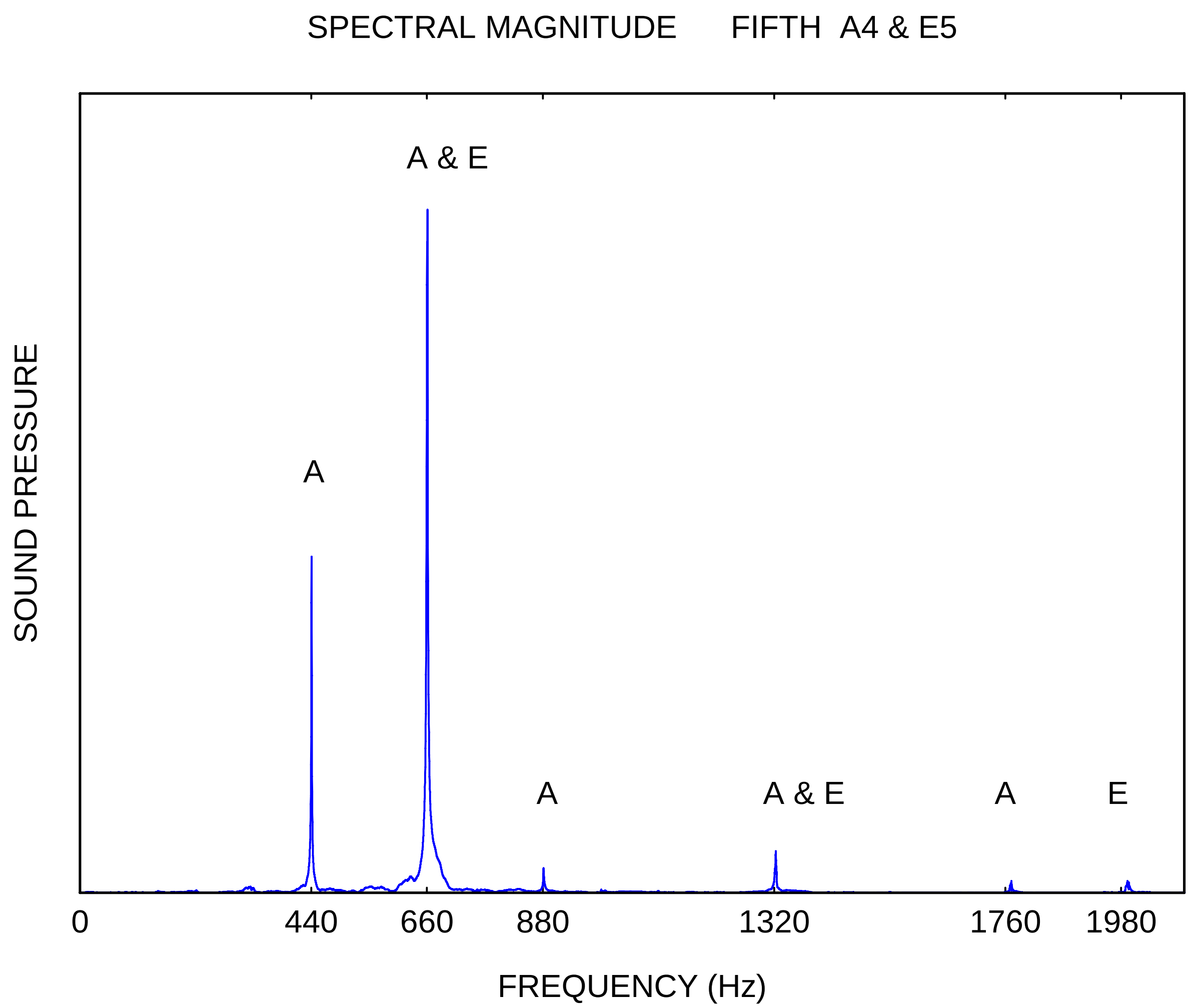


Figure 2.

The second spectral line (or first overtone) of the A note is the same as the fundamental of the E note, with each having a frequency of 660 Hz.

The perfect fifth ratio is 2:3. The perfect fifth interval is responsible for the most consonant, or stable, harmony outside of the unison and octave. It is a valuable interval in chord structure, song development, and western tuning systems.

The perfect fifth is historically significant because it was the first accepted harmony of Gregorian chant besides the octave.

The perfect fifth occurs on the root of all major and minor chords (triads) and their extensions. An example of a perfect fifth is the start of "Twinkle, Twinkle Little Star."

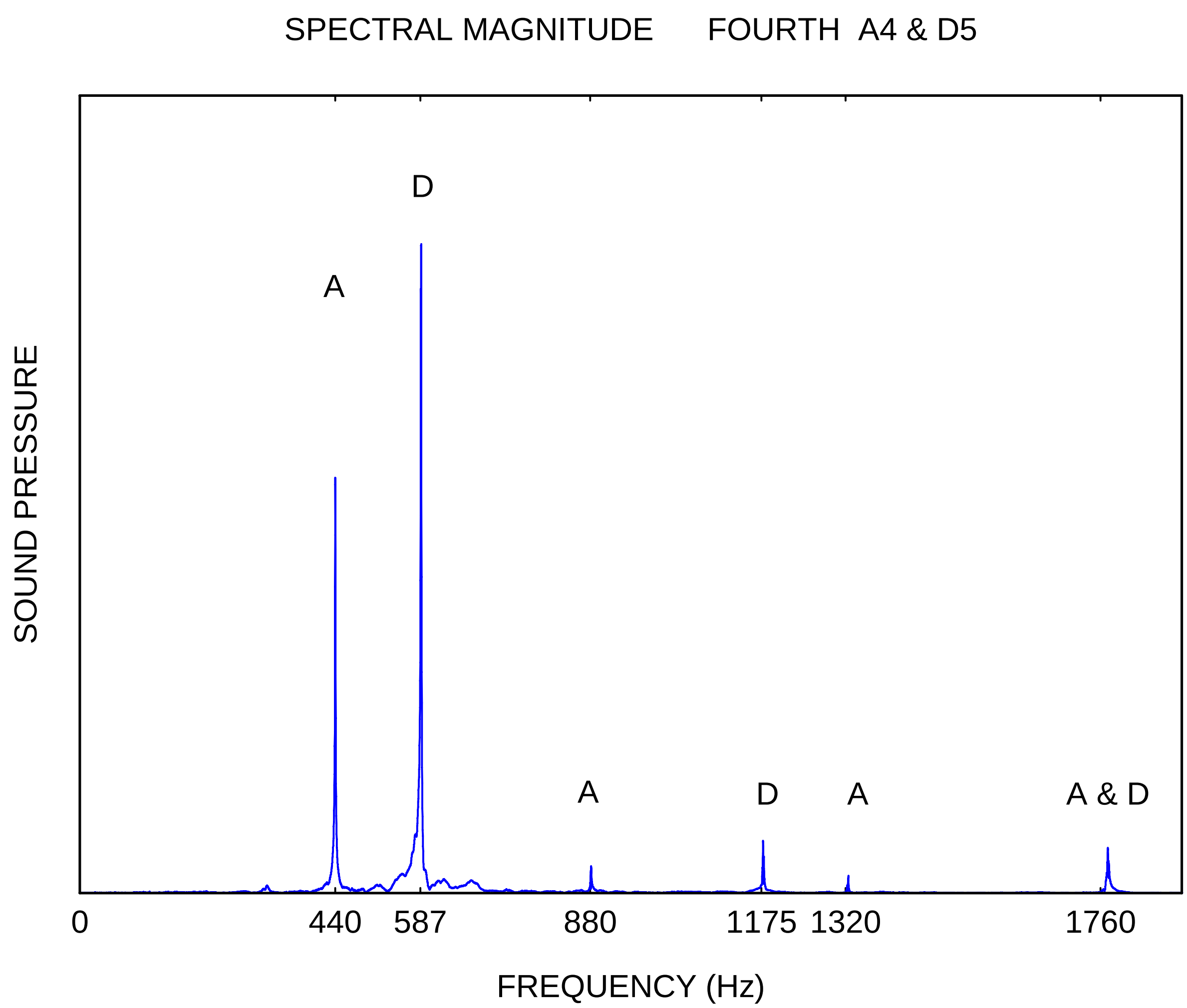


Figure 3.

The fourth spectral line of the A note is the same as the third of the D note, with each having a frequency of 1760 Hz.

The perfect fourth ratio is 3:4. The perfect fourth is a perfect interval like the unison, octave, and perfect fifth. It is a sensory consonance.

An example of a perfect fourth is the start of “Here Comes the Bride.”

Conventionally, the strings of a double bass and a bass guitar are tuned by intervals of perfect fourths.

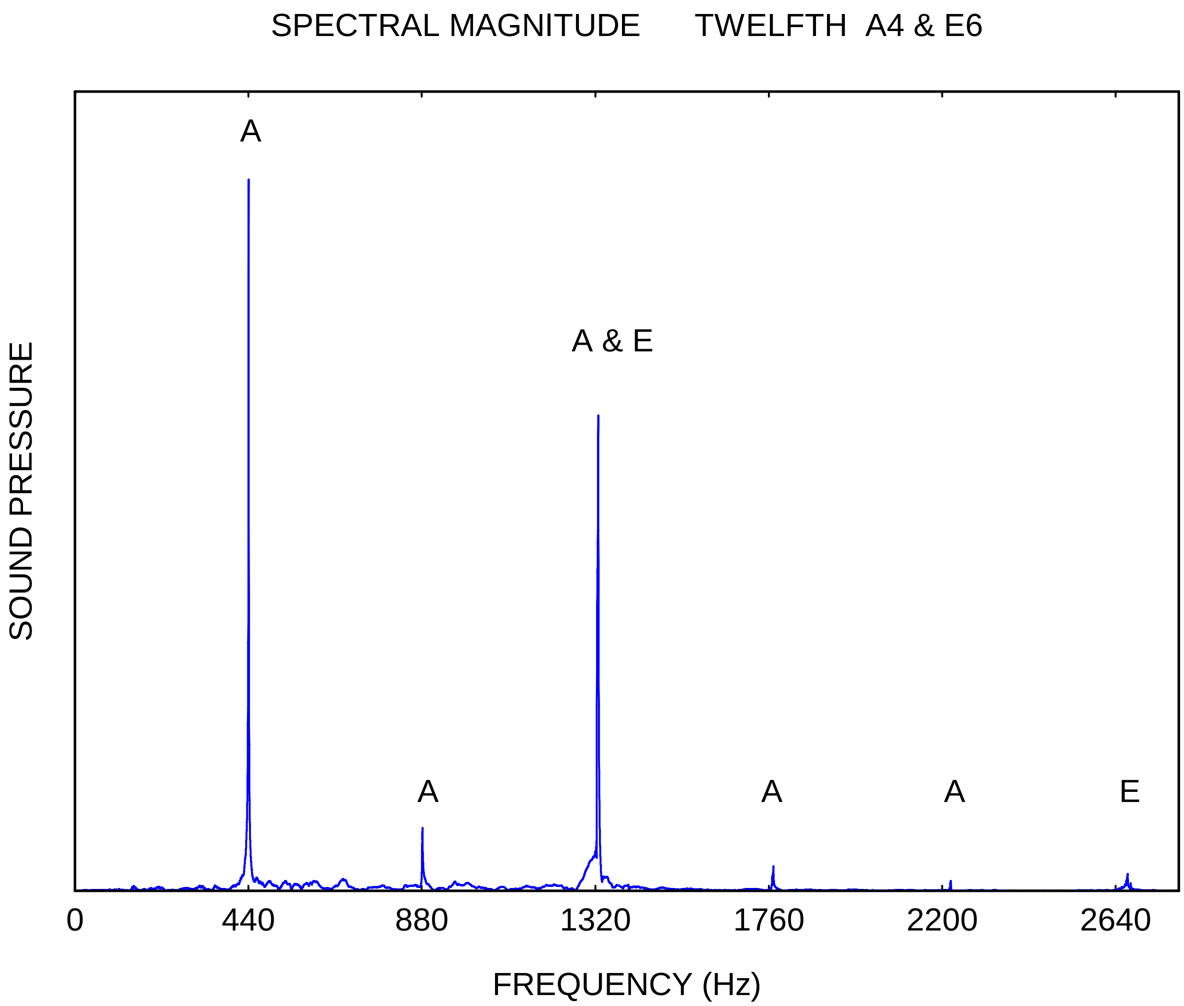


Figure 4.

The perfect twelfth ratio is 1:3.

The third spectral line of the A note coincides with the fundamental of the E note, at 1320 Hz.

The twelfth is the interval of an octave and a fifth. The clarinet and other instruments which behave as a stopped cylindrical pipe overblow at the twelfth.

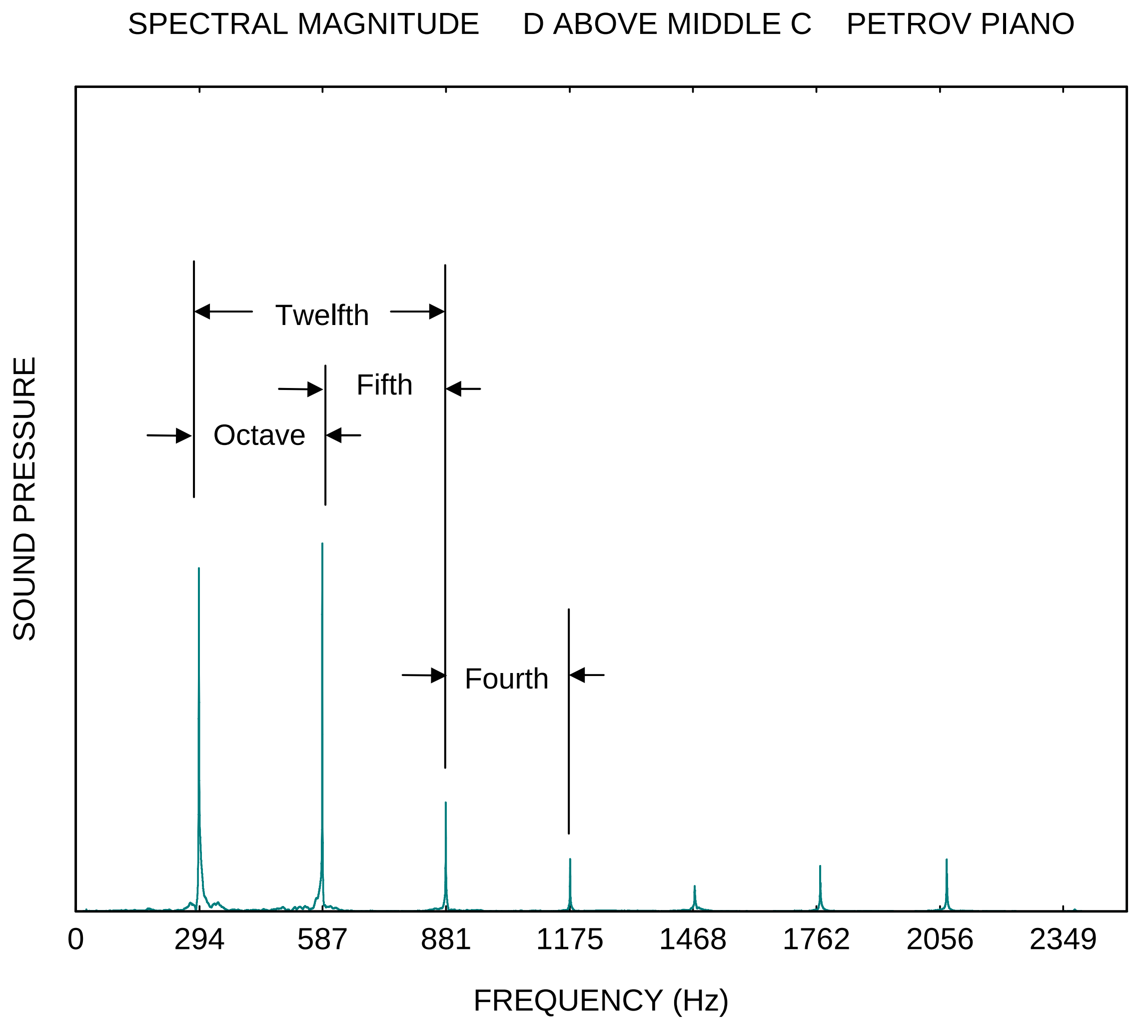

SPECTRAL MAGNITUDE D ABOVE MIDDLE C PETROV PIANO
SOUND PRESSURE
Twelfth
Fifth
Octave
Fourth
0
294
587
881
1175
1468
1762
2056
2349
FREQUENCY (Hz)

Figure 5.