NATURAL FREQUENCIES OF CIRCULAR PLATE BENDING MODES
Revision F

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Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. The method is taken from References 1 through 3. In addition, a Bessel function solution is given in Appendices D and E.

A displacement function is assumed for the Rayleigh method which satisfies the geometric boundary conditions. The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limit of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the circular plate in Figure 1.

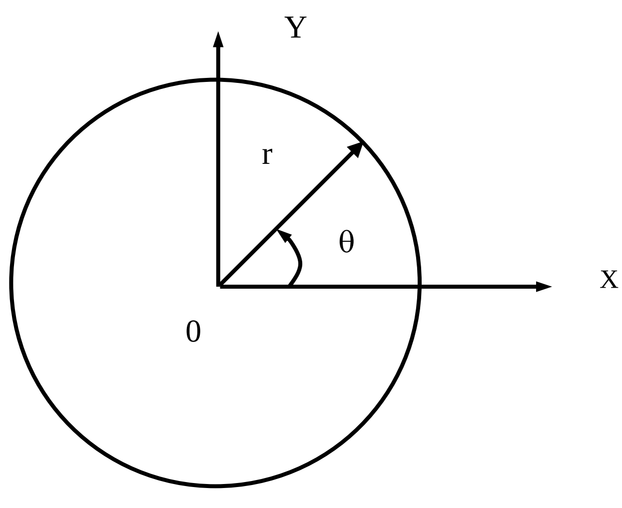

Figure 1.

Let Z represent the out-of-plane displacement.

Table 1.	
Appendix	Topic
A	Strain and kinetic energy
B	Simply Supported Plate, Rayleigh Method
C	Integral Table
D	Solution of Differential Equation via Bessel Functions
E	Simply Supported Plate, Bessel Function Solution
F	Mass Normalization of Eigenvectors
G	Completely Free Circular Plate
H	Fixed Plate, Bessel Function Solution

References

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Wiley-Interscience, New York, 1988.
2. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
3. Arthur W. Leissa, Vibration of Plates, NASA SP-160, National Aeronautics and Space Administration, Washington D.C., 1969.
4. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.
5. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
6. W. Soedel, Vibrations of Shells and Plates, Third Edition, Marcel Dekker, New York, 2004.

APPENDIX A

The total strain energy V of the plate is

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} \right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} \right) \right.$$

$$\left. + 2(1-\mu)\left\{ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \theta} \right) \right\}^2 \right] r \, dr \, d\theta$$

(A-1)

Note that the plate stiffness factor D_e is given by

$$D_e = \frac{Eh^3}{12\left(1-\mu^2\right)} \qquad \text{(A-2)}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio

For a displacement which is symmetric about the center,

$$\frac{\partial}{\partial \theta} Z(r,\theta) = 0 \qquad \text{(A-3)}$$

$$\frac{\partial^2}{\partial \theta^2} Z(r,\theta) = 0 \qquad \text{(A-4)}$$

Substitute equations (A-3) and (A-4) into (A-1).

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^{R}\left[\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta$$

(A-5)

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^{R}\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + 2\left(\frac{\partial^2 Z}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + (-2+2\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta$$

(A-6)

The total strain energy equation for the symmetric case is thus

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^{R}\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + 2\mu\,\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta \qquad \text{(A-7)}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h\,\Omega^2}{2}\int_0^{2\pi}\int_0^{R} Z^2\ r\, dr\, d\theta \qquad \text{(A-8)}$$

where

ρ = mass per volume
Ω = angular natural frequency

APPENDIX B

Simply Supported Plate

Consider a circular plate which is simply supported around its circumference. The plate has a radius a. The displacement perpendicular to the plate is Z. A polar coordinate system is used with the origin at the plate's center.

Seek a displacement function that satisfies the geometric boundary conditions.

The geometric boundary conditions are

$$Z(a,\theta) = 0 \tag{B-1}$$

$$\left.\frac{\partial^2 Z}{\partial^2 r}\right|_{r=a} = 0 \tag{B-2}$$

The following function satisfies the geometric boundary conditions.

$$Z(r,\theta) = Z_o \cos\left(\frac{\pi r}{2a}\right) \tag{B-3}$$

The partial derivatives are

$$\frac{\partial}{\partial \theta} Z(r,\theta) = 0 \tag{B-4}$$

$$\frac{\partial^2}{\partial \theta^2} Z(r,\theta) = 0 \tag{B-5}$$

$$\frac{\partial}{\partial r} Z(r,\theta) = -Z_o\left(\frac{\pi}{2a}\right)\sin\left(\frac{\pi r}{2a}\right) \tag{B-6}$$

$$\frac{\partial^2}{\partial r^2} Z(r,\theta) = -Z_o\left(\frac{\pi}{2a}\right)^2 \cos\left(\frac{\pi r}{2a}\right) \tag{B-7}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_0^a \left[Z_o \cos\left(\frac{\pi r}{2a}\right)\right]^2 r\,dr\,d\theta \quad \text{(B-8)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \int_0^a \left[1 + \cos\left(\frac{\pi r}{a}\right)\right] r\,dr\,d\theta \quad \text{(B-9)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \int_0^a \left[r + r\cos\left(\frac{\pi r}{a}\right)\right] dr\,d\theta \quad \text{(B-10)}$$

Evaluate equation (B-9) using the integral table in Appendix C

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{a\,r}{\pi} \sin\left(\frac{\pi r}{a}\right) + \frac{a^2}{\pi^2} \cos\left(\frac{\pi r}{a}\right)\right]\Bigg|_0^a d\theta \quad \text{(B-11)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{a^2}{2} + \frac{a^2}{\pi^2} \cos(\pi) - \frac{a^2}{\pi^2} \cos(0)\right] d\theta \quad \text{(B-12)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2}{4} \int_0^{2\pi} \left[\frac{a^2}{2} - \frac{2a^2}{\pi^2}\right] d\theta \quad \text{(B-13)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \int_0^{2\pi} \left[\pi^2 - 4\right] d\theta \quad \text{(B-14)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \left[\pi^2 - 4\right] \int_0^{2\pi} d\theta \quad \text{(B-15)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{8\pi^2} \left[\pi^2 - 4\right] \left[2\pi\right] \quad \text{(B-16)}$$

$$T = \frac{\rho h \Omega^2 Z_o{}^2 a^2}{4\pi}\left[\pi^2 - 4\right] \tag{B-17}$$

$$T = \left(0.4671\right)\rho h \Omega^2 Z_o{}^2 a^2 \tag{B-18}$$

Again, the total strain energy for the symmetric case is

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^R\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + 2\mu\,\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\; dr\, d\theta \tag{B-19}$$

$$\begin{aligned} V = {} & \frac{D_e}{2}\int_0^{2\pi}\int_0^R\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2\right] r\; dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^R\left[\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2\right] r\; dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^R\left[\;2\mu\,\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\; dr\, d\theta \end{aligned} \tag{B-20}$$

$$\begin{aligned} V = {} & +\frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2\left[\left(\frac{\pi}{2a}\right)^4 \cos^2\left(\frac{\pi r}{2a}\right)\right] r\; dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2\left[\frac{1}{r^2}\left(\frac{\pi}{2a}\right)^2 \sin^2\left(\frac{\pi r}{2a}\right)\right] r\; dr\, d\theta \\ & + \frac{D_e}{2}\int_0^{2\pi}\int_0^a Z_o{}^2\left[(2\mu)\left(\frac{\pi}{2a}\right)^3 \frac{1}{r}\cos\left(\frac{\pi r}{2a}\right)\sin\left(\frac{\pi r}{2a}\right)\right] r\; dr\, d\theta \end{aligned} \tag{B-21}$$

$$V = \quad + \frac{Z_o{}^2 D}{2} \int_0^{2\pi} \int_0^a \left[\left(\frac{\pi}{2a} \right)^4 \cos^2\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2} \int_0^{2\pi} \int_0^a \left[\frac{1}{r^2} \left(\frac{\pi}{2a} \right)^2 \sin^2\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2} \int_0^{2\pi} \int_0^a \left[(2\mu) \left(\frac{\pi}{2a} \right)^3 \frac{1}{r} \cos\left(\frac{\pi r}{2a} \right) \sin\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

(B-22)

$$V = \quad + \frac{Z_o{}^2 D}{2} \left(\frac{\pi}{2a} \right)^4 \int_0^{2\pi} \int_0^a \left[\cos^2\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2} \left(\frac{\pi}{2a} \right)^2 \int_0^{2\pi} \int_0^a \left[\frac{1}{r^2} \sin^2\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2} (2\mu) \left(\frac{\pi}{2a} \right)^3 \int_0^{2\pi} \int_0^a \left[\frac{1}{r} \cos\left(\frac{\pi r}{2a} \right) \sin\left(\frac{\pi r}{2a} \right) \right] r \; dr \, d\theta$$

(B-23)

$$V = \quad + \frac{Z_o{}^2 \, D}{2} \left(\frac{\pi}{2a} \right)^4 \left(\frac{1}{2} \right) \quad \int_0^{2\pi} \int_0^a \left[1 + \cos\left(\frac{\pi r}{a} \right) \right] r \; dr \, d\theta$$

$$+ \frac{Z_o{}^2 \, D}{2} \left(\frac{\pi}{2a} \right)^2 \int_0^{2\pi} \int_0^a \left[\frac{1}{r} \sin^2\left(\frac{\pi r}{2a} \right) \right] dr \, d\theta$$

$$+ \frac{Z_o{}^2 \, D}{2} \left(\mu \left(\frac{\pi}{2a} \right)^3 \right) \int_0^{2\pi} \int_0^a \left[\sin\left(\frac{\pi r}{a} \right) \right] dr \, d\theta$$

(B-24)

The first and third integrals are evaluating using the tables in Appendix C.

$$V = +\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{r^2}{2}+\frac{a\,r}{\pi}\sin\left(\frac{\pi\,r}{a}\right)+\frac{a^2}{\pi^2}\cos\left(\frac{\pi\,r}{a}\right)\right]\Bigg|_0^a\,d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}0.8242\,d\theta$$

$$-\frac{Z_o{}^2\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^3\right)\left(\frac{a}{\pi}\right)\int_0^{2\pi}\cos\left(\frac{\pi\,r}{a}\right)\Bigg|_0^a\,d\theta$$

(B-25)

$$V = +\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}0.8242\,d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^3\right)\left(\frac{2a}{\pi}\right)\int_0^{2\pi}d\theta$$

(B-26)

$$V = +\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\frac{\pi}{2a}\right)^2\int_0^{2\pi}0.8242\,d\theta$$

$$+\frac{Z_o{}^2\,D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^2\right)\int_0^{2\pi}d\theta$$

(B-27)

$$V = +\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right](2\pi)$$

$$+\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2a}\right)^2(0.8242)(2\pi)$$

$$+\frac{Z_o^2\ D}{2}\left(\mu\left(\frac{\pi}{2a}\right)^2\right)2\pi$$

(B-28)

$$V = +Z_o^2\ D\,\pi\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]$$

$$+Z_o^2\ D\,\pi\left(\frac{\pi}{2a}\right)^2(0.8242)$$

$$+Z_o^2\ D\,\pi\left(\mu\left(\frac{\pi}{2a}\right)^2\right)$$

(B-29)

$$V = +Z_o^2\ D\,\pi\left\{\left(\frac{\pi}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]+\left(\frac{\pi}{2a}\right)^2(0.8242)+\left(\mu\left(\frac{\pi}{2a}\right)^2\right)\right\}$$

(B-30)

$$V = +Z_o^2\ D\,\pi^3\left\{\pi^2\left(\frac{1}{2a}\right)^4\left(\frac{1}{2}\right)\left[\frac{a^2}{2}-\frac{2a^2}{\pi^2}\right]+\left(\frac{1}{2a}\right)^2(0.8242)+\left(\mu\left(\frac{1}{2a}\right)^2\right)\right\}$$

(B-31)

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \pi^2 \left(\frac{1}{16a^4} \right) \left(\frac{1}{2} \right) \left[\frac{a^2}{2} - \frac{2a^2}{\pi^2} \right] + \left(\frac{1}{4a^2} \right) (0.8242) + \left(\mu \left(\frac{1}{4a^2} \right) \right) \right\}$$

(B-32)

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \pi^2 \left(\frac{1}{16a^2} \right) \left(\frac{1}{2} \right) \left[\frac{1}{2} - \frac{2}{\pi^2} \right] + \left(\frac{1}{4a^2} \right) (0.8242) + \left(\mu \left(\frac{1}{4a^2} \right) \right) \right\}$$

(B-33)

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \pi^2 \left(\frac{1}{16} \right) \left(\frac{1}{2} \right) \left[\frac{1}{2} - \frac{2}{\pi^2} \right] + \left(\frac{1}{4} \right) (0.8242) + \left(\frac{\mu}{4} \right) \right\}$$

(B-34)

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \pi^2 \ \left(\frac{1}{32} \right) \left[\frac{1}{2} - \frac{2}{\pi^2} \right] + \left(\frac{1}{4} \right) (0.8242) + \left(\frac{\mu}{4} \right) \right\} \qquad \text{(B-35)}$$

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \ \left(\frac{1}{32} \right) \left[\frac{\pi^2}{2} - 2 \right] + \left(\frac{1}{4} \right) (0.8242) + \left(\frac{\mu}{4} \right) \right\} \qquad \text{(B-36)}$$

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \ \left(\frac{1}{64} \right) \left[\pi^2 - 4 \right] + \left(\frac{1}{4} \right) (0.8242) + \left(\frac{\mu}{4} \right) \right\} \qquad \text{(B-37)}$$

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ \ \left[\pi^2 - 4 \right] + 16 (0.8242) + 16\mu \right\} \qquad \text{(B-38)}$$

$$V = \ + Z_o{}^2 \ D\,\pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ \ \left[\pi^2 - 4 \right] + 16 (0.8242) + 16\mu \right\} \qquad \text{(B-39)}$$

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \frac{1}{64a^2} \right\} \left\{ \quad 19.0568 + 16\mu \right\} \qquad \text{(B-40)}$$

$$V = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-41)}$$

Now equate the total kinetic energy with the total strain energy per Rayleigh's method.

$$\frac{\rho h\,\Omega^2 Z_o^{\,2}\, a^2}{4\pi}\left[\pi^2 - 4\right] = \quad + Z_o^{\,2}\ D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-42)}$$

$$\frac{\rho h\,\Omega^2\, a^2}{4\pi}\left[\pi^2 - 4\right] = \quad D\,\pi^3 \left\{ \frac{1}{a^2} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-43)}$$

$$\frac{\rho h\,\Omega^2}{4\pi}\left[\pi^2 - 4\right] = \quad D\,\pi^3 \left\{ \frac{1}{a^4} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-44)}$$

$$\frac{\rho h\,\Omega^2}{4}\left[\pi^2 - 4\right] = \quad D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-45)}$$

$$\rho h\,\Omega^2\left[\pi^2 - 4\right] = \quad D\,\pi^4 \left\{ \frac{4}{a^4} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-46)}$$

$$\Omega^2 = \quad D\,\pi^4 \left\{ \frac{4}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4\right]} \right\} \left\{ \quad 0.2978 \quad + \ 0.25\mu \right\} \qquad \text{(B-47)}$$

$$\Omega^2 = \quad D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4\right]} \right\} \left\{ \quad 1.1911 \quad + \ \mu \right\} \qquad \text{(B-48)}$$

Let $\mu = 0.3$, which is the typical Poisson's ratio.

$$\Omega^2 = \quad D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4 \right]} \right\} \{ \quad 1.1911 \quad + \; 0.3 \} \tag{B-49}$$

$$\Omega^2 = \quad D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4 \right]} \right\} \{ \quad 1.4911 \quad \} \tag{B-50}$$

$$\Omega^2 = \quad D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4 \right]} \right\} \{ \quad 1.4911 \quad \} \tag{B-51}$$

$$\Omega = \quad \sqrt{ D\,\pi^4 \left\{ \frac{1}{a^4} \right\} \left\{ \frac{1}{\rho h \left[\pi^2 - 4 \right]} \right\} \{ \quad 1.4911 \quad \} } \tag{B-52}$$

$$\Omega = \quad \frac{4.9744}{a^2} \sqrt{\frac{D}{\rho h}} \tag{B-53}$$

The natural frequency f_n is

$$f_n = \frac{1}{2\pi} \Omega \tag{B-54}$$

$$f_n = \frac{4.9744}{2\pi\, a^2} \sqrt{\frac{D_e}{\rho h}} \tag{B-55}$$

APPENDIX C

Integral Table

Equation (C-1) is taken from Reference 1.

$$\int x \cos bx \, dx = \frac{x \sin bx}{b} + \frac{\cos bx}{b^2} \tag{C-1}$$

Now consider

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{1}{2}\int_0^a \frac{1}{r}\left[1 - \cos\left(\frac{\pi r}{a}\right)\right] dr \tag{C-2}$$

Nondimensionalize,

$$x = \frac{\pi r}{a} \tag{C-3}$$

$$\frac{a}{\pi} x = r \tag{C-4}$$

$$dx = \frac{\pi}{a} dr \tag{C-5}$$

$$\frac{a}{\pi} dx = dr \tag{C-6}$$

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{a}{2\pi}\left(\frac{\pi}{a}\right)\int_0^\pi \frac{1}{x}[1 - \cos x] \, dx \tag{C-7}$$

$$\int_0^a \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{1}{2}\int_0^\pi \frac{1}{x}[1 - \cos x] \, dx \tag{C-8}$$

Recall the series

$$\cos x \approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!} \tag{C-9}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \frac{1}{x}\left\{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}\right]\right\}dx \tag{C-10}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \frac{1}{x}\left[\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}-\frac{x^{12}}{12!}\right]dx \tag{C-11}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\int_0^\pi \left[\frac{x^1}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\frac{x^7}{8!}+\frac{x^9}{10!}-\frac{x^{11}}{12!}\right]dx \tag{C-12}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}\left[\frac{x^2}{2\cdot 2!}-\frac{x^4}{4\cdot 4!}+\frac{x^6}{6\cdot 6!}-\frac{x^8}{8\cdot 8!}+\frac{x^{10}}{10\cdot 10!}-\frac{x^{12}}{12\cdot 12!}\right]\Bigg|_0^\pi \tag{C-13}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx \frac{1}{2}(1.6483) \tag{C-14}$$

$$\int_0^a \frac{1}{r}\sin^2\left(\frac{\pi r}{2a}\right)dr \approx 0.8242 \tag{C-15}$$

APPENDIX D

Solution of Differential Equation via Bessel Functions

The governing equation is taken from References 5 and 6.

$$\nabla^4 Z(r,\theta) - \beta^4 Z(r,\theta) = 0 \qquad \text{(D-1)}$$

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \qquad \text{(D-2)}$$

$$\beta = \left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \qquad \text{(D-2)}$$

$$D_e = \frac{Eh^3}{12\left(1-\mu^2\right)} \qquad \text{(D-3)}$$

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \qquad \text{(D-4)}$$

The governing equation may be written as

$$\left\{\nabla^2 + \beta^2\right\}\left\{\nabla^2 - \beta^2\right\} Z(r,\theta) = 0 \qquad \text{(D-5)}$$

Thus the equation is satisfy by

$$\left\{\nabla^2 \pm \beta^2\right\} Z(r,\theta) = 0 \qquad \text{(D-6)}$$

Separate variables

$$Z(r,\theta) = R(r)\Theta(\theta) \qquad \text{(D-7)}$$

By substitution

$$\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)+\beta^2\right\}R(r)\Theta(\theta)=0 \qquad \text{(D-8)}$$

$$\left(\frac{\partial^2}{\partial r^2}R\Theta+\frac{1}{r}\frac{\partial}{\partial r}R\Theta+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R\Theta\right)+\beta^2 R\Theta=0 \qquad \text{(D-9)}$$

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)+\beta^2 R\Theta=0 \qquad \text{(D-10)}$$

Similarly,

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)-\beta^2 R\Theta=0 \qquad \text{(D-11)}$$

Thus,

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}\right)\pm\beta^2 R\Theta=0 \qquad \text{(D-12)}$$

$$\left(\Theta\frac{d^2R}{dr^2}+\Theta\frac{1}{r}\frac{dR}{dr}\right)\pm\beta^2 R\Theta=-\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} \qquad \text{(D-13)}$$

$$\left(\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{R\,r}\frac{dR}{dr}\right)\pm\beta^2=-\frac{1}{\Theta r^2}\frac{d^2\Theta}{d\theta^2} \qquad \text{(D-14)}$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)\ \pm\ \beta^2 r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \qquad \text{(D-15)}$$

The equation can be satisfied if each expression is equal to the same constant k^2.

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right) \pm \beta^2 r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k^2 \tag{D-16}$$

Thus

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right) \pm \beta^2 r^2 = k^2 \tag{D-17}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr} + \left[\pm\beta^2 - \frac{k^2}{r^2}\right]R = 0 \tag{D-18}$$

Define a new variable

$$\xi = \begin{cases} \beta r \\ j\beta r \end{cases} \tag{D-19}$$

$$\xi^2 = \begin{cases} \beta^2 r^2 \\ -\beta^2 r^2 \end{cases} \tag{D-20}$$

$$\frac{\xi^2}{r^2} = \begin{cases} \beta^2 \\ -\beta^2 \end{cases} \tag{D-21}$$

Chain rule

$$d\xi = \beta dr \tag{D-22}$$

$$\frac{d}{dr} = \frac{d}{d\xi}\frac{d\xi}{dr} \tag{D-23}$$

$$\frac{d}{dr} = \beta\frac{d}{d\xi} \tag{D-24}$$

$$\frac{\xi^2}{r^2}\frac{d^2R}{d\xi^2}+\frac{\xi}{r^2}\frac{dR}{d\xi}+\left[\frac{\xi^2}{r^2}-\frac{k^2}{r^2}\right]R = 0 \quad \text{(D-25)}$$

$$\xi^2\frac{d^2R}{d\xi^2}+\xi\frac{dR}{d\xi}+\left[\xi^2-k^2\right]R = 0 \quad \text{(D-26)}$$

$$\frac{d^2R}{d\xi^2}+\frac{1}{\xi}\frac{dR}{d\xi}+\left[1-\frac{k^2}{\xi^2}\right]R = 0 \quad \text{(D-27)}$$

Equation (D-27) is Bessel's equation of fractional order.

The solution for circular plates that are closed in the θ direction is

$$R(\xi) = C\,J_n(\xi)+D\,I_n(\xi)+F\,Y_n(\xi)+G\,K_n(\xi) \quad \text{(D-28)}$$

Equation (D-28) represents Bessel of the first and second kind and modified Bessel of the first and second kind.

Both $Y_n(\xi)$ and $K_n(\xi)$ are singular at $\xi = 0$.

Thus for a plate with no hole, F = G = 0.

$$R(\xi) = C\,J_n(\xi)+D\,I_n(\xi) \quad \text{(D-29)}$$

Furthermore, from equation (D-16),

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = k^2 \quad \text{(D-30)}$$

$$\frac{d^2\Theta}{d\theta^2}+\Theta k^2 = 0 \quad \text{(D-31)}$$

The solution for circular plates that are closed in the θ direction is

$$\Theta = \hat{A}\cos k\theta + \hat{B}\sin k\theta \quad , \quad k = n = 0, 1, 2, 3, \; ..... \tag{D-32}$$

Or equivalently

$$\Theta = A\cos\left[k(\theta-\phi)\right] \quad .. \tag{D-33}$$

The total solution is thus

$$Z(\xi,\theta) = \left\{C\, J_n(\xi) + D\, I_n(\xi)\right\}\left\{A\cos\left[k(\theta-\phi)\right]\right\} \tag{D-34}$$

Set the phase angle ϕ =0.

$$Z(\xi,\theta) = \left\{C\, J_n(\xi) + D\, I_n(\xi)\right\} A\cos(k\theta) \tag{D-35}$$

Set A =1. Note that the mass normalization will be performed using the C and D coefficients.

$$Z(\xi,\theta) = \left\{C\, J_n(\xi) + D\, I_n(\xi)\right\}\cos(k\theta) \tag{D-36}$$

APPENDIX E

Simply Supported Plate, Bessel Function Solution

The boundary conditions are

$$Z(a,\theta) = 0 \tag{E-1}$$

$$M_r = 0 \quad \text{at} \quad r = a \tag{E-2}$$

$$\frac{\partial^2 Z}{\partial \theta^2} = 0 \quad \text{at} \quad r = a \tag{E-3}$$

Note that

$$M_r = -D_e \left[\frac{\partial^2 Z}{\partial r^2} + \mu \left(\frac{1}{r} \frac{\partial Z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 Z}{\partial \theta^2} \right) \right] \tag{E-4}$$

Boundary condition (E-3) requires that

$$M_r = -D_e \left[\frac{\partial^2 Z}{\partial r^2} + \frac{\mu}{r} \frac{\partial Z}{\partial r} \right] \qquad \text{at} \quad r = a \tag{E-5}$$

$$Z(r,\theta) = R(r)\Theta(\theta) \tag{E-6}$$

$$Z(r,\theta) = \left[C\, J_n(\beta r) + D\, I_n(\beta r) \right] \cos(n\theta) \tag{E-7}$$

$$Z(a,\theta) = \left[C\, J_n(\beta a) + D\, I_n(\beta a) \right] \cos(n\theta) = 0 \tag{E-8}$$

$$C\, J_n(\beta a) + D\, I_n(\beta a) = 0 \tag{E-9}$$

$$M_r = -D_e \left[\frac{\partial^2}{\partial r^2} [C\ J_n(\beta r) + D\ I_n(\beta r)] \cos(n\theta) \right]$$

$$- \mu D_e \left[\frac{1}{r} \frac{\partial}{\partial r} [C\ J_n(\beta r) + D\ I_n(\beta r)] \cos(n\theta) \right] \quad \text{at} \quad r = a$$

(E-10)

$$M_r = -D_e\ \cos(k\theta) \left[C\ \frac{d^2}{dr^2} J_n(\beta r) + D \frac{d^2}{dr^2} I_n(\beta r) \right]$$

$$- \mu D_e\ \cos(k\theta) \left[C \frac{1}{r} \frac{d}{dr} J_n(\beta r) + D \frac{1}{r} \frac{d}{dr} I_n(\beta r) \right] \quad \text{at} \quad r = a$$

(E-11)

$$M_r \big|_{r=a} = 0 \qquad \text{(E-12)}$$

$$\left[C\ \frac{d^2}{dr^2} J_n(\beta r) + D \frac{d^2}{dr^2} I_n(\beta r) \right] + \mu \left[C \frac{1}{a} \frac{d}{dr} J_n(\beta r) + D \frac{1}{a} \frac{d}{dr} I_n(\beta r) \right] = 0, \quad \text{at } r = a$$

(E-13)

$$C \left[\frac{d^2}{dr^2} J_n(\beta r) + \frac{\mu}{a} \frac{d}{dr} J_n(\beta r) \right] + D \left[\frac{d^2}{dr^2} I_n(\beta r) + \frac{\mu}{a} \frac{d}{dr} I_n(\beta r) \right] = 0, \quad \text{at } r = a$$

(E-14)

Let

$$\lambda = \beta r \qquad \text{(E-15)}$$

$$C\left[\beta^2 \frac{d^2}{d\lambda^2} J_n(\lambda) + \frac{\mu\beta}{a}\frac{d}{d\lambda} J_n(\lambda)\right] + D\left[\beta^2 \frac{d^2}{d\lambda^2} I_n(\lambda) + \frac{\mu\beta}{a}\frac{d}{d\lambda} I_n(\lambda)\right] = 0 \ , \quad \text{at} \quad \lambda = \beta a \tag{E-36}$$

$$C\left[\beta \frac{d^2}{d\lambda^2} J_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} J_n(\lambda)\right] + D\left[\beta \frac{d^2}{d\lambda^2} I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} I_n(\lambda)\right] = 0 \ , \quad \text{at} \quad \lambda = \beta a \tag{E-37}$$

Recall equation (E-9).

$$C\, J_n(\lambda) + D\, I_n(\lambda) = 0 \,, \quad \text{at} \quad \lambda = \beta a \tag{E-38}$$

$$D\, I_n(\lambda) = -C\, J_n(\lambda) \,, \quad \text{at} \quad \lambda = \beta a \tag{E-39}$$

$$D = -C \frac{J_n(\lambda)}{I_n(\lambda)} \ , \quad \text{at} \quad \lambda = \beta a \tag{E-40}$$

By substitution,

$$C\left[\beta \frac{d^2}{d\lambda^2} J_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} J_n(\lambda)\right] - C\frac{J_n(\lambda)}{I_n(\lambda)}\left[\beta \frac{d^2}{d\lambda^2} I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} I_n(\lambda)\right] = 0 \,, \quad \text{at} \quad \lambda = \beta a \tag{E-41}$$

$$\left[\beta \frac{d^2}{d\lambda^2} J_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} J_n(\lambda)\right] - \frac{J_n(\lambda)}{I_n(\lambda)}\left[\beta \frac{d^2}{d\lambda^2} I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} I_n(\lambda)\right] = 0 \,, \quad \text{at} \quad \lambda = \beta a \tag{E-42}$$

$$\frac{1}{J_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right] - \frac{1}{I_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda}I_n(\lambda)\right] = 0 \ ,$$

at $\lambda = \beta a$

(E-43)

Note the following identities:

$$\frac{d}{d\lambda}J_n(\lambda) = J_{n-1}(\lambda) - \frac{n}{\lambda}J_n(\lambda) = -J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda) \qquad \text{(E-44)}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = -\frac{d}{d\lambda}J_{n+1}(\lambda) + \frac{n}{\lambda}\frac{d}{d\lambda}J_n(\lambda) \qquad \text{(E-45)}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-J_n(\lambda) + \frac{n+1}{\lambda}J_{n+1}(\lambda)\right] + \frac{n}{\lambda}\left[-J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda)\right] \qquad \text{(E-46)}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-J_n(\lambda) + \frac{n+1}{\lambda}J_{n+1}(\lambda)\right] + \frac{n}{\lambda}\left[-J_{n+1}(\lambda) + \frac{n}{\lambda}J_n(\lambda)\right] \qquad \text{(E-47)}$$

$$\frac{d^2}{d\lambda^2}J_n(\lambda) = \left[-1 + \frac{n^2}{\lambda^2}\right]J_n(\lambda) + \left[\frac{1}{\lambda}\right]J_{n+1}(\lambda) \qquad \text{(E-48)}$$

Analyze the first term of equation (E-42).

$$+\frac{1}{J_n(\lambda)}\left[\beta\frac{d^2}{d\lambda^2}J_n(\lambda)+\frac{\mu}{a}\frac{d}{d\lambda}J_n(\lambda)\right]$$

$$=\frac{1}{J_n(\lambda)}\left[\beta\left[-1+\frac{n^2}{\lambda^2}\right]J_n(\lambda)+\beta\left[\frac{1}{\lambda}\right]J_{n+1}(\lambda)\right]+\frac{\mu}{a}\frac{1}{J_n(\lambda)}\left[-J_{n+1}(\lambda)+\frac{n}{\lambda}J_n(\lambda)\right]$$

$$=\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\beta}{\lambda}\right]-\frac{\mu}{a}\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\beta\left[-1+\frac{n^2}{\lambda^2}\right]+\frac{\mu n}{a\lambda}$$

$$=-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\beta\left[-1+\frac{n^2}{\lambda^2}\right]+\frac{\mu n}{a\lambda}$$

(E-49)

Consider the following identities:

$$\frac{d}{d\lambda}I_n(\lambda)=I_{n-1}(\lambda)-\frac{n}{\lambda}I_n(\lambda)=I_{n+1}(\lambda)+\frac{n}{\lambda}I_n(\lambda) \qquad \text{(E-50)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\frac{d}{d\lambda}I_{n+1}(\lambda)+\frac{n}{\lambda}\frac{d}{d\lambda}I_n(\lambda) \qquad \text{(E-51)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\left[I_n(\lambda)-\frac{n+1}{\lambda}I_{n+1}(\lambda)\right]+\frac{n}{\lambda}\left[I_{n+1}(\lambda)+\frac{n}{\lambda}I_n(\lambda)\right] \qquad \text{(E-52)}$$

$$\frac{d^2}{d\lambda^2}I_n(\lambda)=\left[1+\frac{n^2}{\lambda^2}\right]I_n(\lambda)-\left[\frac{1}{\lambda}\right]I_{n+1}(\lambda) \qquad \text{(E-53)}$$

Analyze the second term of equation (E-42).

$$- \frac{1}{I_n(\lambda)}\left[\beta \frac{d^2}{d\lambda^2} I_n(\lambda) + \frac{\mu}{a}\frac{d}{d\lambda} I_n(\lambda) \right]$$

$$= - \frac{1}{I_n(\lambda)}\left\{\beta\left[1+\frac{n^2}{\lambda^2}\right] I_n(\lambda) - \beta\left[\frac{1}{\lambda}\right] I_{n+1}(\lambda)\right\} - \frac{\mu}{a}\frac{1}{I_n(\lambda)}\left[I_{n+1}(\lambda) + \frac{n}{\lambda} I_n(\lambda)\right]$$

$$= \frac{\beta}{\lambda}\frac{I_{n+1}(\lambda)}{I_n(\lambda)} - \frac{\mu}{a}\frac{I_{n+1}(\lambda)}{I_n(\lambda)} - \beta\left[1+\frac{n^2}{\lambda^2}\right] - \frac{\mu n}{a\lambda}$$

$$= -\left[\frac{\mu}{a} - \frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)} - \beta\left[1+\frac{n^2}{\lambda^2}\right] - \frac{\mu n}{a\lambda}$$

(E-54)

By substitution,

$$+\frac{1}{J_n(\lambda)}\left[\frac{d^2}{d\lambda^2} J_n(\lambda)\right] + \frac{1}{J_n(\lambda)}\left[\frac{\mu}{a}\frac{d}{d\lambda} J_n(\lambda)\right]$$
$$- \frac{1}{I_n(\lambda)}\left[\frac{d^2}{d\lambda^2} I_n(\lambda)\right] - \frac{1}{I_n(\lambda)}\left[\frac{\mu}{a}\frac{d}{d\lambda} I_n(\lambda)\right] = 0 \quad ,$$

$$\text{at} \quad \lambda = \beta a$$

(E-55)

$$-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a} - \frac{\beta}{\lambda}\right] + \beta\left[-1+\frac{n^2}{\lambda^2}\right] + \frac{\mu n}{a\lambda} - \left[\frac{\mu}{a} - \frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)} - \beta\left[1+\frac{n^2}{\lambda^2}\right] - \frac{\mu n}{a\lambda} = 0 \, ,$$

$$\text{at} \quad \lambda = \beta a$$

(E-56)

$$-\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]-\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}-2\beta=0 \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-57)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}+2\beta=0 \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-58)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]+\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=-2\beta \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-59)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta}{\left[\frac{\mu}{a}-\frac{\beta}{\lambda}\right]} \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-60)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta}{\left[\frac{\mu}{a}-\frac{1}{a}\right]} \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-61)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\beta a}{-1+\mu} \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-62)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{-2\lambda}{-1+\mu} \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-63)}$$

$$\frac{J_{n+1}(\lambda)}{J_n(\lambda)}+\frac{I_{n+1}(\lambda)}{I_n(\lambda)}=\frac{2\lambda}{1-\mu} \quad , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-64)}$$

The following form is better suited for numerical root-finding purposes.

$$I_n(\lambda)\,J_{n+1}(\lambda)+J_n(\lambda)I_{n+1}(\lambda)=\frac{2\lambda}{1-\mu}\left[J_n(\lambda)\,I_n(\lambda)\right] , \quad \text{at} \quad \lambda=\beta a \qquad \text{(E-65)}$$

The roots of equation (E-65) for $\mu = 0.3$ are expressed in terms of λ^2 as

k	n=0	n=1	n=2	n=3
0	4.9351	13.8982	25.6133	39.9573
1	29.7658	48.5299	70.1170	94.5490
2	74.2302	102.7965	134.2978	168.6749
3	138.3181	176.8012	218.2026	262.6244

The roots were determined using the secant method.

The fundamental natural frequency is thus

$$\lambda = \beta a \tag{E-66}$$

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \tag{E-67}$$

$$\beta = \left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{E-68}$$

$$\lambda = a\left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{E-69}$$

$$\omega^2 = \frac{\lambda^4 D_e}{\rho h\, a^4} \tag{E-70}$$

$$\omega = \frac{\lambda^2}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{E-71}$$

$$\omega = \frac{4.9351}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{E-72}$$

The mode shapes are defined by

$$Z(\beta r, \theta) = \left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta) \qquad \text{(E-72)}$$

Recall

$$D = -C\frac{J_n(\lambda)}{I_n(\lambda)} \quad , \qquad \text{at} \quad \lambda = \beta a \qquad \text{(E-73)}$$

$$Z(\beta r, \theta) = \left\{C\ J_n(\beta r) - C\frac{J_n(\lambda)}{I_n(\lambda)}\ I_n(\beta r)\right\}\cos(n\theta) \qquad \text{(E-74)}$$

$$Z(\beta r, \theta) = C\left\{\ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)}\right]\ I_n(\beta r)\right\}\cos(n\theta) \qquad \text{(E-75)}$$

Again,

$$\beta = \lambda / a \qquad \text{(E-76)}$$

APPENDIX F

Note that the numerical calculations in this appendix are performed via Matlab script: circular_SS.m.

Mass Normalization of Eigenvectors

The eigenvector equation is

$$Z_{kn}(\beta r, \theta) = C_{kn} \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\} \cos(n\theta) \tag{F-1}$$

Again,

$$\beta = \lambda / a \tag{F-2}$$

Normalize the eigenvectors as

$$\rho h \int_0^a \int_0^{2\pi} \left[Z_{kn}(\beta r, \theta) \right]^2 \, r \, d\theta \, dr = 1 \tag{F-3}$$

$$\rho h \int_0^a \int_0^{2\pi} C_{kn}^2 \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 \cos^2(n\theta) \, r \, d\theta \, dr = 1 \tag{F-4}$$

$$C_{kn}^2 \, \rho h \int_0^a \int_0^{2\pi} \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 \left\{ \frac{1}{2} + \frac{1}{2} \cos(n\theta) \right\} r \, d\theta \, dr = 1 \tag{F-5}$$

$$C_{kn}^2 \, \rho h \int_0^a \int_0^{2\pi} \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 r \, d\theta \, dr = 1 \qquad \text{for } n=0 \tag{F-6}$$

$$C_{kn}^2 \left\{ \rho h \int_0^a \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 r dr \right\} \left\{ \left[\frac{\theta}{2} + \frac{1}{2n} \sin(n\theta) \right]_0^{2\pi} \right\} = 1 \quad \text{for } n \geq 1 \tag{F-7}$$

$$\pi C_{kn}^2 \left\{ \rho h \int_0^a \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 r dr \right\} = 1 \quad \text{for } n \geq 1 \tag{F-8}$$

For k=0, n=0,

$$C_{kn}^2 \; \rho h \int_0^a \int_0^{2\pi} \left\{ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\}^2 r \, d\theta \, dr = 1 \tag{F-9}$$

$$2\pi C_{00}^2 \; \rho h \int_0^a \int_0^{2\pi} \left\{ J_0(\beta r) - \left[\frac{J_0(\lambda)}{I_0(\lambda)} \right] I_0(\beta r) \right\}^2 r \, d\theta \, dr = 1 \tag{F-10}$$

$$2\pi \; C_{00}^2 \; \rho h \int_0^a \left\{ J_0(\beta r) - \left[\frac{J_0(\lambda)}{I_0(\lambda)} \right] I_0(\beta r) \right\}^2 r \, dr = 1 \tag{F-11}$$

$$2\pi \; C_{00}^2 \; \rho h \int_0^a \left\{ J_0(\beta r) - \left[\frac{J_0(\lambda)}{I_0(\lambda)} \right] I_0(\beta r) \right\}^2 r \, dr = 1 \tag{F-12}$$

The eigenvalue is

$$\lambda = \sqrt{4.9351} = 2.2215 \quad \text{for k=0, n=0} \tag{F-13}$$

$$\left[\frac{J_0(\lambda)}{I_0(\lambda)} \right] = 0.03686 \quad \text{for} \quad \lambda = 2.2215 \tag{F-14}$$

By substitution

$$2\pi\ C_{00}^{2}\ \rho h\ \int_0^a \{\ J_0(\beta r) - 0.03686\ I_0(\beta r)\}^2\ r\,dr = 1 \qquad \text{(F-15)}$$

Recall

$$\beta = \lambda / a \qquad \text{(F-16)}$$

By substitution,

$$2\pi\ C_{00}^{2}\ \rho h\ \int_0^a \{\ J_0(\lambda\, r/a) - 0.03686\ I_0(\lambda\, r/a)\}^2\ r\,dr = 1 \qquad \text{(F-17)}$$

$$2\pi\ C_{00}^{2}\ \rho h\ \int_0^a \{\ J_0(2.2215\ r/a) - 0.03686\ I_0(2.2215\ r/a)\}^2\ r\,dr = 1 \qquad \text{(F-18)}$$

Let

$$u = 2.2215\ r/a \qquad \text{(F-19)}$$

$$r = a\,u / 2.2215 \qquad \text{(F-20)}$$

$$dr = a\,du / 2.2215 \qquad \text{(F-21)}$$

$$r\,dr = a^2\,u\,du / 4.9351 \qquad \text{(F-22)}$$

By substitution,

$$2\pi\ C_{00}^{2}\ \frac{\rho h\, a^2}{4.9351}\ \int_0^{2.2215} \{\ J_0(u) - 0.03686\ I_0(u)\}^2\ u\,du = 1 \qquad \text{(F-23)}$$

The mass is

$$m = \pi \rho h\, a^2 \qquad \text{(F-24)}$$

$$C_{00}^{2}\ \frac{2m}{4.9351}\ (0.6525) = 1 \qquad \text{(F-25)}$$

$$C_{00}{}^2 \; m(0.2644) = 1 \tag{F-26}$$

$$C_{00} = \frac{1.945}{\sqrt{m}} \tag{F-27}$$

The eigenvector for k=0, n=0 is

$$Z_{00} = \frac{1.945}{\sqrt{m}} \{ \, J_0(2.2215 \; r/a) - 0.03686 \; I_0(2.2215 \; r/a) \} \tag{F-28}$$

<u>Participation Factors</u>

The participation factor Γ_{kn} is

$$\Gamma_{kn} = \rho h \int_0^a \int_0^{2\pi} C_{kn} \left\{ \; J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)} \right] I_n(\beta r) \right\} \cos(n\theta) \; r \, d\theta \, dr \tag{F-29}$$

Note that

$$\Gamma_{kn} = 0 \qquad \text{for } n \geq 1 \tag{F-30}$$

The participation factor for k=0, n=0, is

$$\Gamma_{kn} = \rho h \int_0^a \int_0^{2\pi} \{ Z_{kn}(r, \theta) \} \, r \, d\theta \, dr \tag{F-31}$$

$$\Gamma_{00} = \rho h \int_0^a \int_0^{2\pi} \left\{ \frac{1.945}{\sqrt{m}} \{ \, J_0(2.2215 \; r/a) - 0.03686 \; I_0(2.2215 \; r/a) \} \right\} r \, d\theta \, dr \tag{F-32}$$

$$\Gamma_{00} = \rho h \frac{1.945}{\sqrt{m}} \int_0^a \int_0^{2\pi} \{ \, J_0(2.2215 \; r/a) - 0.03686 \; I_0(2.2215 \; r/a) \} \, r \, d\theta \, dr \tag{F-33}$$

$$\Gamma_{00} = \frac{1.945\, m}{\pi a^2 \sqrt{m}} \int_0^a \int_0^{2\pi} \left\{ J_0(2.2215\ r/a) - 0.03686\ I_0(2.2215\ r/a) \right\} r\, d\theta\, dr \qquad \text{(F-34)}$$

$$\Gamma_{00} = \frac{(2\pi) 1.945 \sqrt{m}}{\pi a^2} \int_0^a \left\{ J_0(2.2215\ r/a) - 0.03686\ I_0(2.2215\ r/a) \right\} r\, dr \qquad \text{(F-35)}$$

$$\Gamma_{00} = \frac{3.890 \sqrt{m}}{a^2} \int_0^a \left\{ J_0(2.2215\ r/a) - 0.03686\ I_0(2.2215\ r/a) \right\} r\, dr \qquad \text{(F-36)}$$

Recall

$$u = 2.2215\ r/a \qquad \text{(F-37)}$$

$$r\, dr = a^2\, u\, du / 4.9351 \qquad \text{(F-38)}$$

By substitution,

$$\Gamma_{00} = \frac{3.890 \sqrt{m}}{a^2} \int_0^{2.2215} \left\{ J_0(u) - 0.03686\ I_0(u) \right\} \frac{a^2\, u}{4.9351}\, du \qquad \text{(F-39)}$$

$$\Gamma_{00} = 3.890 \sqrt{m}\ \int_0^{2.2215} \left\{ J_0(u) - 0.03686\ I_0(u) \right\} \frac{u}{4.9351}\, du \qquad \text{(F-40)}$$

$$\Gamma_{00} = 0.7881 \sqrt{m}\ \int_0^{2.2215} \left\{ J_0(u) - 0.03686\ I_0(u) \right\} u\, du \qquad \text{(F-41)}$$

$$\Gamma_{00} = 0.7881 \sqrt{m}\ (1.0682) \qquad \text{(F-42)}$$

$$\Gamma_{00} = 0.8419 \sqrt{m} \qquad \text{(F-43)}$$

The effective modal mass is

$$M_{00} = \left[0.8419 \sqrt{m} \right]^2 \tag{F-44}$$

$$M_{00} = 0.7088\, m \tag{F-45}$$

Example

Assume a 64 inch diameter, 1 inch thick aluminum plate, with a simply-supported perimeter. Calculate the fundamental frequency and mode shape.

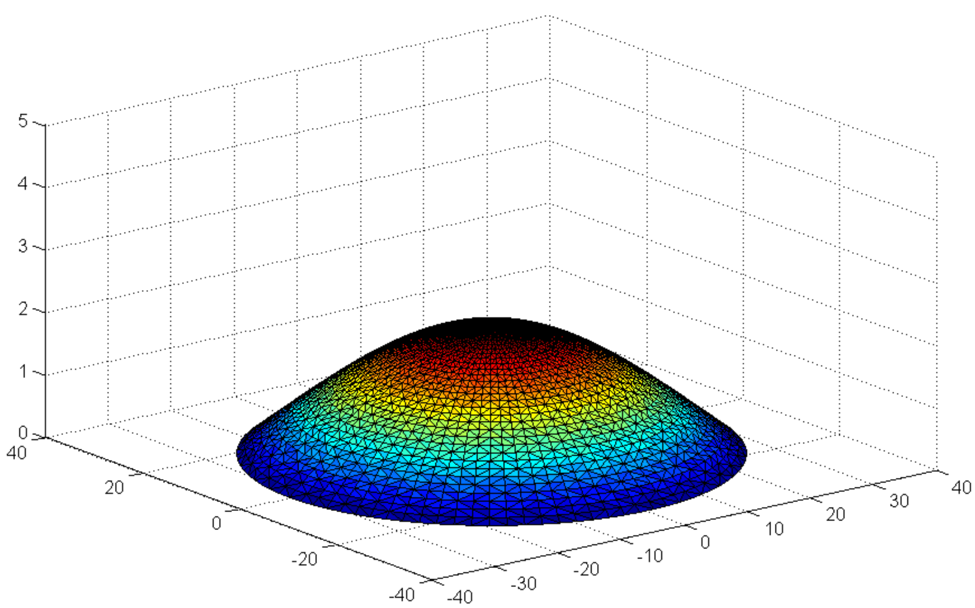

Figure F-1. Fundamental Mode, fn = 45.6 Hz

APPENDIX G

This appendix is unfinished.

Completely Free Circular Plate

Consider a completely free circular plate. The plate has a radius a. The displacement perpendicular to the plate is Z. A polar coordinate system is used with the origin at the plate's center.

The mode shape is

$$Z(r,\theta) = \left[C\ J_n(\beta r) + D\ I_n(\beta\, r)\right]\cos(n\theta) \tag{G-1}$$

The boundary conditions are

$$M_r = -D_e\left[\frac{\partial^2 Z}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial Z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)\right] = 0 \qquad \text{at} \quad r = a \tag{G-2}$$

$$V_r = -D_e\left[\frac{\partial}{\partial r}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)Z\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[(1-\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \theta}\right)\right]\right] = 0 \qquad \text{at} \quad r = a$$

(G-3)

The moment is

$$M_r = -D_e\left[\frac{\partial^2}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\right]\left[\left[C\ J_n(\beta r) + D\ I_n(\beta\, r)\right]\cos(n\theta)\right] \tag{G-4}$$

$$M_r = -D_e \frac{d^2}{dr^2}\left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta)$$

$$-\mu D_e \frac{1}{r}\frac{d}{dr}\left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta)$$

$$+n^2 \mu D_e \frac{1}{r^2}\left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta)$$

(G-5)

$$M_r = -D_e\left\{\left[\frac{d^2}{dr^2} + \mu\frac{1}{r}\frac{d}{dr} - \mu\frac{n^2}{r^2}\right]J_n(\beta r)\right\}C\ \cos(n\theta)$$

$$-D_e\left\{\left[\frac{d^2}{dr^2} + \mu\frac{1}{r}\frac{d}{dr} - n^2\mu\frac{n^2}{r^2}\right]I_n(\beta r)\right\}D\ \cos(n\theta)$$

(G-6)

Let

$$\lambda = \beta r \tag{G-7}$$

$$r = \lambda / \beta \tag{G-8}$$

$$d\lambda = \beta dr \tag{G-9}$$

$$\frac{d}{dr} = \frac{d}{d\lambda}\frac{d\lambda}{dr} \tag{G-10}$$

$$\frac{d}{dr} = \beta\frac{d}{d\lambda} \tag{G-11}$$

$$\frac{d^2}{dr^2} = \beta^2\frac{d^2}{d\lambda^2} \tag{G-12}$$

$$M_r = -D_e \beta^2 \left\{ \left[\frac{d^2}{d\lambda^2} + \mu \frac{1}{\lambda} \frac{d}{d\lambda} - \mu \frac{n^2}{\lambda^2} \right] J_n(\lambda) \right\} C \cos(n\theta)$$

$$- D_e \beta^2 \left\{ \left[\frac{d^2}{d\lambda^2} + \mu \frac{1}{\lambda} \frac{d}{d\lambda} - n^2 \mu \frac{n^2}{\lambda^2} \right] I_n(\lambda) \right\} D \cos(n\theta)$$

(G-13)

Now consider the shear boundary condition.

$$V_r = -D_e \left[\frac{\partial}{\partial r} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) Z \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[(1-\mu) \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial Z}{\partial \theta} \right) \right] \right] = 0 \qquad \text{(G-14)}$$

$$V_r = -D_e \left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{2}{r^3} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^3}{\partial r \partial \theta^2} \right] Z$$

$$- D_e \frac{1}{r} (1-\mu) \left[\frac{1}{-r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial^3}{\partial r \partial \theta^2} \right] Z$$

(G-15)

$$V_r = -D_e \left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{2}{r^3} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^3}{\partial r \partial \theta^2} \right] [C\, J_n(\beta r) + D\, I_n(\beta r)] \cos(n\theta)$$

$$- D_e \frac{1}{r} (1-\mu) \left[\frac{1}{-r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial^3}{\partial r \partial \theta^2} \right] [C\, J_n(\beta r) + D\, I_n(\beta r)] \cos(n\theta)$$

(G-16)

$$
\begin{aligned}
V_r = -D_e &\left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r \partial \theta^2}\right][C\, J_n(\beta r)]\cos(n\theta) \\
&- D_e\left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r \partial \theta^2}\right][D\, I_n(\beta r)]\cos(n\theta) \\
&- D_e \frac{1}{r}(1-\mu)\left[\frac{1}{-r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial^3}{\partial r \partial \theta^2}\right][C\, J_n(\beta r)]\cos(n\theta) \\
&- D_e \frac{1}{r}(1-\mu)\left[\frac{1}{-r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial^3}{\partial r \partial \theta^2}\right][D\, I_n(\beta r)]\cos(n\theta)
\end{aligned}
\tag{G-17}
$$

$$
\begin{aligned}
V_r = -D_e &\left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r \partial \theta^2}\right][C\, J_n(\beta r)]\cos(n\theta) \\
&- D_e\left[\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r \partial \theta^2}\right][D\, I_n(\beta r)]\cos(n\theta) \\
&- D_e \frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^3}{\partial r \partial \theta^2}\right][C\, J_n(\beta r)]\cos(n\theta) \\
&- D_e \frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^3}{\partial r \partial \theta^2}\right][D\, I_n(\beta r)]\cos(n\theta)
\end{aligned}
\tag{G-18}
$$

$V_r =$

$$-D_e\left[\frac{\partial^3}{\partial r^3}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^3}{\partial r\partial \theta^2}+\frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^3}{\partial r\partial \theta^2}\right]\right]C J_n(\beta r)\cos(n\theta)$$

$$-D_e\left[\frac{\partial^3}{\partial r^3}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^3}{\partial r\partial \theta^2}+\frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^3}{\partial r\partial \theta^2}\right]\right]D I_n(\beta r)\cos(n\theta)$$

(G-19)

$V_r =$

$$-D_e\left[\frac{\partial^3}{\partial r^3}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^3}{\partial r\partial \theta^2}+\frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^3}{\partial r\partial \theta^2}\right]\right]C J_n(\beta r)\cos(n\theta)$$

$$-D_e\left[\frac{\partial^3}{\partial r^3}-\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2}{\partial \theta^2}+\frac{1}{r^2}\frac{\partial^3}{\partial r\partial \theta^2}+\frac{1}{r^2}(1-\mu)\left[\frac{1}{-r}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^3}{\partial r\partial \theta^2}\right]\right]D I_n(\beta r)\cos(n\theta)$$

(G-20)

Let

$$\lambda = \beta r \qquad \text{(G-21)}$$

$$V_r =$$

$$-D_e\beta^3\left[\frac{\partial^3}{\partial\lambda^3}-\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}+\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}-\frac{2}{\lambda^3}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\lambda^2}\frac{\partial^3}{\partial\lambda\partial\theta^2}+\frac{1}{\lambda^2}(1-\mu)\left[\frac{1}{-\lambda}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^3}{\partial\lambda\partial\theta^2}\right]\right]C\,J_n(\lambda)\cos(n\theta)$$

$$-D_e\beta^3\left[\frac{\partial^3}{\partial\lambda^3}-\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}+\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}-\frac{2}{\lambda^3}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\lambda^2}\frac{\partial^3}{\partial\lambda\partial\theta^2}+\frac{1}{\lambda^2}(1-\mu)\left[\frac{1}{-\lambda}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^3}{\partial\lambda\partial\theta^2}\right]\right]D\,I_n(\lambda)\cos(n\theta)$$

(G-22)

$$V_r =$$

$$-D_e\beta^3\left[\frac{\partial^3}{\partial\lambda^3}-\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}+\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}+\frac{2n^2}{\lambda^3}-\frac{n^2}{\lambda^2}\frac{\partial}{\partial\lambda}-\frac{n^2}{\lambda^2}(1-\mu)\left[\frac{1}{-\lambda}+\frac{\partial}{\partial\lambda}\right]\right]C\,J_n(\lambda)\cos(n\theta)$$

$$-D_e\beta^3\left[\frac{\partial^3}{\partial\lambda^3}-\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}+\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}+\frac{2n^2}{\lambda^3}-\frac{n^2}{\lambda^2}\frac{\partial}{\partial\lambda}-\frac{n^2}{\lambda^2}(1-\mu)\left[\frac{1}{-\lambda}+\frac{\partial}{\partial\lambda}\right]\right]D\,I_n(\lambda)\cos(n\theta)$$

(G-23)

APPENDIX H

Fixed Plate, Bessel Function Solution

The boundary conditions are

$$Z(a,\theta) = 0 \qquad \text{(H-1)}$$

$$\frac{\partial Z}{\partial r} = 0 \quad \text{at} \quad r = a \qquad \text{(H-2)}$$

The displacement is

$$Z(r,\theta) = \left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta) \qquad \text{(H-3)}$$

The slope is

$$\frac{\partial Z}{\partial r} = \frac{\partial}{\partial r}\left\{\left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta)\right\} \qquad \text{(H-4)}$$

$$\frac{dZ}{dr} = \left[C\ \frac{d}{dr} J_n(\beta r) + D\ \frac{d}{dr} I_n(\beta r)\right]\cos(n\theta) \qquad \text{(H-5)}$$

Let

$$\lambda = \beta r \qquad \text{(H-6)}$$

$$\frac{d}{dr} = \beta \frac{d}{d\lambda} \qquad \text{(H-7)}$$

The displacement is

$$Z(r,\theta) = \left[C\ J_n(\lambda) + D\ I_n(\lambda)\right]\cos(n\theta) \qquad \text{(H-8)}$$

The slope is

$$\frac{dZ}{dr} = \beta\left[C\ \frac{d}{d\lambda} J_n(\lambda) + D\ \frac{d}{d\lambda} I_n(\lambda)\right]\cos(n\theta) \qquad \text{(H-9)}$$

Apply the boundary conditions

$$[C\, J_n(\lambda) + D\, I_n(\lambda)]\cos(n\theta) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-10)}$$

$$C\, J_n(\lambda) + D\, I_n(\lambda) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-11)}$$

$$\beta\left[C\, \frac{d}{d\lambda} J_n(\lambda) + D\, \frac{d}{d\lambda} I_n(\lambda)\right]\cos(n\theta) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-12)}$$

$$C\, \frac{d}{d\lambda} J_n(\lambda) + D\, \frac{d}{d\lambda} I_n(\lambda) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-13)}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} J_n(\lambda) & I_n(\lambda) \\ \frac{d}{d\lambda} J_n(\lambda) & \frac{d}{d\lambda} I_n(\lambda) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{at } \lambda = \beta a \qquad \text{(H-14)}$$

The roots are found by setting the determinant of the coefficient matrix equal to zero.

$$J_n(\lambda) \frac{d}{d\lambda} I_n(\lambda) - I_n(\lambda) \frac{d}{d\lambda} J_n(\lambda) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-15)}$$

$$J_n(\lambda)\left[I_{n+1}(\lambda) + \frac{n}{\lambda} I_n(\lambda)\right] - I_n(\lambda)\left[-J_{n+1}(\lambda) + \frac{n}{\lambda} J_n(\lambda)\right] = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-16)}$$

$$J_n(\lambda) I_{n+1}(\lambda) + J_{n+1}(\lambda) I_n(\lambda) = 0 \qquad \text{at } \lambda = \beta a \qquad \text{(H-17)}$$

The roots of equation (H-17) for $\mu = 0.3$ are expressed in terms of λ^2 as

k	n=0	n=1	n=2	n=3
0	10.2158	21.2609	34.8770	51.0334
1	39.7711	60.8287	84.5837	111.0214
2	89.1041	120.0792	153.8151	190.3038
3	158.1842	199.0534	242.7285	289.1799

The roots were determined using the secant method.

The fundamental natural frequency is thus

$$\lambda = \beta a \tag{H-18}$$

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \tag{H-19}$$

$$\beta = \left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{H-20}$$

$$\lambda = a\left[\frac{\omega^2 \rho h}{D_e}\right]^{1/4} \tag{H-21}$$

$$\omega^2 = \frac{\lambda^4 D_e}{\rho h\, a^4} \tag{H-22}$$

$$\omega = \frac{\lambda^2}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{H-23}$$

$$\omega = \frac{10.2158}{a^2}\sqrt{\frac{D_e}{\rho h}} \tag{H-24}$$

The mode shapes are defined by

$$Z(\beta r,\theta) = \left[C\ J_n(\beta r) + D\ I_n(\beta r)\right]\cos(n\theta) \tag{H-25}$$

Recall

$$D = -C\frac{J_n(\lambda)}{I_n(\lambda)} \quad , \qquad \text{at} \quad \lambda = \beta a \tag{H-26}$$

$$Z(\beta r,\theta) = \left\{C\ J_n(\beta r) - C\frac{J_n(\lambda)}{I_n(\lambda)}\ I_n(\beta r)\right\}\cos(n\theta) \tag{H-27}$$

$$Z(\beta r,\theta) = C\left\{\ J_n(\beta r) - \left[\frac{J_n(\lambda)}{I_n(\lambda)}\right]\ I_n(\beta r)\right\}\cos(n\theta) \tag{H-28}$$

Let

$$\lambda = \beta r \tag{H-29}$$

$$Z(\lambda,\theta) = C\left\{\ J_n(\lambda) - \left[\frac{J_n(\lambda)}{I_n(\lambda)}\right]\ I_n(\beta r)\right\}\cos(n\theta) \tag{H-30}$$