MAE/CE 671 Continuum Mechanics

HW #1 (Problems 1-6 are based on Hjelmstad)

1. Compute the values of the following expressions
(a) $\delta_{ij}\delta_{ij}$
(b) $\sigma_{ij}\delta_{ik}\delta_{jk}$
(c) $\delta_{ab}\delta_{bc}\delta_{cd}\ldots\delta_{yz}$ (enough to exhaust the whole alphabet)

a) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

b) $\sigma_{ij}\delta_{ik}\delta_{jk} = \sigma_{ij}\,\delta_{ij} = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$

c) $\delta_{ab}\delta_{bc}\delta_{cd}\ldots\delta_{yz} = \delta_{az}$

2. Let two vectors, $\mathbf{u}$ and $\mathbf{v}$, have components relative to some basis as $\mathbf{u} = (5, -2, 1)$ and $\mathbf{v} = (1, 2, 1)$. Compute the lengths of the vectors and the angle between them. Find the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$.

$$|u| = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}$$

$$|v| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\theta = \arccos\left(\frac{u \bullet v}{|u||v|}\right) = \arccos\left(\frac{(5)(1) + (-2)(2) + (1)(1)}{\sqrt{30}\sqrt{6}}\right) = \arccos\left(\frac{2}{\sqrt{180}}\right) = 1.421\text{rad}$$

Area= $|u \times v|$

$$\text{Area}=\left\|\begin{matrix} i & j & k \\ 5 & -2 & 1 \\ 1 & 2 & 1 \end{matrix}\right\| = |(-2-2)i+(1-5)j+(10+2)k| = |(-4)i+(-4)j+(12)k|$$

$$\text{Area}=\sqrt{(-4)^2+(-4)^2+(12)^2} = \sqrt{16+16+144}+\sqrt{176} = 13.27$$

$$\text{Area}=|u\|v|\sin\theta = \sqrt{180}\sin(1.421) = 13.27$$

3. Demonstrate that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \varepsilon_{ijk}u_i v_j w_k$ from basic operations on the base vectors.

$$(u \times v)\bullet w = (u_i v_j \varepsilon_{ijk} w_k)$$

$$(u \times v)\bullet w = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \bullet (w_1 i + w_2 j + w_3 k)$$

$$= [(u_2v_3 - u_3v_2)i + (u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k] \bullet (w_1 i + w_2 j + w_3 k)$$

$$(u \times v)\bullet w = [(u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3]$$

$$(u \times v)\bullet w = [(u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3]$$

$$= (\varepsilon_{231}u_2v_3 + \varepsilon_{321}u_3v_2)w_1 + (\varepsilon_{312}u_3v_1 + \varepsilon_{132}u_1v_3)w_2 + (\varepsilon_{132}u_1v_2 + \varepsilon_{213}u_2v_1)w_3$$

$$= (u_i v_j \varepsilon_{ijk} w_k)$$

4. The vertices of a triangle are given by the position vectors $\mathbf{a} = (0,0,0)$, $\mathbf{b} = (1,4,3)$, and $\mathbf{c} = (2,3,1)$. Using a vector approach, compute the area of the triangle. Find the area of the triangle projected onto the plane with normal $\mathbf{n} = (0,0,1)$.

v = b – a = (1-0)i+(4-0)j+(3-0)k=1i+4j+3k
w = c – a = (2-0)i+(3-0)j+(1-0)k=2i+3j+1k

$$\text{triangle area} = \frac{1}{2}|u\,x\,v| = \frac{1}{2}|(4-9)i + (6-1)j + (3-8)k|$$

$$= \frac{1}{2}|u\,x\,v| = \frac{1}{2}|-5i + 5j - 5k| = \frac{1}{2}\sqrt{5^2 + 5^2 + 5^2} = \frac{5}{2}\sqrt{3} = 4.33$$

$$P \equiv I - n \otimes n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Pv = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \quad , \quad Pw = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{Triangle area projected} = \frac{1}{2}|Pv\,x\,Pw| = \frac{1}{2}\left\| \begin{matrix} i & j & k \\ 1 & 4 & 0 \\ 2 & 3 & 0 \end{matrix} \right\| = \frac{1}{2}|(3-8)k| = \frac{5}{2}$$

<u>Alternate method</u>

$$\text{Triangle area projected} = \left| \frac{1}{2}(-5i + 5j - 5k) \bullet 1k \right| = \frac{5}{2}$$

5. Use the observation that $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$ to show that

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2\right)$$

$$u \bullet v = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

$$u \bullet v = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - (u-v)\bullet(u-v)\right)$$

$$u \bullet v = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|u\|^2 - \|v\|^2 + 2(u \bullet v)\right)$$

$$u \bullet v = \frac{1}{2}(2(u \bullet v)) = u \bullet v$$

6. Prove the Schwarz inequality: $\mathbf{u}\cdot\mathbf{v} \le \|\mathbf{u}\|\|\mathbf{v}\|$.

$$u \bullet v = |u\|v|\cos\theta$$

$$\cos\theta \le 1$$

$$u \bullet v \le |u\|v|$$

7. Text, Problem 1.1

$$A = 3i_1 + 2i_2 + 4i_3$$

$$|A| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{9+4+16} = \sqrt{29}$$

$$\cos\theta_1 = \frac{3}{\sqrt{29}} \quad , \quad \theta_1 = \arccos\left(\frac{3}{\sqrt{29}}\right)$$

$$\cos\theta_2 = \frac{2}{\sqrt{29}} \quad , \quad \theta_2 = \arccos\left(\frac{2}{\sqrt{29}}\right)$$

$$\cos\theta_3 = \frac{4}{\sqrt{29}} \quad , \quad \theta_3 = \arccos\left(\frac{4}{\sqrt{29}}\right)$$