Derivation of the Cayley-Hamilton equation using the Lagrange multiplier Method

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Position Vectors

γ_{ij}	strain components on a plane arbitrarily inclined from the principal planes
γ_i	principle strains
$\boldsymbol{\gamma}$	Strain tensor
n_j	direction cosines that identify the inclined plane
λ	Lagrange multiplier
F	Deformation tensor
C	Green deformation tensor or metric tensor

The relationship between the principal strain components on the inclined plain is

$$\gamma_i = \gamma_{ij} n_j \tag{1}$$

Multiply by n_i

$$\gamma(n) = \gamma_i n_i = n_i n_j \gamma_{ij} \tag{2}$$

Let $\gamma = \lambda$ a scalar for one of the principal strain components

$$\gamma_i = \lambda n_i = \lambda \delta_{ij} n_j \tag{3}$$

Substitute equation (3) into (1).

$$\gamma_{ij} n_j = \lambda \delta_{ij} n_j \tag{4}$$

Multiply equation (4) by n_i

$$\gamma_{ij}\, n_i n_j = \lambda \delta_{ij}\, n_i n_j = \lambda\, n_i n_i \tag{5}$$

Direction cosine property

$$\mathbf{n} \bullet \mathbf{n} = n_i\, n_i = 1 \tag{6}$$

Now construct a scalar function *f*.

$$f = \gamma_{ij}\, n_i n_j - \lambda \left(n_i n_i - 1 \right) \tag{7}$$

f is the work required for satisfying equation (2).

The goal is to find the direction of unit vector **n** for which $\gamma(n)$ is maximum.

Take the derivative of equation (7).

$$\frac{\partial}{\partial n_k}\left(\gamma_{ij}\, n_i n_j\right) = \gamma_{ij} \frac{\partial}{\partial n_k}\left(n_i n_j\right) = \gamma_{ij}\left[n_j \frac{\partial n_i}{\partial n_k} + n_i \frac{\partial n_j}{\partial n_k} \right] = \gamma_{ij}\left[n_j \delta_{ik} + n_i\, \delta_{jk} \right]$$

$$= \gamma_{kj} n_j + \gamma_{ik} n_i = 2\gamma_{kj} n_j \quad \text{(by symmetry)} \tag{8}$$

Also,

$$\frac{\partial}{\partial n_j}\left(\lambda n_i n_i\right) = \lambda \frac{\partial}{\partial n_j}\left(n_i n_i\right) = \lambda\left[n_i \frac{\partial n_i}{\partial n_j} + n_i \frac{\partial n_i}{\partial n_j} \right] = \lambda\left[n_i \delta_{ij} + n_i\, \delta_{ij} \right] = 2\lambda \delta_{ij} n_j \tag{9}$$

Thus

$$\frac{\partial}{\partial n_i} f = 2\gamma_{ij} n_j - 2\lambda \delta_{ij} n_j = 0 \tag{10}$$

$$\left(\gamma_{ij} - \lambda \delta_{ij}\right) n_j = 0 \tag{11}$$

Since n_j is arbitrary,

$$\det\left(\gamma_{ij} - \lambda\delta_{ij}\right) = 0 \tag{12}$$

The Cayley-Hamilton equation is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{13}$$

The standard eigenvalue problem is

$$\left(\gamma_{ij} - \lambda\delta_{ij}\right)n_j = 0 \tag{14}$$

$$\det\left(\gamma_{ij} - \lambda\delta_{ij}\right) = 0 \tag{15}$$

$$\begin{vmatrix} \gamma_{11} - \lambda & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} - \lambda & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} - \lambda \end{vmatrix} = 0 \tag{16}$$

$$\begin{aligned} &+\left(\gamma_{11} - \lambda\right)\left(\gamma_{22} - \lambda\right)\left(\gamma_{33} - \lambda\right) - \left(\gamma_{11} - \lambda\right)\gamma_{32}\gamma_{23} \\ &+\gamma_{12}\gamma_{23}\gamma_{31} - \gamma_{12}\gamma_{21}\left(\gamma_{33} - \lambda\right) \\ &+\gamma_{13}\gamma_{21}\gamma_{32} - \gamma_{31}\gamma_{13}\left(\gamma_{22} - \lambda\right) = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} &+\left(\gamma_{11} - \lambda\right)\left(\gamma_{22}\gamma_{33} - \lambda\left(\gamma_{22} + \gamma_{33}\right) + \lambda^2\right) \\ &-\gamma_{11}\gamma_{32}\gamma_{23} + \lambda\gamma_{32}\gamma_{23} \\ &+\gamma_{12}\gamma_{23}\gamma_{31} - \gamma_{12}\gamma_{21}\gamma_{33} + \lambda\gamma_{12}\gamma_{21} \\ &+\gamma_{13}\gamma_{21}\gamma_{32} - \gamma_{31}\gamma_{13}\gamma_{22} + \lambda\gamma_{31}\gamma_{13} = 0 \end{aligned} \tag{18}$$

$$
\begin{aligned}
&-\lambda\gamma_{22}\gamma_{33} + \lambda^2(\gamma_{22} + \gamma_{33}) - \lambda^3 \\
&\gamma_{11}\gamma_{22}\gamma_{33} - \lambda(\gamma_{11}\gamma_{22} + \gamma_{11}\gamma_{33}) + \gamma_{11}\lambda^2 \\
&- \gamma_{11}\gamma_{32}\gamma_{23} + \lambda\gamma_{32}\gamma_{23} \\
&+ \gamma_{12}\gamma_{23}\gamma_{31} - \gamma_{12}\gamma_{21}\gamma_{33} + \lambda\gamma_{12}\gamma_{21} \\
&+ \gamma_{13}\gamma_{21}\gamma_{32} - \gamma_{31}\gamma_{13}\gamma_{22} + \lambda\gamma_{31}\gamma_{13} = 0
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&-\lambda^3 + (\gamma_{11} + \gamma_{22} + \gamma_{33})\lambda^2 - (\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} + \gamma_{11}\gamma_{33} - \gamma_{31}\gamma_{13} + \gamma_{22}\gamma_{33} - \gamma_{32}\gamma_{23})\lambda \\
&+ \gamma_{11}\gamma_{22}\gamma_{33} - \gamma_{11}\gamma_{32}\gamma_{23} + \gamma_{12}\gamma_{23}\gamma_{31} - \gamma_{12}\gamma_{21}\gamma_{33} + \gamma_{13}\gamma_{21}\gamma_{32} - \gamma_{31}\gamma_{13}\gamma_{22} = 0
\end{aligned}
\tag{20}
$$

$$I_1 = \gamma_{11} + \gamma_{22} + \gamma_{33} = \gamma_{ii} \tag{21}$$

$$I_2 = \frac{1}{2}\left(\gamma_{ii}\gamma_{jj} - \gamma_{ij}\gamma_{ij}\right) \tag{22}$$

$$I_3 = \left|\gamma_{ij}\right| \tag{23}$$

The eigenvalues λ are the principle strain invariants.

The eigenvectors represent the a_{ij} coefficients which are the principal direction cosines

An alternate derivation is given in Appendix A.

<u>References</u>

1. Chung, General Continuum Mechanics, Cambridge University Press, 2007.

2. Hjelmstad, Fundamentals of Structral Mechanics, Second Edition, Springer, New York, 2005.

APPENDIX A

Alternate Derivation

The stretch can be expressed in terms of the deformation gradient.

$$\beta^2(\mathbf{n}) = \mathbf{Fn} \bullet \mathbf{Fn} = \mathbf{n} \bullet \mathbf{F}^{\mathrm{T}}\mathbf{Fn} \tag{A-1}$$

The deformation tensor is

$$\mathbf{C} \equiv \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{A-2}$$

$$\beta^2(\mathbf{n}) = \mathbf{n} \bullet \mathbf{Cn} \tag{A-3}$$

The strain tensor is

$$\boldsymbol{\gamma}(\mathbf{n}) = \left[\beta^2(\mathbf{n}) - 1\right] \equiv \mathbf{n} \bullet \boldsymbol{\gamma}\,\mathbf{n} \tag{A-4}$$

$$\beta^2 = \mathbf{n} \bullet \boldsymbol{\gamma}\,\mathbf{n} + 1 \tag{A-5}$$

Now construct a scalar function *f*.

$$f = \mathbf{n} \bullet \boldsymbol{\gamma}\,\mathbf{n} + 1 - \lambda(\mathbf{n} \bullet \mathbf{n} - 1) \tag{A-6}$$

The constraint equation is

$$\mathbf{n} \bullet \mathbf{n} = 1 \tag{A-7}$$

Take the derivative and set equal to zero.

$$\frac{\partial}{\partial \mathbf{n}} f = \frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\,\mathbf{n} + 1 - \lambda(\mathbf{n} \bullet \mathbf{n} - 1)\right] = 0 \tag{A-8}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1 - \lambda\left(\mathbf{n} \bullet \mathbf{n} - 1\right)\right] = 0 \tag{A-9}$$

$$\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} = \mathrm{n}_{k}\, \boldsymbol{\gamma}_{kj}\, \mathrm{n}_{j} = \boldsymbol{\gamma}_{kj}\, \mathrm{n}_{j} \mathrm{n}_{k} \tag{A-10}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = \frac{\partial}{\partial \mathrm{n}_{i}}\left[\boldsymbol{\gamma}_{kj}\, \mathrm{n}_{j} \mathrm{n}_{k}\right] \tag{A-11}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = \boldsymbol{\gamma}_{kj} \mathrm{n}_{k} \frac{\partial \mathrm{n}_{j}}{\partial \mathrm{n}_{i}} + \boldsymbol{\gamma}_{kj}\, \mathrm{n}_{j} \frac{\partial \mathrm{n}_{k}}{\partial \mathrm{n}_{i}} \tag{A-12}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = \boldsymbol{\gamma}_{kj} \mathrm{n}_{k} \delta_{ij} + \boldsymbol{\gamma}_{kj}\, \mathrm{n}_{j} \delta_{ik} \tag{A-13}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = \boldsymbol{\gamma}_{ki} \mathrm{n}_{k} + \boldsymbol{\gamma}_{ij}\, \mathrm{n}_{j} \tag{A-14}$$

Change index k to j.

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = \boldsymbol{\gamma}_{ji} \mathrm{n}_{j} + \boldsymbol{\gamma}_{ij}\, \mathrm{n}_{j} \tag{A-15}$$

By symmetry,

$$\boldsymbol{\gamma}_{ji} = \boldsymbol{\gamma}_{ij} \tag{A-16}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\mathbf{n} \bullet \boldsymbol{\gamma}\, \mathbf{n} + 1\right] = 2\boldsymbol{\gamma}_{ji} \mathrm{n}_{j} \tag{A-17}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\lambda\left(\mathbf{n}\bullet\mathbf{n}-1\right)\right]=\lambda n_i \frac{\partial n_i}{\partial n_j}+\lambda n_i \frac{\partial n_i}{\partial n_j} \tag{A-18}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\lambda\left(\mathbf{n}\bullet\mathbf{n}-1\right)\right]=2\lambda n_i \frac{\partial n_i}{\partial n_j} \tag{A-19}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\lambda\left(\mathbf{n}\bullet\mathbf{n}-1\right)\right]=2\lambda n_i \delta_{ij} \tag{A-20}$$

$$\frac{\partial}{\partial \mathbf{n}}\left[\lambda\left(\mathbf{n}\bullet\mathbf{n}-1\right)\right]=2\lambda n_i = 2\lambda n_j \tag{A-21}$$

$$\frac{\partial}{\partial \mathbf{n}} f = 2\boldsymbol{\gamma}_{ji} n_j - 2\lambda n_j = 0 \tag{A-22}$$

$$\boldsymbol{\gamma}_{ji} n_j - \lambda n_j = 0 \tag{A-23}$$

$$\left(\boldsymbol{\gamma}_{ij} - \lambda\right) n_j = 0 \tag{A-24}$$

$$\left(\boldsymbol{\gamma} - \lambda\right) \mathbf{n} = 0 \tag{A-25}$$