MAE/CE 671 Continuum Mechanics

Text

1.2 Let x_i be rotated 30 deg counterclockwise about the x_1 axis to x_i' through transformation a_{ijk} in problem 1.1 such that

$$A_i' = a_{ij}A_j \ , \qquad (i\ ,j = 1,2,3)$$

A= [3, 2, 4]

$$A_i' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \sin 30^\circ \\ 0 & -\sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ \sqrt{3}+2 \\ -1+2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 3.7321 \\ 2.4641 \end{bmatrix}$$

1.3 Let x_i be rotated 60 deg clockwise successively about the x_2, x_1' and x_3'' axes. Calculate the components of A in terms of the final set of coordinates if the components of A based on the original coordinates x_i are (1,1,1)

$$A = \begin{bmatrix} \cos(-60^\circ) & \sin(-60^\circ) & 0 \\ -\sin(-60^\circ) & \cos(-60^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-60^\circ) & \sin(-60^\circ) \\ 0 & -\sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix} \begin{bmatrix} \cos(-60^\circ) & 0 & -\sin(-60^\circ) \\ 0 & 1 & 0 \\ \sin(-60^\circ) & 0 & \cos(-60^\circ) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A=\begin{bmatrix}1/2 & -\sqrt{3}/2 & 0\\ \sqrt{3}/2 & 1/2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & -\sqrt{3}/2\\ 0 & \sqrt{3}/2 & 1/2\end{bmatrix}\begin{bmatrix}1/2 & 0 & \sqrt{3}/2\\ 0 & 1 & 0\\ -\sqrt{3}/2 & 0 & 1/2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & -\sqrt{3}\\ 0 & \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1 & 0 & \sqrt{3}\\ 0 & 2 & 0\\ -\sqrt{3} & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & -\sqrt{3}\\ 0 & \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}1+\sqrt{3}\\ 2\\ -\sqrt{3}+1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2+2\sqrt{3}\\ 2-\sqrt{3}\left(-\sqrt{3}+1\right)\\ 2\sqrt{3}-\sqrt{3}+1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2+2\sqrt{3}\\ 2+3-\sqrt{3}\\ \sqrt{3}+1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2+2\sqrt{3}\\ 5-\sqrt{3}\\ \sqrt{3}+1\end{bmatrix}$$

$$A=\frac{1}{8}\begin{bmatrix}2+2\sqrt{3}-\sqrt{3}\left(5-\sqrt{3}\right)\\ \sqrt{3}\left(2+2\sqrt{3}\right)+5-\sqrt{3}\\ 2\sqrt{3}+2\end{bmatrix}=\frac{1}{8}\begin{bmatrix}2+2\sqrt{3}-5\sqrt{3}+3\\ 2\sqrt{3}+6+5-\sqrt{3}\\ 2\sqrt{3}+2\end{bmatrix}=\frac{1}{8}\begin{bmatrix}5-3\sqrt{3}\\ 11+\sqrt{3}\\ 2\sqrt{3}+2\end{bmatrix}=\begin{bmatrix}-0.0245\\ 1.5915\\ 0.6830\end{bmatrix}$$

norm(A)= 1.732 = $\sqrt{3}$

1.5

(a) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(b)

$$|A||B| = |A^T||B| = |A^T B|$$

$$\varepsilon_{inm}\varepsilon_{jpq} = \left|\begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \end{bmatrix}\begin{vmatrix} \delta_{j1} & \delta_{p1} & \delta_{q1} \\ \delta_{j2} & \delta_{p2} & \delta_{q2} \\ \delta_{j3} & \delta_{p3} & \delta_{q3} \end{vmatrix}\right|$$

$$= \begin{vmatrix} \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3} & \delta_{i1}\delta_{p1} + \delta_{i2}\delta_{p2} + \delta_{i3}\delta_{p3} & \delta_{i1}\delta_{q1} + \delta_{i2}\delta_{q2} + \delta_{i3}\delta_{q3} \\ \delta_{n1}\delta_{j1} + \delta_{n2}\delta_{j2} + \delta_{n3}\delta_{j3} & \delta_{n1}\delta_{p1} + \delta_{n2}\delta_{p2} + \delta_{n3}\delta_{p3} & \delta_{n1}\delta_{q1} + \delta_{n2}\delta_{q2} + \delta_{n3}\delta_{q3} \\ \delta_{m1}\delta_{j1} + \delta_{m2}\delta_{j2} + \delta_{m3}\delta_{j3} & \delta_{m1}\delta_{p1} + \delta_{m2}\delta_{p2} + \delta_{m3}\delta_{p3} & \delta_{m1}\delta_{q1} + \delta_{m2}\delta_{q2} + \delta_{m3}\delta_{q3} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ij} & \delta_{ip} & \delta_{iq} \\ \delta_{nj} & \delta_{np} & \delta_{nq} \\ \delta_{mj} & \delta_{mp} & \delta_{mq} \end{vmatrix}$$

$$\varepsilon_{inm}\varepsilon_{jpq} = \delta_{ij}\delta_{np}\delta_{mq} + \delta_{ip}\delta_{nq}\delta_{mj} + \delta_{iq}\delta_{nj}\delta_{mp} - \delta_{ij}\delta_{mp}\delta_{nq} - \delta_{ip}\delta_{nj}\delta_{mq} - \delta_{iq}\delta_{np}\delta_{mj}$$

(c) $\varepsilon_{inm}\varepsilon_{jpq} = \delta_{ij}\delta_{np}\delta_{mq} + \delta_{ip}\delta_{nq}\delta_{mj} + \delta_{iq}\delta_{nj}\delta_{mp} - \delta_{ij}\delta_{mp}\delta_{nq} - \delta_{ip}\delta_{nj}\delta_{mq} - \delta_{iq}\delta_{np}\delta_{mj}$

Change j to i

$$\varepsilon_{inm}\varepsilon_{ipq} = \delta_{ii}\delta_{np}\delta_{mq} + \delta_{ip}\delta_{nq}\delta_{mi} + \delta_{iq}\delta_{ni}\delta_{mp} - \delta_{ii}\delta_{mp}\delta_{nq} - \delta_{ip}\delta_{ni}\delta_{mq} - \delta_{iq}\delta_{np}\delta_{mi}$$

$$\varepsilon_{inm}\varepsilon_{ipq} = 3\delta_{np}\delta_{mq} + \delta_{mp}\delta_{nq} + \delta_{nq}\delta_{mp} - 3\delta_{mp}\delta_{nq} - \delta_{np}\delta_{mq} - \delta_{mq}\delta_{np}$$

$$\varepsilon_{inm}\varepsilon_{ipq} = 3\delta_{np}\delta_{mq} - \delta_{np}\delta_{mq} - \delta_{mq}\delta_{np} \quad + \quad -3\delta_{mp}\delta_{nq} + \delta_{mp}\delta_{nq} + \delta_{nq}\delta_{mp}$$

$$\varepsilon_{inm}\varepsilon_{ipq} = \delta_{np}\delta_{mq} \quad - \delta_{mp}\delta_{nq}$$

(d) $\varepsilon_{inm}\varepsilon_{ipq} = \delta_{np}\delta_{mq} \quad - \delta_{mp}\delta_{nq}$

Change p to n

$$\varepsilon_{inm}\varepsilon_{inq} = \delta_{nn}\delta_{mq} \quad - \delta_{mn}\delta_{nq}$$

$$\varepsilon_{inm}\varepsilon_{inq} = 3\delta_{mq} \quad - \delta_{mq} = 2\delta_{mq}$$

(e) $\varepsilon_{inm}\varepsilon_{inq} = 2\delta_{mq}$

Change q to m

$$\varepsilon_{inm}\varepsilon_{inq} = 2\delta_{mm} = 6$$

1.6

a) Show $(V \bullet \nabla)\phi = V_i \, \phi,_i$

$$(V \bullet \nabla)\phi = \left(V_i \frac{\partial}{\partial x_j} \delta_{ij} \right)\phi$$

$$(V \bullet \nabla)\phi = \left(V_i \frac{\partial}{\partial x_i} \right)\phi = V_i \, \phi,_i$$

b) Show $\nabla(\nabla \bullet V) = V_{j,ji} \, \mathbf{i}_i$

$$\nabla(\nabla \bullet V) = \nabla\left(\frac{\partial}{\partial x_i} V_j \delta_{ij} \right)$$

$$\nabla(\nabla \bullet V) = \nabla\left(\frac{\partial}{\partial x_j} V_j \right)$$

$$\nabla(\nabla \bullet V) = \nabla\left(V_{j,j}\right)$$

$$\nabla(\nabla \bullet V) = \frac{\partial}{\partial x_i}\left(V_{j,j}\right)\mathbf{i}_i = V_{j,ji} \, \mathbf{i}_i$$

1.7 Using index notation, prove the relation

$$(\mathrm{V} \bullet \nabla)\mathrm{V} = \frac{1}{2}\nabla(\mathrm{V} \bullet \mathrm{V}) - \mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) \tag{1.7-1}$$

$$(\mathrm{V} \bullet \nabla)\mathrm{V} = \left(\mathrm{V_i} \frac{\partial}{\partial x_j} \delta_{ij} \right) \mathrm{V_k} \mathbf{i}_k = \mathrm{V_i} \frac{\partial}{\partial x_i} \mathrm{V_k} \mathbf{i}_k = \mathrm{V_i} \mathrm{V_{k,i}} \mathbf{i}_k \tag{1.7-2}$$

$$\frac{1}{2}\nabla(\mathrm{V} \bullet \mathrm{V}) = \frac{1}{2}\nabla\left(\mathrm{V_i V_j} \delta_{ij}\right) = \frac{1}{2}\nabla(\mathrm{V_i V_i}) = \frac{1}{2}\nabla\left(\mathrm{V_i}^2\right) \tag{1.7-3}$$

$$\frac{1}{2}\nabla(\mathrm{V} \bullet \mathrm{V}) = \frac{1}{2}\nabla\left(\mathrm{V_i}^2\right) = \frac{1}{2}\frac{\partial}{\partial x_k}\left(\mathrm{V_i}^2\right)\mathbf{i}_k = \mathrm{V_i V_{i,k}} \mathbf{i}_k \tag{1.7-4}$$

$$\mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) = \mathrm{V}\,\mathrm{x}\left(\varepsilon_{mnp} \frac{\partial}{\partial x_m} \mathrm{V_n}\, \mathbf{i}_p \right) = \mathrm{V}\,\mathrm{x}\left(\varepsilon_{mnp} \mathrm{V_{n,m}}\, \mathbf{i}_p\right) \tag{1.7-5}$$

$$\mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) = \varepsilon_{ipk}\varepsilon_{mnp} \mathrm{V_i V_{n,m}} \mathbf{i}_k = \varepsilon_{pki}\varepsilon_{pmn} \mathrm{V_i V_{n,m}} \mathbf{i}_k \tag{1.7-6}$$

$$\mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) = \varepsilon_{pki}\varepsilon_{pmn} \mathrm{V_i V_{n,m}} \mathbf{i}_k = \left(\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}\right)\mathrm{V_i V_{n,m}} \mathbf{i}_k \tag{1.7-7}$$

$$\mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) = \left(\delta_{km}\delta_{in} \mathrm{V_i V_{n,m}} \mathbf{i}_k - \delta_{kn}\delta_{im} \mathrm{V_i V_{n,m}} \mathbf{i}_k\right) \tag{1.7-8}$$

$$\mathrm{V}\,\mathrm{x}\,(\nabla\,\mathrm{x}\,\mathrm{V}) = \left(\mathrm{V_i V_{i,k}} \mathbf{i}_k - \mathrm{V_i V_{k,i}} \mathbf{i}_k\right) = \left(\mathrm{V_i V_{i,k}} - \mathrm{V_i V_{k,i}}\right)\mathbf{i}_k \tag{1.7-9}$$

Substitute equations (1-7.2), (1.7-4) and (1.7-9) into (1.7-1).

$$(\mathrm{V} \bullet \nabla)\mathrm{V} = \mathrm{V_i V_{k,i}} \mathbf{i}_k = \mathrm{V_i V_{i,k}} \mathbf{i}_k - \left(\mathrm{V_i V_{i,k}} - \mathrm{V_i V_{k,i}}\right)\mathbf{i}_k \tag{1.7-10}$$

$$\left(V \bullet \nabla\right)V = V_i V_{k,i}\mathbf{i}_k = V_i V_{k,i}\mathbf{i}_k \qquad (1.7\text{-}11)$$

1.9 Consider an inclined boundary surface identified by the coordinates at

A (4,4,3) B(0,4,2) and C(4,2,4)

Determine the unit vector normal to the surface ABC satisfying $n \bullet n = 1$

V=B-A =(0-4)i+(4-4)j+(2-3)k=-4i-1k

W=C-A=(4-4)i+(2-4)j+(4-3)k=-2j+1k

$$VxW = \begin{vmatrix} i & j & k \\ -4 & 0 & -1 \\ 0 & -2 & 1 \end{vmatrix} = -2i + 4j + 8k$$

$$n = \frac{VxW}{\left|VxW\right|} = \frac{-2i+4j+8k}{\sqrt{2^2+4^2+8^2}} = \frac{1}{\sqrt{4+16+64}}\left(-2i+4j+8k\right) = \frac{1}{\sqrt{84}}\left(-2i+4j+8k\right)$$