MAE/CE 671 Continuum Mechanics

Text

1. Show that the Jacobian is equal to the square root of the determinant of the metric tensor.

Let F be the displacement gradient:

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \\ \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{bmatrix}$$

$$G_{ij} = \mathbf{G}_i \bullet \mathbf{G}_j = \frac{\partial \mathbf{R}}{\partial x_i} \bullet \frac{\partial \mathbf{R}}{\partial x_j} = \frac{z_i\, \mathbf{i}_i}{\partial x_i} \bullet \frac{z_i\, \mathbf{i}_i}{\partial x_j} = \frac{\partial z_i}{\partial x_i} \frac{\partial z_i}{\partial x_j}$$

$$G_{ij} = F_{mi}\, F_{mj}$$

$$\mathbf{G = F^T\, F}$$

$$\det[G] = \det\left[F^T\, F\right] = \det[F]\det[F] = J^2$$

$$J = \sqrt{\det[G]}$$

2. Let the motion of a fluid in steady-state flow be given in the Eulerian coordinates by

$$V_1 = k\,z_1$$
$$V_2 = -k\,z_2$$
$$V_3 = 0$$

a) Show that the motions and velocities in Lagrangian coordinates are

$$z_1 = x_1 \exp[k(t - t_o)] \qquad v_1 = kx_1 \exp[k(t - t_o)]$$

$$z_2 = x_2 \exp[-k(t - t_o)] \qquad v_2 = -kx_2 \exp[-k(t - t_o)]$$

$$z_3 = x_3 \qquad v_3 = 0$$

Pfaffian Form

$\mathbf{R} = z_i\, \mathbf{i}_i$

$$\frac{dz_1}{V_1(\mathbf{R},t)} = \frac{dz_2}{V_2(\mathbf{R},t)} = \frac{dz_3}{V_3(\mathbf{R},t)} = dt$$

$$V_1 = k\,z_1$$

$$\int_{z_1(t_o)}^{z_1(t)} \frac{dz_1}{k\,z_1} = \int_{t_o}^{t} dt \qquad \frac{dz_1}{dt} = kz_1$$

$$\frac{1}{k}\{\ln[z_1(t)] - \ln[z_1(t_o)]\} = t - t_o \qquad \frac{dz_1}{dt} - kz_1 = 0$$

$$\ln[z_1(t)/z_1(t_o)] = k(t - t_o) \qquad sZ_1(s) - z_1(0) - kZ_1(s) = 0$$

$$z_1(t)/z_1(t_o) = \exp[k(t - t_o)] \qquad (s - k)Z_1(s) - z_1(0) = 0$$

$$z_1(t) = z_1(t_o)\exp[k(t - t_o)] \qquad Z_1(s) = \frac{z_1(0)}{(s - k)}$$

$$z_1(t_o) = x_1$$

$$z_1(t) = x_1 \exp[k(t - t_o)]$$

$$v_1(t) = k\, x_1 \exp[k(t - t_o)]$$

$$z_1(t) = z_1(0) \exp[k(t - t_o)]$$

$$z_1(t_o) = x_1$$

$$z_1(t) = x_1 \exp[k(t - t_o)]$$

alternate

$$v_1 = \frac{dz_1}{dt} = kz_1$$

Let

$$z_1 = A \exp(-\omega t)$$

$$\frac{dz_1}{dt} = -\omega A \exp(-\omega t)$$

$$-\omega A \exp(-\omega t) = k\, A \exp(-\omega t)$$

$$\omega = -k$$

$$z_1 = A \exp(kt)$$

$$z_1(0) = x_1$$

$$z_1 = x_1 \exp(k(t - t_o))$$

$$V_2 = -k z_2$$

$$-\int_{z_2(t_o)}^{z_2(t)} \frac{dz_2}{k z_2} = \int_{t_o}^{t} dt$$

$$-\frac{1}{k}\{\ln[z_2(t)] - \ln[z_2(t_o)]\} = t - t_o$$

$$\ln[z_1(t)/z_1(t_o)] = -k(t - t_o)$$

$$z_2(t)/z_2(t_o) = \exp[-k(t - t_o)]$$

$$z_2(t) = z_2(t_o)\exp[-k(t - t_o)]$$

$$z_2(t_o) = x_2$$

$$z_2(t) = x_2 \exp[-k(t - t_o)]$$

$$v_2(t) = -k x_2 \exp[-k(t - t_o)]$$

$$dz_3 = V_3(\mathbf{R}, t)dt$$

$$V_3 = 0$$

$$dz_3 = 0$$

$$z_3(t) - z_3(t_o) = 0$$

$$z_3(t) = z_3(t_o) = x_3$$

b) Transform the Lagrangian velocities into Eulerian coordinates to prove that the results in part (a) are recovered.

$$z_1 = x_1 \exp[k(t - t_o)] \qquad x_1 = z_1 / \exp[k(t - t_o)]$$

$$z_2 = x_2 \exp[-k(t - t_o)] \qquad x_2 = z_2 / \exp[-k(t - t_o)]$$

$$z_3 = x_3 \qquad x_3 = z_3$$

$$v_1 = kx_1 \exp[k(t - t_o)] \qquad V_1 = kz_1 \exp[k(t - t_o)] / \exp[k(t - t_o)] = kz_1$$

$$v_2 = -kx_2 \exp[-k(t - t_o)] \qquad V_2 = -kz_2 \exp[-k(t - t_o)] / \exp[-k(t - t_o)] = -kz_2$$

$$v_3 = 0 \qquad V_3 = 0$$

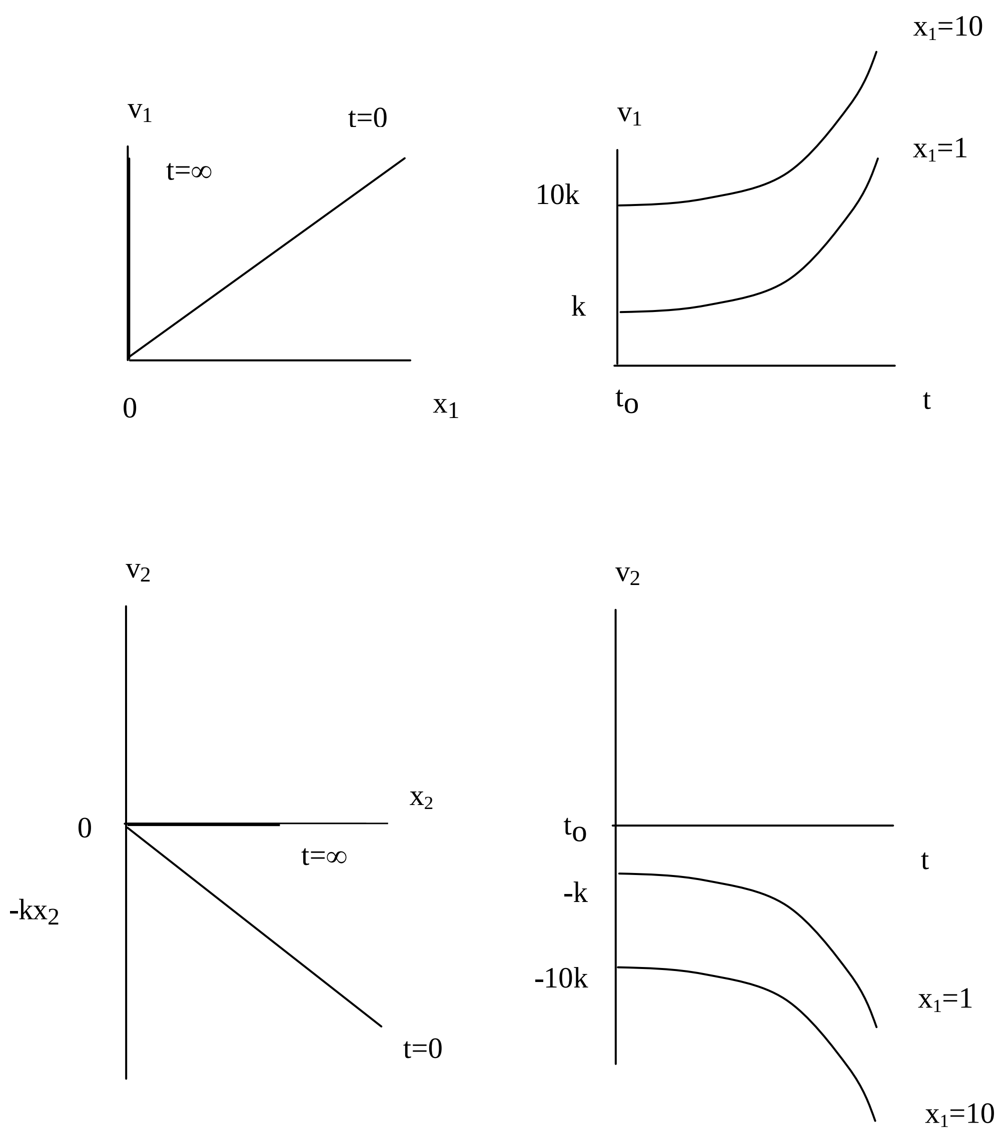
c) Plot the Eulerian and Lagrangian distributions

Lagragian

$$v_1 = kx_1 \exp[k(t - t_o)]$$

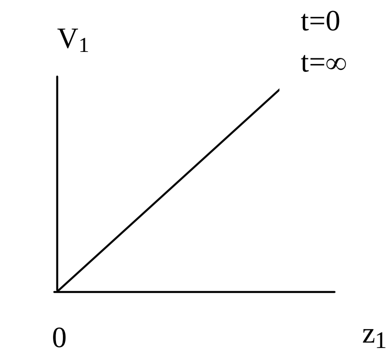
$$v_2 = -kx_2 \exp[-k(t - t_o)]$$

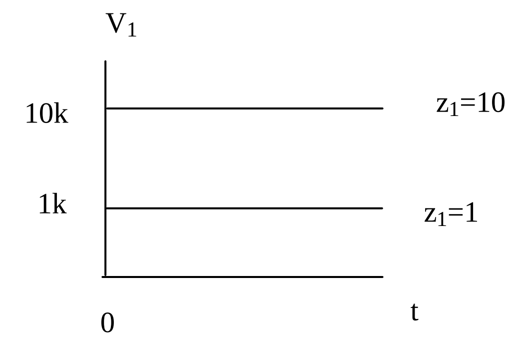
$$v_3 = 0$$

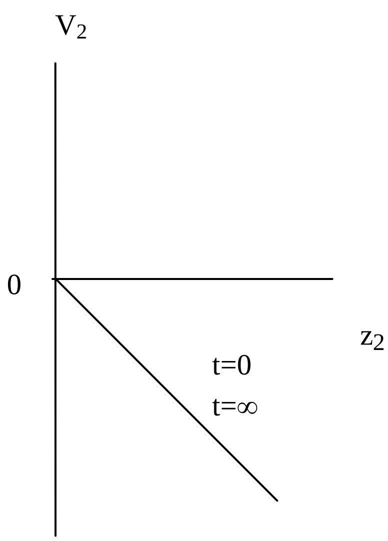


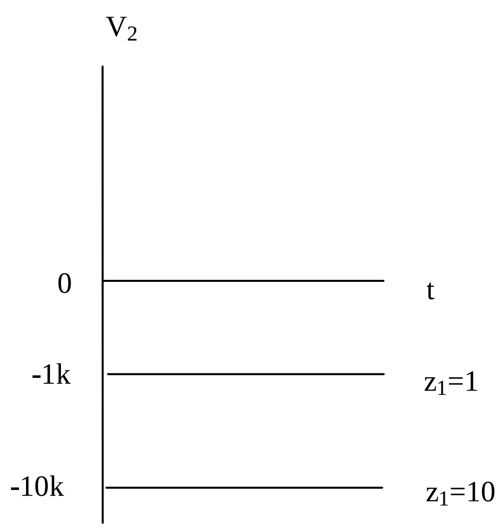
Eulerian

$$V_1 = k z_1$$
$$V_2 = -k z_2$$
$$V_3 = 0$$









$$V_1 = k\,z_1$$
$$V_2 = -k\,z_2$$
$$V_3 = 0$$

Equations for streamlines

$$\frac{dz_1}{d\lambda} = V_1 = k\,z_1 \qquad \frac{dz_2}{d\lambda} = V_2 = -k\,z_2$$

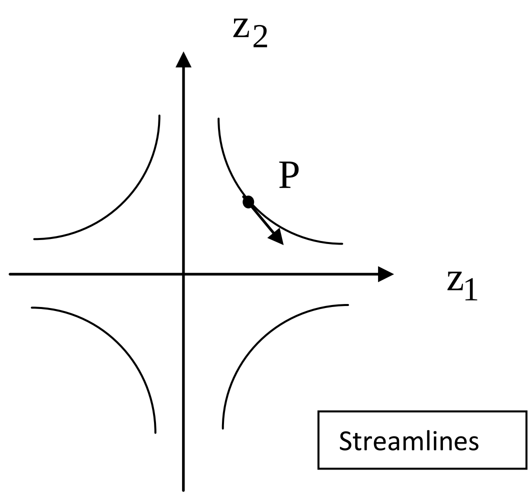
Pfaffian form

$$\frac{dz_1}{V_1} = \frac{dz_2}{V_2} = d\lambda$$

$$\frac{dz_1}{k\,z_1} = \frac{dz_2}{-k\,z_2}$$

$$\frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0$$

$z_1 z_2 = C$, a constant



The tangent of the streamline is given by

$$\frac{dz_2}{dz_1} = \frac{C}{-z_2^{\,2}} = -\frac{z_2}{z_1}$$

The fluid flows along the tangent of the streamlines, and there can be no flow perpendicular to the streamlines.

3. Derive the Cayley-Hamilton equation using the Lagrange multiplier method and identify the principle strain invariants

γ_{ij}	strain components on a plane arbitrarily inclined from the principal planes
n_j	direction cosines that identify the inclined plane
γ_i	principle strains

The relationship between the principal strain components on the inclined plain is

$$\gamma_i = \gamma_{ij} n_j$$

Multiply by n_i

$$\gamma(n) = \gamma_i n_i = n_i n_j \gamma_{ij}$$

Let $\gamma = \lambda$ a scalar for one of the principal strain components

$$\gamma_i = \lambda n_i = \lambda \delta_{ij} n_j$$

By substitution,

$$\gamma_{ij} n_j = \lambda \delta_{ij} n_j$$

Multiply by n_i

$$\gamma_{ij} n_i n_j = \lambda \delta_{ij} n_i n_j = \lambda n_i n_i$$

Direction cosine property

$$\mathbf{n} \bullet \mathbf{n} = n_i n_i = 1$$

Now construct a scalar function *f*.

$$f = \gamma_{ij}\, n_i n_j - \lambda\left(n_i n_i - 1\right)$$

λ is the Lagrange multiplier

f is the work required for satisfying: $\gamma(n) = \gamma_i n_i = n_i\, n_j \gamma_{ij}$

The goal is to find the direction of unit vector **n** for which $\gamma(n)$ is maximum.

Take the derivative and set to zero.

$$\frac{\partial}{\partial n_k}\left(\gamma_{ij}\, n_i n_j\right) = \gamma_{ij}\frac{\partial}{\partial n_k}\left(n_i n_j\right) = \gamma_{ij}\left[n_j \frac{\partial n_i}{\partial n_k} + n_i \frac{\partial n_j}{\partial n_k}\right] = \gamma_{ij}\left[n_j \delta_{ik} + n_i\, \delta_{jk}\right]$$

$$= \gamma_{kj} n_j + \gamma_{ik} n_i = 2\gamma_{kj} n_j \quad \text{(by symmetry)}$$

Also,

$$\frac{\partial}{\partial n_j}\left(\lambda n_i n_i\right) = \lambda \frac{\partial}{\partial n_j}\left(n_i n_i\right) = \lambda\left[n_i \frac{\partial n_i}{\partial n_j} + n_i \frac{\partial n_i}{\partial n_j}\right] = \lambda\left[n_i \delta_{ij} + n_i\, \delta_{ij}\right] = 2\lambda\delta_{ij} n_j$$

Thus

$$\frac{\partial}{\partial n_i} f = 2\gamma_{ij} n_j - 2\lambda\delta_{ij} n_j = 0$$

$$\left(\gamma_{ij} - \lambda\delta_{ij}\right) n_j = 0$$

Since n_j is arbitrary,

$$\det\left(\gamma_{ij} - \lambda\delta_{ij}\right) = 0$$

Cayley-Hamilton equation

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

The standard eigenvalue problem is

$$\left(\gamma_{ij} - \lambda\delta_{ij}\right)n_j = 0$$

$$\det\left(\gamma_{ij} - \lambda\delta_{ij}\right) = 0$$

$$\begin{vmatrix} \gamma_{11}-\lambda & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22}-\lambda & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33}-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} &+(\gamma_{11}-\lambda)(\gamma_{22}-\lambda)(\gamma_{33}-\lambda)-(\gamma_{11}-\lambda)\gamma_{32}\gamma_{23} \\ &+\gamma_{12}\gamma_{23}\gamma_{31}-\gamma_{12}\gamma_{21}(\gamma_{33}-\lambda) \\ &+\gamma_{13}\gamma_{21}\gamma_{32}-\gamma_{31}\gamma_{13}(\gamma_{22}-\lambda)=0 \end{aligned}$$

$$\begin{aligned} &+(\gamma_{11}-\lambda)\left(\gamma_{22}\gamma_{33}-\lambda(\gamma_{22}+\gamma_{33})+\lambda^2\right) \\ &-\gamma_{11}\gamma_{32}\gamma_{23}+\lambda\gamma_{32}\gamma_{23} \\ &+\gamma_{12}\gamma_{23}\gamma_{31}-\gamma_{12}\gamma_{21}\gamma_{33}+\lambda\gamma_{12}\gamma_{21} \\ &+\gamma_{13}\gamma_{21}\gamma_{32}-\gamma_{31}\gamma_{13}\gamma_{22}+\lambda\gamma_{31}\gamma_{13}=0 \end{aligned}$$

$$-\lambda\gamma_{22}\gamma_{33}+\lambda^2(\gamma_{22}+\gamma_{33})-\lambda^3$$
$$\gamma_{11}\gamma_{22}\gamma_{33}-\lambda(\gamma_{11}\gamma_{22}+\gamma_{11}\gamma_{33})+\gamma_{11}\lambda^2$$
$$-\gamma_{11}\gamma_{32}\gamma_{23}+\lambda\gamma_{32}\gamma_{23}$$
$$+\gamma_{12}\gamma_{23}\gamma_{31}-\gamma_{12}\gamma_{21}\gamma_{33}+\lambda\gamma_{12}\gamma_{21}$$
$$+\gamma_{13}\gamma_{21}\gamma_{32}-\gamma_{31}\gamma_{13}\gamma_{22}+\lambda\gamma_{31}\gamma_{13}=0$$

$$-\lambda^3+(\gamma_{11}+\gamma_{22}+\gamma_{33})\lambda^2-(\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21}+\gamma_{11}\gamma_{33}-\gamma_{31}\gamma_{13}+\gamma_{22}\gamma_{33}-\gamma_{32}\gamma_{23})\lambda$$
$$+\gamma_{11}\gamma_{22}\gamma_{33}-\gamma_{11}\gamma_{32}\gamma_{23}+\gamma_{12}\gamma_{23}\gamma_{31}-\gamma_{12}\gamma_{21}\gamma_{33}+\gamma_{13}\gamma_{21}\gamma_{32}-\gamma_{31}\gamma_{13}\gamma_{22}=0$$

$$I_1=\gamma_{11}+\gamma_{22}+\gamma_{33}=\gamma_{ii}$$
$$I_2=\frac{1}{2}\left(\gamma_{ii}\gamma_{jj}-\gamma_{ij}\gamma_{ij}\right)$$
$$I_3=\left|\gamma_{ij}\right|$$

The eigenvalues λ are the principle strain invariants.

4. From the laboratory measurements the following strain tensor components are obtained.

$$\gamma_{ij} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

a) Determine the principle strains invariants

$$I_1 = \mathrm{trace}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = 5$$

$$I_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 + 2 - 1 + 4 - 1 = 6$$

$$I_3 = \det\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = 4 - 1 - 2 = 1$$

b) Determine the principle strains

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Eigenvalues λ via Matlab

0.1981
1.555
3.247

The eigenvalues are the principles strains

c) Determine the principle direction matrix

Eigenvectors via Matlab

V =

-0.5910 0.3280 0.7370

0.3280 -0.7370 0.5910

0.7370 0.5910 0.3280

The eigenvectors represent the a_{ij}, which are the principal direction cosines

Verify your results by recalculating the principle strains from principal direction cosines.

Calculated using Matlab.

$$V^T \gamma V = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.1981 & 0 & 0 \\ 0 & 1.555 & 0 \\ 0 & 0 & 3.247 \end{bmatrix}$$

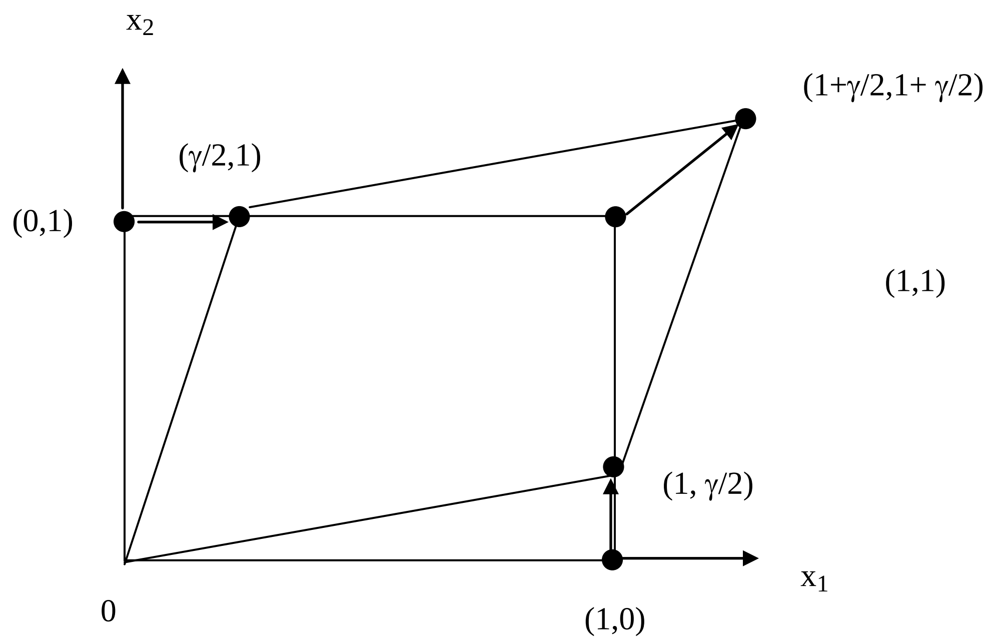
5. Sketch the displaced positions in the x_1x_2 – plane of the points initially on the sides of the square bounded by $x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1$.

(a) $\mathbf{u} = \frac{1}{2}\gamma x_2 \mathbf{i}_1 + \frac{1}{2}\gamma x_1 \mathbf{i}_2$

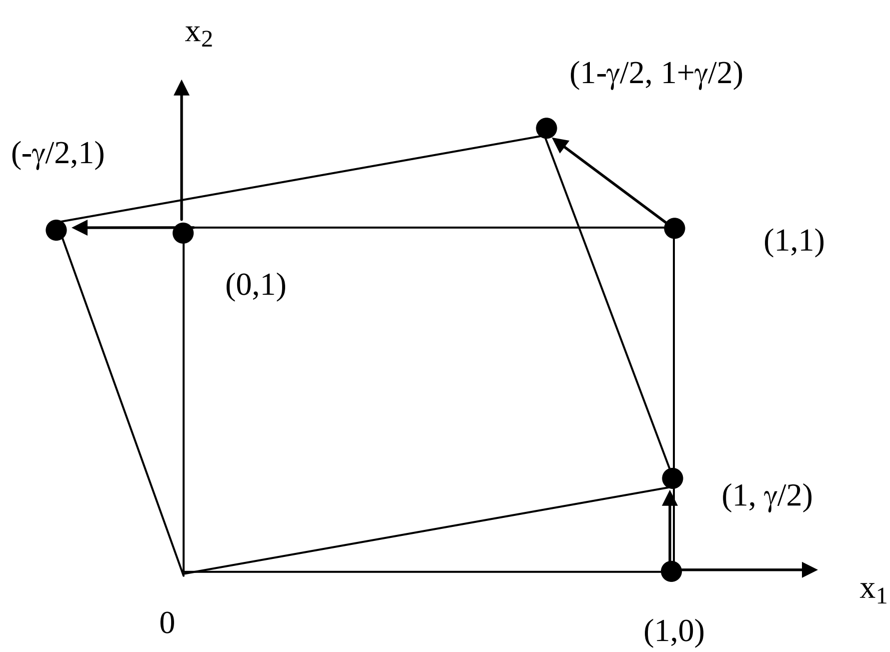
(b) $\mathbf{u} = -\frac{1}{2}\gamma x_2 \mathbf{i}_1 + \frac{1}{2}\gamma x_1 \mathbf{i}_2$

(c) $\mathbf{u} = \gamma x_1 \mathbf{i}_2$

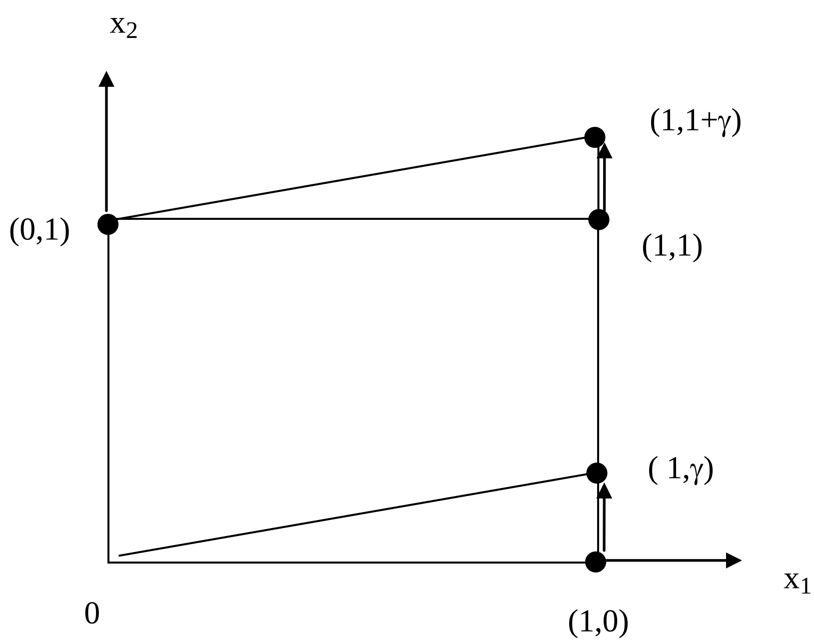
a)



b)



c)



6. For each of the displacement fields in P.5, determine the finite and infinitesimal strain tensors.

5. Sketch the displaced positions in the x_1x_2 – plane of the points initially on the sides of the square bounded by $x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1$.

(a) $\mathbf{u} = \frac{1}{2}\gamma x_2 \mathbf{i}_1 + \frac{1}{2}\gamma x_1 \mathbf{i}_2$

(b) $\mathbf{u} = -\frac{1}{2}\gamma x_2 \mathbf{i}_1 + \frac{1}{2}\gamma x_1 \mathbf{i}_2$

(c) $\mathbf{u} = \gamma x_1 \mathbf{i}_2$

a)

$$\mathbf{u} = \frac{1}{2}\gamma x_2 \mathbf{i}_1 + \frac{1}{2}\gamma x_1 \mathbf{i}_2$$

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{linear}} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathbf{T}}\right) = \frac{1}{2}\left\{\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{finite}} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathbf{T}} + \nabla \mathbf{u}^{\mathbf{T}} \nabla \mathbf{u}\right) = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{finite}} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}} + \nabla\mathbf{u}^{\mathbf{T}}\nabla\mathbf{u}\right) = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8}\gamma^2 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & \frac{1}{8}\gamma^2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{4}\gamma^2 & 0 \\ 0 & \frac{1}{4}\gamma^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{8}\gamma^2 & 0 \\ 0 & \frac{1}{8}\gamma^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8}\gamma^2 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & \frac{1}{8}\gamma^2 \end{bmatrix}$$

b)

$$\mathbf{u} = -\frac{1}{2}\gamma x_2\mathbf{i}_1 + \frac{1}{2}\gamma x_1\mathbf{i}_2$$

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ -\frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{linear}} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}}\right) = \frac{1}{2}\left\{\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ -\frac{1}{2}\gamma & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{finite}} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}} + \nabla\mathbf{u}^{\mathbf{T}}\nabla\mathbf{u}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -\frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}\gamma \\ -\frac{1}{2}\gamma & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8}\gamma^2 & 0 \\ 0 & \frac{1}{8}\gamma^2 \end{bmatrix}$$

C) $\mathbf{u} = \gamma x_1 \mathbf{i}_2$

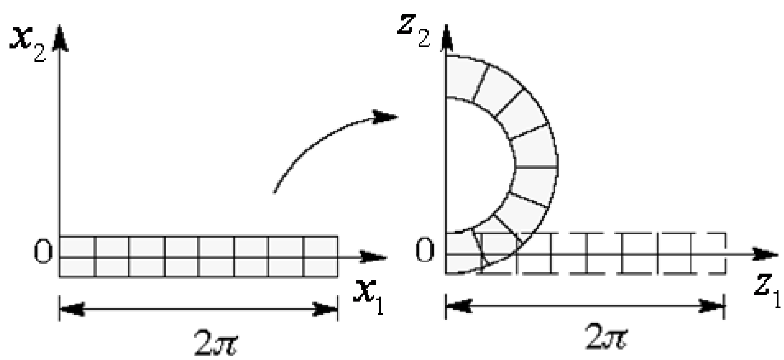
$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{linear}} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}}\right) = \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$

$$\mathbf{E}_{\text{finite}} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}} + \nabla\mathbf{u}^{\mathbf{T}}\nabla\mathbf{u}\right) = \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \gamma^2 & \gamma \\ \gamma & \gamma^2 \end{bmatrix}$$

7. **(Based on Hjelmstad's book)** Consider the large deformation of a flexible beam (strip) under bending that deforms the strip of length 2π (say inches) into a semicircular arc, as shown below. Assume the beam is inextensible; that is, the length of the middle line (given by $x_2 = 0$), 2π, remains unchanged after the bending (the top surface and bottom surface respectively becomes shorter and longer than 2π, as shown in the figure). Let the depth (in the x_2 direction) be $2c$. For a narrow strip ($c << \pi$), the change in the depth can be neglected.



a. Construct the deformation map, that is, find the mathematical expression

$\mathbf{R} = \mathbf{R}(\mathbf{r}, t)$, or, $z_i = z_i(x_j, t)$

What is the location of the point originally located at $\mathbf{r} = 2\pi\mathbf{i}_1$?

b. Compute the deformation gradient tensor $\mathbf{F}$;

c. Determine the Lagrangian (finite) strain tensor.

diameter = 4

radius=2

$$\underline{\phi}(\mathbf{x}) = [(2 - x_2)\sin(x_1/2)]\mathbf{i}_1 + [2 - (2 - x_2)\cos(x_1/2)]\mathbf{i}_2 + x_3\mathbf{i}_2$$

The point at r = $2\pi\mathbf{i}_i$ maps to (0, 4, 0) in the new coordinate system.

Or $4\mathbf{i}_2$

b)

$$\mathbf{F} = \nabla \underline{\phi} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x3} \\ \frac{\partial \phi_3}{\partial x_1} & \frac{\partial \phi_3}{\partial x_2} & \frac{\partial \phi_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(2-x_2)\cos(x_1/2) & -\sin(x_1/2) & 0 \\ -\frac{1}{2}(2-x_2)\sin(x_1/2) & \cos(x_1/2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c)

The Green deformation tensor is:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}$$

$$= \begin{bmatrix} \frac{1}{2}(2-x_2)\cos(x_1/2) & -\frac{1}{2}(2-x_2)\sin(x_1/2) & 0 \\ -\sin(x_1/2) & \cos(x_1/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(2-x_2)\cos(x_1/2) & -\sin(x_1/2) & 0 \\ -\frac{1}{2}(2-x_2)\sin(x_1/2) & \cos(x_1/2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(2-x_2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \left(1-\frac{1}{2}x_2\right)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Lagrangian strain tensor **E** is defined to be have the difference between the Green deformation tensor and the identity tensor **I** as

$$\mathbf{E} \equiv \frac{1}{2}[\mathbf{C}-\mathbf{I}] = \frac{1}{2}\left\{\begin{bmatrix} \frac{1}{4}(2-x_2)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} \frac{1}{4}(2-x_2)^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{1}{4}\left(4-4x_2+x_2{}^2\right) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \left(1-x_2+\frac{1}{4}x_2{}^2\right) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -x_2+\frac{1}{4}x_2{}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}x_2+\frac{1}{8}x_2{}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$