NATURAL FREQUENCIES OF CIRCULAR PLATE BENDING MODES
Revision A

By Tom Irvine
Email: tomirvine@aol.com

August 6, 2009

Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. The method is taken from References 1 through 3. In addition, a Bessel function solution is given in Appendices D and E.

A displacement function is assumed for the Rayleigh method which satisfies the geometric boundary conditions. The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limit of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the circular plate in Figure 1.

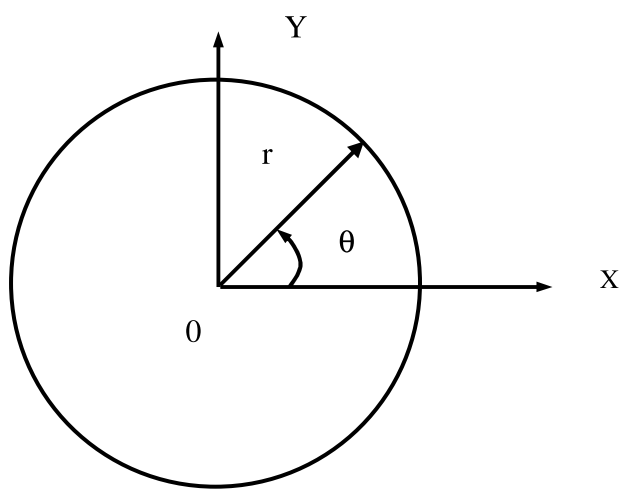

Figure 1.

Let Z represent the out-of-plane displacement.

Table 1.	
Appendix	Topic
A	Strain and kinetic energy
B	Simply Supported Plate, Rayleigh Method
C	Integral Table
D	Solution of Differential Equation via Bessel Functions
E	Simply Supported Plate, Bessel Function Solution

References

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Wiley-Interscience, New York, 1988.
2. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
3. Arthur W. Leissa, Vibration of Plates, NASA SP-160, National Aeronautics and Space Administration, Washington D.C., 1969.
4. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.
5. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
6. W. Soedel, Vibrations of Shells and Plates, Third Edition, Marcel Dekker, New York, 2004.

APPENDIX A

The total strain energy V of the plate is

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right) + 2(1-\mu)\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \theta}\right)\right\}^2\right] r\, dr\, d\theta \tag{A-1}$$

Note that the plate stiffness factor D_e is given by

$$D_e = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{A-2}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio

For a displacement which is symmetric about the center,

$$\frac{\partial}{\partial \theta} Z(r,\theta) = 0 \tag{A-3}$$

$$\frac{\partial^2}{\partial \theta^2} Z(r,\theta) = 0 \tag{A-4}$$

Substitute equations (A-3) and (A-4) into (A-1).

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\,d\theta$$

(A-5)

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + 2\left(\frac{\partial^2 Z}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + (-2+2\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\,d\theta$$

(A-6)

The total strain energy equation for the symmetric case is thus

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + 2\mu\,\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\,d\theta \qquad \text{(A-7)}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h\,\Omega^2}{2}\int_0^{2\pi}\int_0^R Z^2\ r\,dr\,d\theta \qquad \text{(A-8)}$$

where

ρ = mass per volume
Ω = angular natural frequency

APPENDIX B

Simply Supported Plate

Consider a circular plate which is simply supported around its circumference. The plate has a radius a. The displacement perpendicular to the plate is Z. A polar coordinate system is used with the origin at the plate's center.

Seek a displacement function that satisfies the geometric boundary conditions.

The geometric boundary conditions are

$$Z(a, \theta) = 0 \tag{B-1}$$

$$\left.\frac{\partial^2 Z}{\partial^2 r}\right|_{r=a} = 0 \tag{B-2}$$

The following function satisfies the geometric boundary conditions.

$$Z(r, \theta) = Z_o \cos\left(\frac{\pi r}{2a}\right) \tag{B-3}$$

The partial derivatives are

$$\frac{\partial}{\partial \theta} Z(r, \theta) = 0 \tag{B-4}$$

$$\frac{\partial^2}{\partial \theta^2} Z(r, \theta) = 0 \tag{B-5}$$

$$\frac{\partial}{\partial r} Z(r, \theta) = -Z_o \left(\frac{\pi}{2a}\right) \sin\left(\frac{\pi r}{2a}\right) \tag{B-6}$$

$$\frac{\partial^2}{\partial r^2} Z(r, \theta) = -Z_o \left(\frac{\pi}{2a}\right)^2 \cos\left(\frac{\pi r}{2a}\right) \tag{B-7}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_0^{a} \left[Z_o \cos\left(\frac{\pi r}{2a} \right) \right]^2 r \, dr \, d\theta \tag{B-8}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \int_0^{a} \left[1 + \cos\left(\frac{\pi r}{a} \right) \right] r \, dr \, d\theta \tag{B-9}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \int_0^{a} \left[r + r\cos\left(\frac{\pi r}{a} \right) \right] dr \, d\theta \tag{B-10}$$

Evaluate equation (B-9) using the integral table in Appendix C

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{a\,r}{\pi} \sin\left(\frac{\pi r}{a} \right) + \frac{a^2}{\pi^2} \cos\left(\frac{\pi r}{a} \right) \right] \Bigg|_0^a d\theta \tag{B-11}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{a^2}{2} + \frac{a^2}{\pi^2} \cos(\pi) - \frac{a^2}{\pi^2} \cos(0) \right] d\theta \tag{B-12}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{a^2}{2} - \frac{2a^2}{\pi^2} \right] d\theta \tag{B-13}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \int_0^{2\pi} \left[\pi^2 - 4 \right] d\theta \tag{B-14}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \left[\pi^2 - 4 \right] \int_0^{2\pi} d\theta \tag{B-15}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \left[\pi^2 - 4 \right] [2\pi] \tag{B-16}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{4\pi}\left[\pi^2 - 4\right] \qquad \text{(B-17)}$$

$$T = \left(0.4671\right)\rho h \Omega^2 Z_o{}^2 a^2 \qquad \text{(B-18)}$$

Again, the total strain energy for the symmetric case is

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + 2\mu \frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\, dr\, d\theta \qquad \text{(B-19)}$$

$$\begin{aligned} V = {} & \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2\right] r\, dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2\right] r\, dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[2\mu \frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\, dr\, d\theta \end{aligned}$$

(B-20)

$$\begin{aligned} V = {} & + \frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2 \left[\left(\frac{\pi}{2a}\right)^4 \cos^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2 \left[\frac{1}{r^2}\left(\frac{\pi}{2a}\right)^2 \sin^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2 \left[(2\mu)\left(\frac{\pi}{2a}\right)^3 \frac{1}{r}\cos\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta \end{aligned}$$

(B-21)

$$V = +\frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_0^a\left[\left(\frac{\pi}{2a}\right)^4 \cos^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_0^a\left[\frac{1}{r^2}\left(\frac{\pi}{2a}\right)^2 \sin^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_0^a\left[(2\mu)\left(\frac{\pi}{2a}\right)^3 \frac{1}{r}\cos\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

(B-22)

$$V = +\frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2a}\right)^4\int_0^{2\pi}\int_0^a\left[\cos^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}\int_0^a\left[\frac{1}{r^2}\sin^2\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}(2\mu)\left(\frac{\pi}{2a}\right)^3\int_0^{2\pi}\int_0^a\left[\frac{1}{r}\cos\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{2a}\right)\right] r\, dr\, d\theta$$

(B-23)

$$V = +\frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\int_0^a\left[1+\cos\left(\frac{\pi r}{a}\right)\right] r\, dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}\int_0^a\left[\frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)\right] dr\, d\theta$$

$$+\frac{Z_o{}^2 D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^3\right)\int_0^{2\pi}\int_0^a\left[\sin\left(\frac{\pi r}{a}\right)\right] dr\, d\theta$$

(B-24)

The first and third integrals are evaluating using the tables in Appendix C.

$$V = +\frac{Z_o^{\,2}\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{r^2}{2}+\frac{a\,r}{\pi}\sin\left(\frac{\pi r}{a}\right)+\frac{a^2}{\pi^2}\cos\left(\frac{\pi r}{a}\right)\right]\Bigg|_0^a d\theta$$

$$+\frac{Z_o^{\,2}\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi} 0.8242\,d\theta$$

$$-\frac{Z_o^{\,2}\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^3\right)\left(\frac{a}{\pi}\right)\int_0^{2\pi}\cos\left(\frac{\pi r}{a}\right)\Bigg|_0^a d\theta$$

(B-25)

$$V = +\frac{Z_o^{\,2}\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]d\theta$$

$$+\frac{Z_o^{\,2}\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi} 0.8242\,d\theta$$

$$+\frac{Z_o^{\,2}\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^3\right)\left(\frac{2a}{\pi}\right)\int_0^{2\pi} d\theta$$

(B-26)

$$V = +\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}0.8242\,d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^2\right)\int_0^{2\pi}d\theta$$

(B-27)

$$V = +\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right](2\pi)$$

$$+\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^2(0.8242)(2\pi)$$

$$+\frac{Z_o{}^2\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^2\right)2\pi$$

(B-28)

$$V = +Z_o{}^2\,D\,\pi\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]$$

$$+Z_o{}^2\,D\,\pi\left(\frac{\pi}{2a}\right)^2(0.8242)$$

$$+Z_o{}^2\,D\,\pi\left(\mu\left(\frac{\pi}{2a}\right)^2\right)$$

(B-29)

$$V = \quad + Z_o^{\,2}\ D\,\pi \left\{ \left(\frac{\pi}{2a}\right)^4 \left(\frac{1}{2}\right)\left[\frac{a^2}{2} - \frac{2a^2}{\pi^2}\right] + \left(\frac{\pi}{2a}\right)^2 (0.8242) + \left(\mu \left(\frac{\pi}{2a}\right)^2\right)\right\}$$

(B-30)

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \pi^2 \left(\frac{1}{2a}\right)^4 \left(\frac{1}{2}\right)\left[\frac{a^2}{2} - \frac{2a^2}{\pi^2}\right] + \left(\frac{1}{2a}\right)^2 (0.8242) + \left(\mu \left(\frac{1}{2a}\right)^2\right)\right\}$$

(B-31)

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \pi^2 \left(\frac{1}{16a^4}\right) \left(\frac{1}{2}\right)\left[\frac{a^2}{2} - \frac{2a^2}{\pi^2}\right] + \left(\frac{1}{4a^2}\right) (0.8242) + \left(\mu \left(\frac{1}{4a^2}\right)\right)\right\}$$

(B-32)

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \pi^2 \left(\frac{1}{16a^2}\right) \left(\frac{1}{2}\right)\left[\frac{1}{2} - \frac{2}{\pi^2}\right] + \left(\frac{1}{4a^2}\right) (0.8242) + \left(\mu \left(\frac{1}{4a^2}\right)\right)\right\}$$

(B-33)

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{\frac{1}{a^2}\right\} \left\{ \pi^2 \left(\frac{1}{16}\right) \left(\frac{1}{2}\right)\left[\frac{1}{2} - \frac{2}{\pi^2}\right] + \left(\frac{1}{4}\right) (0.8242) + \left(\frac{\mu}{4}\right)\right\}$$

(B-34)

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{\frac{1}{a^2}\right\} \left\{ \pi^2 \ \left(\frac{1}{32}\right)\left[\frac{1}{2} - \frac{2}{\pi^2}\right] + \left(\frac{1}{4}\right) (0.8242) + \left(\frac{\mu}{4}\right)\right\} \qquad \text{(B-35)}$$

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{\frac{1}{a^2}\right\} \left\{ \ \left(\frac{1}{32}\right)\left[\frac{\pi^2}{2} - 2\right] + \left(\frac{1}{4}\right) (0.8242) + \left(\frac{\mu}{4}\right)\right\} \qquad \text{(B-36)}$$

$$V = +Z_o^2 D \pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \left(\frac{1}{64} \right) \left[\pi^2 - 4 \right] + \left(\frac{1}{4} \right) (0.8242) + \left(\frac{\mu}{4} \right) \right\} \quad \text{(B-37)}$$

$$V = +Z_o^2 D \pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ \left[\pi^2 - 4 \right] + 16(0.8242) + 16\mu \right\} \quad \text{(B-38)}$$

$$V = +Z_o^2 D \pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ \left[\pi^2 - 4 \right] + 16(0.8242) + 16\mu \right\} \quad \text{(B-39)}$$

$$V = +Z_o^2 D \pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ 19.0568 + 16\mu \right\} \quad \text{(B-40)}$$

$$V = +Z_o^2 D \pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ 0.2978 + 0.25\mu \right\} \quad \text{(B-41)}$$

Now equate the total kinetic energy with the total strain energy per Rayleigh's method.

$$\frac{\rho h \Omega^2 Z_o^2 a^2}{4\pi} \left[\pi^2 - 4 \right] = +Z_o^2 D \pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ 0.2978 + 0.25\mu \right\} \quad \text{(B-42)}$$

$$\frac{\rho h \Omega^2 a^2}{4\pi} \left[\pi^2 - 4 \right] = D \pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ 0.2978 + 0.25\mu \right\} \quad \text{(B-43)}$$

$$\frac{\rho h \Omega^2}{4\pi} \left[\pi^2 - 4 \right] = D \pi^3 \left\{ \frac{1}{a^4} \right\} \left\{ 0.2978 + 0.25\mu \right\} \quad \text{(B-44)}$$

$$\frac{\rho h \Omega^2}{4} \left[\pi^2 - 4 \right] = D \pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ 0.2978 + 0.25\mu \right\} \quad \text{(B-45)}$$

$$\rho h\,\Omega^2\left[\pi^2-4\right]= \quad D\,\pi^4\left\{\frac{4}{a^4}\right\}\{\quad 0.2978 \quad + \; 0.25\mu\} \qquad \text{(B-46)}$$

$$\Omega^2 = \quad D\,\pi^4\left\{\frac{4}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 0.2978 \quad + \; 0.25\mu\} \qquad \text{(B-47)}$$

$$\Omega^2 = \quad D\,\pi^4\left\{\frac{1}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 1.1911 \quad + \; \mu\} \qquad \text{(B-48)}$$

Let $\mu = 0.3$, which is the typical Poisson's ratio.

$$\Omega^2 = \quad D\,\pi^4\left\{\frac{1}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 1.1911 \quad + \; 0.3\} \qquad \text{(B-49)}$$

$$\Omega^2 = \quad D\,\pi^4\left\{\frac{1}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 1.4911 \quad \} \qquad \text{(B-50)}$$

$$\Omega^2 = \quad D\,\pi^4\left\{\frac{1}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 1.4911 \quad \} \qquad \text{(B-51)}$$

$$\Omega = \quad \sqrt{D\,\pi^4\left\{\frac{1}{a^4}\right\}\left\{\frac{1}{\rho h\left[\pi^2-4\right]}\right\}\{\quad 1.4911 \;\}} \qquad \text{(B-52)}$$

$$\Omega = \quad \frac{4.9744}{a^2}\sqrt{\frac{D}{\rho h}} \qquad \text{(B-53)}$$

The natural frequency f_n is

$$f_n = \frac{1}{2\pi}\Omega \qquad \text{(B-54)}$$

$$f_n = \frac{4.9744}{2\pi\, a^2}\sqrt{\frac{D_e}{\rho h}} \qquad \text{(B-55)}$$

APPENDIX C

Integral Table

Equation (C-1) is taken from Reference 1.

$$\int x \cos bx \, dx = \frac{x \sin bx}{b} + \frac{\cos bx}{b^2} \tag{C-1}$$

Now consider

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{1}{2}\int_0^a \frac{1}{r}\left[1 - \cos\left(\frac{\pi r}{a}\right)\right] dr \tag{C-2}$$

Nondimensionalize,

$$x = \frac{\pi r}{a} \tag{C-3}$$

$$\frac{a}{\pi} x = r \tag{C-4}$$

$$dx = \frac{\pi}{a} dr \tag{C-5}$$

$$\frac{a}{\pi} dx = dr \tag{C-6}$$

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{a}{2\pi}\left(\frac{\pi}{a}\right)\int_0^\pi \frac{1}{x}[1 - \cos x] \, dx \tag{C-7}$$

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{1}{2}\int_0^\pi \frac{1}{x}[1 - \cos x] \, dx \tag{C-8}$$

Recall the series

$$\cos x \approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!} \tag{C-9}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \frac{1}{x}\left\{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}\right]\right\}dx \tag{C-10}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \frac{1}{x}\left[\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}-\frac{x^{12}}{12!}\right]dx \tag{C-11}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \left[\frac{x^1}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\frac{x^7}{8!}+\frac{x^9}{10!}-\frac{x^{11}}{12!}\right]dx \tag{C-12}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\left[\frac{x^2}{2\cdot 2!}-\frac{x^4}{4\cdot 4!}+\frac{x^6}{6\cdot 6!}-\frac{x^8}{8\cdot 8!}+\frac{x^{10}}{10\cdot 10!}-\frac{x^{12}}{12\cdot 12!}\right]\Bigg|_0^\pi \tag{C-13}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}(1.6483) \tag{C-14}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx 0.8242 \tag{C-15}$$

APPENDIX D

Solution of Differential Equation via Bessel Functions

The governing equation is taken from References 5 and 6.

$$\nabla^4 Z(r,\theta) - \beta^4 Z(r,\theta) = 0 \tag{D-1}$$

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \tag{D-2}$$

$$\beta = \left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{D-2}$$

$$D_e = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{D-3}$$

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \tag{D-4}$$

The governing equation may be written as

$$\left\{\nabla^2 + \beta^2\right\}\left\{\nabla^2 - \beta^2\right\} Z(r,\theta) = 0 \tag{D-5}$$

Thus the equation is satisfy by

$$\left\{\nabla^2 \pm \beta^2\right\} Z(r,\theta) = 0 \tag{D-6}$$

Separate variables

$$Z(r,\theta) = R(r)\Theta(\theta) \tag{D-7}$$

By substitution

$$\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)+\beta^2\right\}R(r)\Theta(\theta)=0 \tag{D-8}$$

$$\left(\frac{\partial^2}{\partial r^2}R\Theta+\frac{1}{r}\frac{\partial}{\partial r}R\Theta+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}R\Theta\right)+\beta^2 R\Theta=0 \tag{D-9}$$

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)+\beta^2 R\Theta=0 \tag{D-10}$$

Similarly,

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)-\beta^2 R\Theta=0 \tag{D-11}$$

Thus,

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)\pm\beta^2 R\Theta=0 \tag{D-12}$$

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}\right)\pm\beta^2 R\Theta=-\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} \tag{D-13}$$

$$\left(\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{R\,r}\frac{dR}{dr}\right)\pm\beta^2=-\frac{1}{\Theta r^2}\frac{d^2\Theta}{d\theta^2} \tag{D-14}$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)\ \pm\ \beta^2 r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \tag{D-15}$$

The equation can be satisfied if each expression is equal to the same constant k^2.

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right) \pm \beta^2 r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k^2 \tag{D-16}$$

Thus

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right) \pm \beta^2 r^2 = k^2 \tag{D-17}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr} + \left[\pm\beta^2 - \frac{k^2}{r^2}\right]R = 0 \tag{D-18}$$

Define a new variable

$$\xi = \begin{cases} \beta r \\ j\beta r \end{cases} \tag{D-19}$$

$$\xi^2 = \begin{cases} \beta^2 r^2 \\ -\beta^2 r^2 \end{cases} \tag{D-20}$$

$$\frac{\xi^2}{r^2} = \begin{cases} \beta^2 \\ -\beta^2 \end{cases} \tag{D-21}$$

Chain rule

$$d\xi = \beta dr \tag{D-22}$$

$$\frac{d}{dr} = \frac{d}{d\xi}\frac{d\xi}{dr} \tag{D-23}$$

$$\frac{d}{dr} = \beta\frac{d}{d\xi} \tag{D-24}$$

$$\frac{\xi^2}{r^2}\frac{d^2R}{d\xi^2}+\frac{\xi}{r^2}\frac{dR}{d\xi} \quad +\left[\frac{\xi^2}{r^2} - \frac{k^2}{r^2}\right]R \quad = \ 0 \tag{D-25}$$

$$\xi^2\frac{d^2R}{d\xi^2}+\xi\frac{dR}{d\xi} \quad +\left[\ \xi^2 - \ k^2\right]R \quad = \ 0 \tag{D-26}$$

$$\frac{d^2R}{d\xi^2} \ + \ \frac{1}{\xi}\frac{dR}{d\xi} \quad +\left[1-\frac{k^2}{\xi^2}\right]R \quad = \ 0 \tag{D-27}$$

Equation (D-27) is Bessel's equation of fractional order.

The solution for circular plates that are closed in the θ direction is

$$R(\xi) = C\ J_n(\xi) + D\ I_n(\xi) + F\,Y_n(\xi) + G\,K_n(\xi) \tag{D-28}$$

Equation (D-28) represents Bessel of the first and second kind and modified Bessel of the first and second kind.

Both $Y_n(\xi)$ and $K_n(\xi)$ are singular at $\xi = 0$.

Thus for a plate with no hole, F = G = 0.

$$R(\xi) = C\ J_n(\xi) + D\ I_n(\xi) \tag{D-29}$$

Furthermore, from equation (D-16),

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \ = \ k^2 \tag{D-30}$$

$$\frac{d^2\Theta}{d\theta^2} \ + \Theta k^2 = 0 \tag{D-31}$$

The solution for circular plates that are closed in the θ direction is

$$\Theta = \hat{A}\cos k\theta + \hat{B}\sin k\theta \quad , \quad k = n = 0, 1, 2, 3, \; ..... \tag{D-32}$$

Or equivalently

$$\Theta = A\cos\left[k(\theta-\phi)\right] \quad .. \tag{D-33}$$

The total solution is thus

$$Z(\xi,\theta) = \left\{C\ J_n\left(\xi\right) + D\ I_n\left(\xi\right)\right\}\left\{A\cos\left[k(\theta-\phi)\right]\right\} \tag{D-34}$$

Set the phase angle ϕ =0.

$$Z(\xi,\theta) = \left\{C\ J_n\left(\xi\right) + D\ I_n\left(\xi\right)\right\} A\cos(k\theta) \tag{D-35}$$

Set A =1. Note that the mass normalization will be performed using the C and D coefficients.

$$Z(\xi,\theta) = \left\{C\ J_n\left(\xi\right) + D\ I_n\left(\xi\right)\right\}\cos(k\theta) \tag{D-36}$$

APPENDIX E

Simply Supported Plate, Bessel Function Solution

The boundary conditions are

$$Z(a,\theta) = 0 \tag{E-1}$$

$$M_r = 0 \quad \text{at} \quad r = a \tag{E-2}$$

$$\frac{\partial^2 Z}{\partial \theta^2} = 0 \quad \text{at} \quad r = a \tag{E-3}$$

Note that

$$M_r = -D\left[\frac{\partial^2 Z}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)\right] \tag{E-4}$$

Boundary condition (E-3) requires that

$$M_r = -D\left[\frac{\partial^2 Z}{\partial r^2} + \frac{\mu}{r}\frac{\partial Z}{\partial r}\right] \qquad \text{at} \quad r = a \tag{E-5}$$

$$Z(r,\theta) = R(r)\Theta(\theta) \tag{E-6}$$

$$Z(r,\theta) = \left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(k\theta) \tag{E-7}$$

$$Z(a,\theta) = \left[C\ J_n(\beta a) + D\ I_n(\beta a)\right]\cos(k\theta) = 0 \tag{E-8}$$

$$C\ J_n(\beta a) + D\ I_n(\beta a) = 0 \tag{E-9}$$

$$M_r = -D\left[\frac{\partial^2}{\partial r^2}\left[C\, J_n(\beta r) + D\, I_n(\beta r)\right]\cos(k\theta)\right]$$
$$-\mu D\left[\frac{1}{r}\frac{\partial}{\partial r}\left[C\, J_n(\beta r) + D\, I_n(\beta r)\right]\cos(k\theta)\right] \quad \text{at} \quad r = a$$
(E-10)

$$M_r = -D_E \cos(k\theta)\left[C\,\frac{d^2}{dr^2} J_n(\beta r) + D\frac{d^2}{dr^2} I_n(\beta r)\right]$$
$$-\mu D_E \cos(k\theta)\left[C\frac{1}{r}\frac{d}{dr} J_n(\beta r) + D\frac{1}{r}\frac{d}{dr} I_n(\beta r)\right] \quad \text{at} \quad r = a$$
(E-11)

$$M_r\big|_{r=a} = 0 \tag{E-12}$$

$$\left[C\,\frac{d^2}{dr^2} J_n(\beta r) + D\frac{d^2}{dr^2} I_n(\beta r)\right] + \mu\left[C\frac{1}{a}\frac{d}{dr} J_n(\beta r) + D\frac{1}{a}\frac{d}{dr} I_n(\beta r)\right] = 0, \quad \text{at } r = a$$
(E-13)

$$C\left[\frac{d^2}{dr^2} J_n(\beta r) + \frac{\mu}{a}\frac{d}{dr} J_n(\beta r)\right] + D\left[\frac{d^2}{dr^2} I_n(\beta r) + \frac{\mu}{a}\frac{d}{dr} I_n(\beta r)\right] = 0\,, \quad \text{at } r = a$$
(E-14)

Let

$$\lambda = \beta r \tag{E-15}$$

$$C\left[\beta^2\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu\beta}{a}\frac{d}{d\lambda}J_n(\lambda)\right]+D\left[\beta^2\frac{d^2}{d\lambda^2}I_n(\lambda)+\frac{\mu\beta}{a}\frac{d}{d\lambda}I_n(\lambda)\right]=0\ ,\quad \text{at}\quad \lambda=\beta a \tag{E-36}$$

$$C\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]+D\left[\beta\frac{d^2}{d\lambda^2}I_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right]=0\ ,\quad \text{at}\quad \lambda=\beta a \tag{E-37}$$

Recall equation (E-9).

$$C\,J_n(\lambda)+D\,I_n(\lambda)=0\,,\quad \text{at}\quad \lambda=\beta a \tag{E-38}$$

$$D\,I_n(\lambda)=-C\,J_n(\lambda)\ ,\quad \text{at}\quad \lambda=\beta a \tag{E-39}$$

$$D=-C\frac{J_n(\lambda)}{I_n(\lambda)}\ ,\quad \text{at}\quad \lambda=\beta a \tag{E-40}$$

By substitution,

$$C\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]-C\frac{J_n(\lambda)}{I_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}I_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right]=0\,,\quad \text{at}\quad \lambda=\beta a \tag{E-41}$$

$$\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]-\frac{J_n(\lambda)}{I_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}I_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right]=0\,,\quad \text{at}\quad \lambda=\beta a \tag{E-42}$$

$$\frac{1}{J_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right] - \frac{1}{I_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right] = 0 \quad ,$$

$$\text{at} \quad \lambda = \beta a$$

(E-43)

Note the following identities:

$$\frac{d}{d\lambda}J_n(\lambda) = J_{n-1}(\lambda) - \frac{n}{\lambda}J_n(\lambda) = -J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda) \tag{E-44}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = -\frac{d}{d\lambda}J_{n+1}(\lambda) + \frac{n}{\lambda}\frac{d}{d\lambda}J_n(\lambda) \tag{E-45}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-J_n(\lambda) + \frac{n+1}{\lambda}J_{n+1}(\lambda)\right] + \frac{n}{\lambda}\left[-J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda)\right] \tag{E-46}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-J_n(\lambda) + \frac{n+1}{\lambda}J_{n+1}(\lambda)\right] + \frac{n}{\lambda}\left[-J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda)\right] \tag{E-47}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-1 + \frac{n^2}{\lambda^2}\right]J_n(\lambda) + \left[\frac{1}{\lambda}\right]J_{n+1}(\lambda) \tag{E-48}$$

Analyze the first term of equation (E-42).

$$+\frac{1}{J_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]$$

$$=\frac{1}{J_n(\lambda)}\left[\beta\left[-1+\frac{n^2}{\lambda^2}\right]J_n(\lambda)+\beta\left[\frac{1}{\lambda}\right]J_{n+1}(\lambda)\right]+\frac{\mu}{a}\frac{1}{J_n(\lambda)}\left[-J_{n+1}(\lambda)+\frac{n}{\lambda}J_n(\lambda)\right]$$

$$=\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\beta}{\lambda}\right]-\frac{\mu}{a}\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\beta\left[-1+\frac{n^2}{\lambda^2}\right]+\frac{\mu n}{a\lambda}$$

$$=-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\beta\left[-1+\frac{n^2}{\lambda^2}\right]+\frac{\mu n}{a\lambda}$$

(E-49)

Consider the following identities:

$$\frac{d}{d\lambda}I_n(\lambda) = I_{n-1}(\lambda)-\frac{n}{\lambda}I_n(\lambda) = I_{n+1}(\lambda)+\frac{n}{\lambda}I_n(\lambda) \quad \text{(E-50)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\frac{d}{d\lambda}I_{n+1}(\lambda)+\frac{n}{\lambda}\frac{d}{d\lambda}I_n(\lambda) \quad \text{(E-51)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\left[I_n(\lambda)-\frac{n+1}{\lambda}I_{n+1}(\lambda)\right]+\frac{n}{\lambda}\left[I_{n+1}(\lambda)+\frac{n}{\lambda}I_n(\lambda)\right] \quad \text{(E-52)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\left[1+\frac{n^2}{\lambda^2}\right]I_n(\lambda)-\left[\frac{1}{\lambda}\right]I_{n+1}(\lambda) \quad \text{(E-53)}$$

Analyze the second term of equation (E-42).

$$-\frac{1}{I_n(\lambda)}\left[\ \beta\frac{d^2}{d\lambda^2}I_n(\lambda)\ +\ \frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\ \right]$$

$$=-\frac{1}{I_n(\lambda)}\left\{\beta\left[1+\frac{n^2}{\lambda^2}\right]I_n(\lambda)-\beta\left[\frac{1}{\lambda}\right]I_{n+1}(\lambda)\right\}-\frac{\mu}{a}\frac{1}{I_n(\lambda)}\left[\ I_{n+1}(\lambda)+\frac{n}{\lambda}I_n(\lambda)\right]$$

$$=\frac{\beta}{\lambda}\frac{I_{n+1}(\lambda)}{I_n(\lambda)}-\frac{\mu}{a}\frac{I_{n+1}(\lambda)}{I_n(\lambda)}-\beta\left[1+\frac{n^2}{\lambda^2}\right]-\frac{\mu n}{a\lambda}$$

$$=-\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}\ -\beta\left[1+\frac{n^2}{\lambda^2}\right]-\frac{\mu n}{a\lambda}$$

(E-54)

By substitution,

$$+\frac{1}{J_n(\lambda)}\left[\frac{d^2}{d\lambda^2}J_n(\lambda)\right]+\frac{1}{J_n(\lambda)}\left[\ \frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]$$
$$-\frac{1}{I_n(\lambda)}\left[\frac{d^2}{d\lambda^2}I_n(\lambda)\right]-\frac{1}{I_n(\lambda)}\left[\ \frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right]\ =\ 0\quad ,$$

at $\lambda=\beta a$

(E-55)

$$-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\beta\left[-1+\frac{n^2}{\lambda^2}\right]+\frac{\mu n}{a\lambda}-\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}\ -\beta\left[1+\frac{n^2}{\lambda^2}\right]-\frac{\mu n}{a\lambda}=0\,,$$

at $\lambda=\beta a$

(E-56)

$$-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]-\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}-2\beta=0 \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-57}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}+2\beta=0 \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-58}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=-2\beta \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-59}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta}{\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]} \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-60}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta}{\left[\frac{\mu}{a}-\frac{1}{a}\right]} \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-61}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta a}{-1+\mu} \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-62}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\lambda}{-1+\mu} \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-63}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{2\lambda}{1-\mu} \quad , \quad \text{at} \quad \lambda=\beta a \tag{E-64}$$

The following form is better suited for numerical root-finding purposes.

$$I_n(\lambda)\,J_{n+1}(\lambda)+J_n(\lambda)I_{n+1}(\lambda)=\frac{2\lambda}{1-\mu}\left[J_n(\lambda)\,I_n(\lambda)\right] , \quad \text{at} \quad \lambda=\beta a \tag{E-65}$$

The roots of equation (E-65) for $\mu = 0.3$ are

k	n=0	n=1	n=2	n=3
0	4.9351	13.8982	25.6133	39.9573
1	29.9844	48.7391	70.1170	95.2930
2	74.9211	103.4057	134.4289	168.8374
3	138.7787	176.8456	218.2026	264.0849

The roots were determined using the secant method.

The fundamental natural frequency is thus

$$\lambda = \beta a \tag{E-66}$$

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \tag{E-67}$$

$$\lambda = a\left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{E-68}$$

$$\omega^2 = \frac{\lambda^4 D_e}{\rho h a^4} \tag{E-69}$$

$$\omega = \frac{\lambda^2}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{E-70}$$

$$\omega = \frac{4.9351}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{E-71}$$

The mode shapes are defined by

$$Z(r,\theta) = [C\, J_n(\beta r) + D\, I_n(\beta r)]\cos(k\theta) \tag{E-72}$$

Recall

$$D = -C\frac{J_n(\lambda)}{I_n(\lambda)} \quad , \qquad \text{at} \quad \lambda = \beta a \tag{E-73}$$

$$Z(r,\theta) = \left\{C\ J_n(r) + -C\frac{J_n(\lambda)}{I_n(\lambda)}\ I_n(r)\right\}\cos(k\theta) \tag{E-74}$$

$$Z(r,\theta) = C\left\{\ J_n(r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)}\right] I_n(r)\right\}\cos(k\theta) \tag{E-75}$$