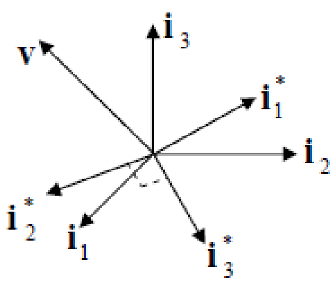
1. Two Cartesian bases, $(\mathbf{i}_i)$, and $(\mathbf{i}_i^*)$ are given as below, with $\mathbf{i}_1^* = (2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3)/3$ and $\mathbf{i}_2^* = (\mathbf{i}_1 - \mathbf{i}_2)/\sqrt{2}$.

(a) Express $\mathbf{i}_3^*$ in terms of $(\mathbf{i}_i)$.

(b) Express $(\mathbf{i}_i)$ in terms of the $(\mathbf{i}_i^*)$.

(c) Let $\mathbf{v} = 6\mathbf{i}_1 - 6\mathbf{i}_2 + 12\mathbf{i}_3$. Find the v_i^*, the components of the vector $\mathbf{v}$ in the $(\mathbf{i}_i^*)$ basis.

(d) verify your answer of part (c).



a)

$$i_3^* = i_1^* \times i_2^* = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix} = \left(\frac{1}{3\sqrt{2}}\right)\mathbf{i} + \left(\frac{1}{3\sqrt{2}}\right)\mathbf{j} + \left(\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right)\mathbf{k}$$

$$= \left(\frac{1}{3\sqrt{2}}\right)\mathbf{i} + \left(\frac{1}{3\sqrt{2}}\right)\mathbf{j} + \left(\frac{-4}{3\sqrt{2}}\right)\mathbf{k} = \left(\frac{1}{3\sqrt{2}}\right)(1\mathbf{i} + 1\mathbf{j} - 4\mathbf{k})$$

b)

$$\begin{bmatrix} \mathbf{i}^* \\ \mathbf{j}^* \\ \mathbf{k}^* \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & 1/3\sqrt{2} \\ 1/3 & 0 & -4/\sqrt{6} \end{bmatrix} \begin{bmatrix} \mathbf{i}^* \\ \mathbf{j}^* \\ \mathbf{k}^* \end{bmatrix}$$

(c) Let $\mathbf{v} = 6\mathbf{i}_1 - 6\mathbf{i}_2 + 12\mathbf{i}_3$. Find the v_i^*, the components of the vector $\mathbf{v}$ in the $(\mathbf{i}_i^*)$ basis.

$$\begin{bmatrix} V_i\,^* \\ V_j\,^* \\ V_k\,^* \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & 1/3\sqrt{2} & -4/3\sqrt{2} \end{bmatrix} \begin{bmatrix} 6 \\ -6 \\ 12 \end{bmatrix}$$

$$V_i{}^* = 4 - 4 + 4 = 4$$
$$V_j{}^* = 6/\sqrt{2} + 6/\sqrt{2} = 12/\sqrt{2}$$
$$V_k{}^* = 6/3\sqrt{2} - 6/3\sqrt{2} - 16\sqrt{2} = -16/\sqrt{2}$$

d) check norm

norm[6 -6 12]= 14.6969

norm[4 $12/\sqrt{2}$ $-16/\sqrt{2}$]= 14.6969

$$\begin{bmatrix} \mathbf{i}^* \\ \mathbf{j}^* \\ \mathbf{k}^* \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -4/\sqrt{6} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

2. Recall the following definitions for *divergence*, *curl*, and *gradient* of a *vector* field $\mathbf{v} = v_i\left(x_j\right)\mathbf{i}_i$:

$\nabla \cdot \mathbf{v} = v_{i,i} = trace\nabla\mathbf{v}$

$\nabla \times \mathbf{v} = \varepsilon_{ijk} v_{k,j}\,\mathbf{i}_i$

$\nabla\mathbf{v} = v_{i,j}\,\mathbf{i}_i \otimes \mathbf{i}_j$ [note the order of the indices; it's how we define the gradient tensor]

Let the position vector be $\mathbf{r} = x_i\mathbf{i}_i$. Compute

(a) $\nabla\mathbf{r}$,

(b) $\nabla \cdot \mathbf{r}$

(c) $\nabla \times \mathbf{r}$

a) $$\nabla\mathbf{r} = x_{i,j}\,\mathbf{i}_i \otimes \mathbf{i}_j = \begin{bmatrix} x_{i,i} & x_{i,j} & x_{i,k} \\ x_{j,i} & x_{j,j} & x_{j,k} \\ x_{k,i} & x_{k,j} & x_{k,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $$\nabla \bullet \mathbf{r} = \frac{\partial}{\partial x_i}\mathbf{i}_i \bullet x_j\mathbf{i}_j = \frac{\partial}{\partial x_i} x_j\delta_{ij} = x_{i,i} = 3$$

c) $$\nabla \text{x}\, \mathbf{r} = \frac{\partial}{\partial x_i}\mathbf{i}_i \text{ x } \left(x_j\mathbf{i}_j\right) = \varepsilon_{ijk}\frac{\partial}{\partial x_i} x_j\,\mathbf{i}_k = \varepsilon_{ijk} x_{j,i}\,\mathbf{i}_k = 0$$

3. For an arbitrary *scalar* function $\phi = \phi(x_i)$, and an arbitrary *vector* function $\mathbf{v} = \mathbf{v}(x_i)$, prove that

(a) $\nabla \times \nabla \phi = 0$;
(b) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$.

The curl of the gradient of a scalar is zero.

a) $$\nabla x \nabla \phi = \nabla x \frac{\partial \phi}{\partial x_j} \mathbf{i}_j = \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right) \varepsilon_{ijk} \mathbf{i}_k$$

$$\nabla x \nabla \phi = \left(\frac{\partial^2 \phi}{\partial x_2 \partial x_3} - \frac{\partial^2 \phi}{\partial x_3 \partial x_2}\right) \mathbf{i}_1 + \left(\frac{\partial^2 \phi}{\partial x_3 \partial x_1} - \frac{\partial^2 \phi}{\partial x_1 \partial x_3}\right) \mathbf{i}_2 + \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2} - \frac{\partial^2 \phi}{\partial x_2 \partial x_1}\right) \mathbf{i}_3$$

If the scalar field ϕ is twice continuously differentiable, then its second derivatives are independent of the order in which the derivatives are applied.

$$\frac{\partial^2 \phi}{\partial x_j \partial x_k} - \frac{\partial^2 \phi}{\partial x_k \partial x_j} = 0$$

$$\nabla x \nabla \phi = 0 \mathbf{i}_k = 0$$

b)

$$\nabla \bullet \nabla x V = \nabla \bullet \left(\frac{\partial}{\partial x_i} V_j \, \varepsilon_{ijk} \mathbf{i}_k\right) = \frac{\partial}{\partial x_m} \bullet \left(\frac{\partial}{\partial x_i} V_j \, \varepsilon_{ijk}\right) \delta_{km}$$

$$\nabla \bullet \nabla x V = \nabla \bullet \left[\left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) i_1 + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) i_2 + \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) i_3\right]$$

$$\nabla \bullet \nabla x V = \left[\frac{\partial}{\partial x_1}\mathbf{i}_1 + \frac{\partial}{\partial x_2}\mathbf{i}_2 + \frac{\partial}{\partial x3}\mathbf{i}_3\right] \bullet \left[\left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right)\mathbf{i}_1 + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right)\mathbf{i}_2 + \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)\mathbf{i}_3\right]$$

$$\nabla \bullet \nabla x V = \left[\frac{\partial}{\partial x_1}\left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)\right]$$

$$\nabla \bullet \nabla x V = \left[\frac{\partial^2 V_3}{\partial x_1 \partial x_2} - \frac{\partial^2 V_2}{\partial x_1 \partial x_3} + \frac{\partial^2 V_1}{\partial x_2 \partial x_3} - \frac{\partial^2 V_3}{\partial x_1 \partial x_2} + \frac{\partial^2 V_2}{\partial x_1 \partial x_3} - \frac{\partial^2 V_1}{\partial x_2 \partial x_3}\right]$$

$$\nabla \bullet \nabla x V = 0$$

4. Show that the *triple scalar product* is skew-symmetric with respect to changing the order in which the vector appear in the product. For example, show that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$$

To generalize this notion, any cyclic permutation (e.g., $\mathbf{u}, \mathbf{v}, \mathbf{w} \rightarrow \mathbf{w}, \mathbf{u}, \mathbf{v}$) of the order of the vectors leaves the sign of the product unchanged, while any acyclic permutation (e.g., $\mathbf{u}, \mathbf{v}, \mathbf{w} \rightarrow \mathbf{u}, \mathbf{v}, \mathbf{w}$) of the order of the vectors changes the sign. How does this observation relate to the swapping rows of a matrix in the computation of the determinant of that matrix?

$$(u \, x \, v) \bullet w = -(v \, x \, u) \bullet w$$

$$(\varepsilon_{ijk} \, u_i v_j \, i_k) \bullet w_m i_m = -(\varepsilon_{jik} \, v_j u_i \, i_k) \bullet w_m i_m$$

$$(\varepsilon_{ijk} \, u_i v_j \, i_k) \bullet w_m i_m = -(\varepsilon_{jik} \, u_i v_j \, i_k) \bullet w_m i_m$$

$$\varepsilon_{ijk} \, u_i v_j \, w_m \delta_{km} = -\varepsilon_{jik} \, u_i v_j \, w_m \delta_{km}$$

$$\varepsilon_{ijk} \, u_i v_j \, w_k = -\varepsilon_{jik} \, u_i v_j \, w_k$$

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

5. Let $\phi(\mathbf{r}) = x_2x_3 + x_1x_3 + x_1x_2$ be a *scalar* field.

(a) Compute the gradient $\nabla\phi$ of the field;

(b) Verify $\nabla\times\nabla\phi = 0$.

a)

$$\phi(r) = x_2x_3 + x_1x_3 + x_1x_2$$

$$\nabla\phi = \frac{\partial}{\partial x_1}(x_2x_3 + x_1x_3 + x_1x_2)i_1 + \frac{\partial}{\partial x_2}(x_2x_3 + x_1x_3 + x_1x_2)i_2 + \frac{\partial}{\partial x_3}(x_2x_3 + x_1x_3 + x_1x_2)i_3$$

$$\nabla\phi = (x_3 + x_2)i_1 + (x_3 + x_1)i_2 + (x_2 + x_1)i_3$$

b) $\nabla x\nabla\phi = \nabla x\{(x_3 + x_2)i_1 + (x_3 + x_1)i_2 + (x_2 + x_1)i_3\}$

$$\nabla x\nabla\phi = \begin{vmatrix} i_1 & i_2 & i_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ (x_3 + x_2) & (x_3 + x_1) & (x_2 + x_1) \end{vmatrix}$$

$\nabla x\nabla\phi$

$$= \left(\frac{\partial}{\partial x_2}(x_2 + x_1) - \frac{\partial}{\partial x_3}(x_3 + x_1)\right)i_1 + \left(\frac{\partial}{\partial x_3}(x_3 + x_2) - \frac{\partial}{\partial x_1}(x_2 + x_1)\right)i_2 + \left(\frac{\partial}{\partial x_1}(x_3 + x_1) + \frac{\partial}{\partial x_2}(x_3 + x_2)\right)i_3$$

$$\nabla x\nabla\phi = (1-1)i_1 + (1-1)i_2 + (1-1)i_3 = 0$$

6. Let $\mathbf{v}(\mathbf{r}) = x_2x_3\mathbf{i}_1 + x_1x_3\mathbf{i}_2 + x_1x_2\mathbf{i}_3$ be a *vector* field. Compute

(a) the gradient $\nabla\mathbf{v}$ of the field;

(b) the divergence $\nabla\cdot\mathbf{v}$ of the field.

a)

$$v(r) = x_2x_3i_1 + x_1x_3i_2 + x_1x_2i_3$$

$$\nabla v = \begin{bmatrix} V_{1,1} & V_{1,2} & V_{1,3} \\ V_{2,1} & V_{2,2} & V_{2,3} \\ V_{3,1} & V_{3,2} & V_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{bmatrix}$$

b) $$\nabla \bullet v(r) = \left(\frac{\partial}{\partial x_1}\mathbf{i}_1 + \frac{\partial}{\partial x_2}\mathbf{i}_2 + \frac{\partial}{\partial x3}\mathbf{i}_3\right) \bullet \left(x_2x_3i_1 + x_1x_3i_2 + x_1x_2i_3\right)$$

$$\nabla \bullet v(r) = \left(\frac{\partial}{\partial x_1}x_2x_3 + \frac{\partial}{\partial x_2}x_1x_3 + \frac{\partial}{\partial x3}x_1x_2\right) = 0$$

7. Let $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$, $\mathbf{w}(\mathbf{r})$ be vector fields and let $\mathbf{T}(\mathbf{r})$ be a tensor (2^{nd}-order) field. Compute the component forms of the following derivatives

(a) $\nabla(\mathbf{u}\cdot\mathbf{v})$, (b) $\nabla\cdot(\mathbf{u}\times\mathbf{v})$, (c) $\nabla(\mathbf{u}\times\mathbf{v})$,

(d) $\nabla\cdot(\mathbf{T}\mathbf{v})$, (e) $\nabla(\mathbf{u}\cdot\mathbf{T}\mathbf{v})$, (f) $\nabla(\mathbf{T}\mathbf{v})$,

(g) $\nabla\cdot(\mathbf{u}\otimes\mathbf{v})$, (h) $\nabla\cdot([\mathbf{u}\otimes\mathbf{v}]\mathbf{w})$, (i) $\nabla[(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}]$.

$$(u \times v) = (u_i v_j \varepsilon_{ijk}\, i_k) = \begin{vmatrix} i_1 & i_2 & i_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)i_1 + (u_3v_1 - u_1v_3)i_2 + (u_1v_2 - u_2v_1)i_3$$

a) $\nabla(u \bullet v) = \nabla(u_i v_j \delta_{ij}) = \nabla(u_i\, v_i) = \dfrac{\partial}{\partial x_k}(u_i\, v_i)\, i_k = (u_i\, v_i)_{,k}\, i_k = (u_{i,k}\, v_i + u_i\, v_{i,k})\, i_k$

b)

$$\nabla \bullet (u \times v) = \nabla \bullet (u_i v_j \varepsilon_{ijk}\, i_k) = i_m \frac{\partial}{\partial x_m} \bullet (u_i v_j \varepsilon_{ijk}\, i_k) = \frac{\partial}{\partial x_k}(u_i v_j \varepsilon_{ijk}) = \varepsilon_{ijk}(u_i v_j)_{,k}$$
$$= \varepsilon_{ijk}(u_{i,k} v_j + u_i v_{j,k})$$

c) $$\nabla(u \times v) = \begin{vmatrix} \frac{\partial}{\partial x_1}(u_2v_3 - u_3v_2) & \frac{\partial}{\partial x_2}(u_2v_3 - u_3v_2) & \frac{\partial}{\partial x_3}(u_2v_3 - u_3v_2) \\ \frac{\partial}{\partial x_1}(u_3v_1 - u_1v_3) & \frac{\partial}{\partial x_2}(u_3v_1 - u_1v_3) & \frac{\partial}{\partial x_3}(u_3v_1 - u_1v_3) \\ \frac{\partial}{\partial x_1}(u_1v_2 - u_2v_1) & \frac{\partial}{\partial x_2}(u_1v_2 - u_2v_1) & \frac{\partial}{\partial x_3}(u_1v_2 - u_2v_1) \end{vmatrix}$$

d)

$$
\begin{aligned}
\nabla \bullet (\mathrm{T\,v}) &= \nabla \bullet \left(T_{ij}\, i_i \otimes i_j\right) v_j\, i_j \\
&= \nabla \bullet \left(T_{ij} v_j\right)\left(i_i \left(i_j \bullet i_j\right)\right) \\
&= \nabla \bullet \left(T_{ij} v_j\right)\left(i_i \left(i_j \bullet i_j\right)\right) \\
&= \nabla \bullet T_{ij} v_j\, i_i \\
&= \frac{\partial}{\partial x_k} T_{ij} v_j\, \delta_{ik} \\
&= \frac{\partial}{\partial x_i} T_{ij} v_j \\
&= \left(T_{ij} v_j\right)_{,i} = \left(T_{ij,i} v_j + T_{ij} v_{j,i}\right)
\end{aligned}
$$

e) $\nabla(\mathrm{u} \bullet \mathrm{T\,v})$

$$\nabla(\mathrm{u} \bullet \mathrm{T\,v}) = \nabla\left(\mathrm{u} \bullet T_{ij} v_j\, i_i\right)$$

$$\nabla(\mathrm{u} \bullet \mathrm{T\,v}) = \nabla\left(u_i T_{ij} v_j\right)$$

$$\nabla(\mathrm{u} \bullet \mathrm{T\,v}) = \frac{\partial}{\partial x_k}\left(u_i T_{ij} v_j\right) i_k$$

$$\nabla(\mathrm{u} \bullet \mathrm{T\,v}) = \left(u_i T_{ij} v_j\right)_{,k} i_k = \left(u_{i,k} T_{ij} v_j + u_i T_{ij,k} v_j + u_i T_{ij} v_{j,k}\right) i_k$$

f)

$$\nabla(\mathrm{T\,v}) = \nabla\left(\mathrm{T_{ij} v_j i_i}\right) = \nabla\left(\mathrm{T_{1j} v_j i_1 + T_{2j} v_j i_2 + T_{3j} v_j i_3}\right)$$

$$= \begin{vmatrix} \frac{\partial}{\partial x_1} \mathrm{T_{1j} v_j} & \frac{\partial}{\partial x_2} \mathrm{T_{1j} v_j} & \frac{\partial}{\partial x_3} \mathrm{T_{1j} v_j} \\ \frac{\partial}{\partial x_1} \mathrm{T_{2j} v_j} & \frac{\partial}{\partial x_2} \mathrm{T_{2j} v_j} & \frac{\partial}{\partial x_3} \mathrm{T_{2j} v_j} \\ \frac{\partial}{\partial x_1} \mathrm{T_{3j} v_j} & \frac{\partial}{\partial x_2} \mathrm{T_{3j} v_j} & \frac{\partial}{\partial x_3} \mathrm{T_{3j} v_j} \end{vmatrix} = \frac{\partial}{\partial x_k} \mathrm{T_{ij} v_j \; i_i \otimes i_k} = \left(\mathrm{T_{ij,k} v_j + T_{ij} v_{j,k}}\right) \mathrm{i_i \otimes i_k}$$

g) $\nabla \bullet (\mathrm{u} \otimes \mathrm{v}) = \frac{\partial}{\partial x_k} \mathrm{i_k} \bullet \left[\mathrm{u_i v_j i_i \otimes i_j}\right] = \frac{\partial}{\partial x_j}\left[\mathrm{u_i v_j}\right] \mathrm{i_i} = \left[\mathrm{u_{i,j} v_j + u_i v_{j,j}}\right] \mathrm{i_i}$

h)

$$\nabla \bullet ([u \otimes v]w) = \nabla \bullet (u(v \bullet w))$$
$$= \nabla \bullet u\,(v_i w_i)$$
$$= \frac{\partial}{\partial x_k}[u_k\,(v_i w_i)]$$
$$= [u_k\,(v_i w_i)]_{,k} = u_{k,k} v_i w_i + u_k v_{i,k} w_i + u_k v_i w_{i,k}$$

i)

$$\nabla((u \, x \, v) \bullet w) = \nabla(u_i v_j \varepsilon_{ijk} i_k \bullet w) = \nabla(u_i v_j w_k \varepsilon_{ijk})$$
$$= \frac{\partial}{\partial x_m}(u_i v_j w_k \varepsilon_{ijk}) i_m$$
$$= \varepsilon_{ijk}(u_{i,m} v_j w_k + u_i v_{j,m} w_k + u_i v_j w_{k,m}) i_m$$