MAE/CE 671 Continuum Mechanics
HW #5

1. Let the principal stresses and principal stress directions be $\sigma_1, \sigma_2, \sigma_3$ and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, respectively. Suppose you are given $(\sigma_i, \mathbf{p}_i)$, $i = 1,2,3$ by an analyst running FEA (Ansys and/or Abaqus).

(a) Sketch the principal stresses and principal (stress) planes;

Consider an octahedral plane with the normal of

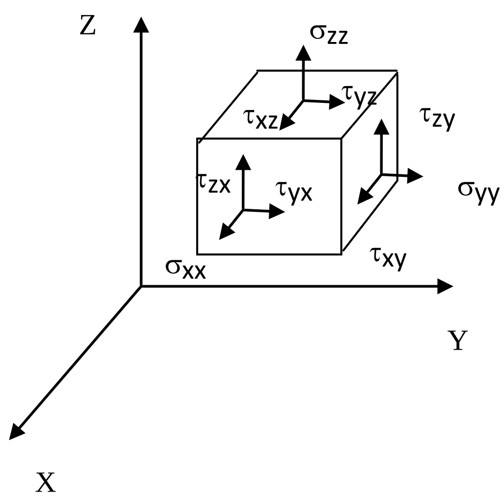
$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).$$

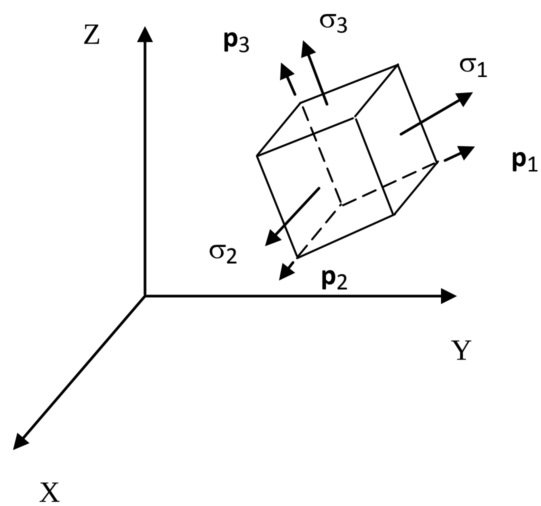
(b) Sketch the octahedral plane;
(c) Calculate the traction vector on the octahedral plane;
(d) Calculate the angle between the traction vector and the normal to the plane
(e) Show that the octahedral normal stress σ_n (the normal component of the traction vector on the octahedral plane) is $\frac{Tr\sigma}{3}$.
(f) Calculate the tangential component of the traction vector on the octahedral plane;
(g) Show that the octahedral shear stress σ_{oct} (or τ_o), the magnitude of the tangential component of the traction vector on the octahedral plane is related to the 2nd invariant of the stress tensor by

$$\sigma_{oct} = \sqrt{\frac{2}{3}\hat{J}_2} = \sqrt{\frac{1}{3}tr\hat{\boldsymbol{\sigma}}^2}$$

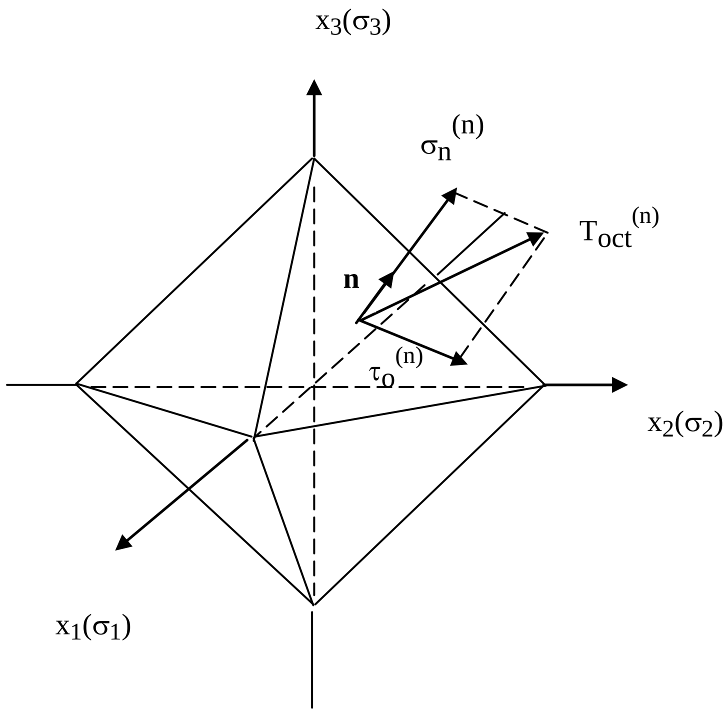
where $\hat{\boldsymbol{\sigma}}$ is the deviatoric stress tensor.
(h) specialize your results for uniaxial stress along $\mathbf{p}_1$;

a)





b)



(c) Calculate the traction vector on the octahedral plane;

The normal traction on the octahedral plane is

$$\sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

The projected shear traction on the octahedral plane is given by

$$\tau_o = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$

$$\mathbf{t}_o = \frac{\sigma_1}{\sqrt{3}}\mathbf{p}_1 + \frac{\sigma_2}{\sqrt{3}}\mathbf{p}_2 + \frac{\sigma_3}{\sqrt{3}}\mathbf{p}_3 = \sigma_n\mathbf{n} + \tau_o\mathbf{m}$$

(d) Calculate the angle between the traction vector and the normal to the plane

$$\cos\theta = \frac{\mathbf{t}\bullet\mathbf{n}}{|\mathbf{t}||\mathbf{n}|} = \frac{\left[\frac{\sigma_1}{\sqrt{3}}\mathbf{p}_1 + \frac{\sigma_2}{\sqrt{3}}\mathbf{p}_2 + \frac{\sigma_3}{\sqrt{3}}\mathbf{p}\right]\bullet\left[\frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\right]}{\left|\frac{\sigma_1}{\sqrt{3}}\mathbf{p}_1 + \frac{\sigma_2}{\sqrt{3}}\mathbf{p}_2 + \frac{\sigma_3}{\sqrt{3}}\mathbf{p}\right|\left|\frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\right|}$$

$$\cos\theta = \frac{\mathbf{t}\bullet\mathbf{n}}{|\mathbf{t}||\mathbf{n}|} = \frac{\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)}{\frac{1}{\sqrt{3}}\sqrt{\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2}} = \frac{1}{\sqrt{3}}\frac{(\sigma_1 + \sigma_2 + \sigma_3)}{\sqrt{\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2}}$$

$$\theta = \arccos\left\{\frac{1}{\sqrt{3}}\frac{(\sigma_1 + \sigma_2 + \sigma_3)}{\sqrt{\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2}}\right\}$$

e) Show that the octahedral normal stress σ_n (the normal component of the traction vector on the octahedral plane) is $\frac{1}{3}\mathrm{Tr}\sigma$

The normal traction on the octahedral plane is

$$\sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{2}\,\mathrm{trace}\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}$$

(f) Calculate the tangential component of the traction vector on the octahedral plane;

The projected shear traction on the octahedral plane is given by

$$\tau_o = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

(g) Show that the octahedral shear stress σ_{oct} (or τ_o), the magnitude of the tangential component of the traction vector on the octahedral plane is related to the 2nd invariant of the stress tensor by

$$\sigma_{oct} = \sqrt{\frac{2}{3}\hat{J}_2} = \sqrt{\frac{1}{3}\mathrm{tr}\hat{\boldsymbol{\sigma}}^2}$$

where $\hat{\boldsymbol{\sigma}}$ is the deviatoric stress tensor.

The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the Cauchy stress tensor.

$$\hat{\boldsymbol{\sigma}} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$$

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}, \ \sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_2 - \sigma_m & 0 \\ 0 & 0 & \sigma_3 - \sigma_m \end{bmatrix},$$

$$\hat{\mathbf{J}}_{\mathbf{1}} = (S_1 + S_2 + S_3) = 0$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -(S_1S_2 + S_1S_3 + S_2S_3)$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\frac{1}{2}S_1(S_2 + S_3) - \frac{1}{2}S_2(S_1 + S_3) - \frac{1}{2}S_3(S_1 + S_2)$$

$$S_1 = -(S_2 + S_3)$$

$$S_2 = -(S_1 + S_3)$$

$$S_3 = -(S_1 + S_2)$$

$$\begin{aligned}\hat{\mathbf{J}}_{\mathbf{2}} &= \frac{1}{2}S_1{}^2 + \frac{1}{2}S_2{}^2 + \frac{1}{2}S_3{}^2 = \frac{1}{2}\left\{(\sigma_1 - \sigma_m)^2 + (\sigma_2 - \sigma_m)^2 + (\sigma_3 - \sigma_m)^2\right\} \\ &= \frac{1}{2}\mathrm{tr}\,\hat{\boldsymbol{\sigma}}^2\end{aligned}$$

$\hat{\mathbf{J}}_{\mathbf{2}}$ can also be written as

$$\begin{aligned}\hat{\mathbf{J}}_{\mathbf{2}} = &-(\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m) \\ &-(\sigma_1 - \sigma_m)(\sigma_3 - \sigma_m) \\ &-(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m)\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{J}}_{\mathbf{2}} = &-\sigma_1\sigma_2 + \sigma_1\sigma_m + \sigma_2\sigma_m - \sigma_m{}^2 \\ &-\sigma_1\sigma_3 + \sigma_1\sigma_m + \sigma_3\sigma_m - \sigma_m{}^2 \\ &-\sigma_2\sigma_3 + \sigma_2\sigma_m + \sigma_3\sigma_m - \sigma_m{}^2\end{aligned}$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + 2\sigma_m(\sigma_1 + \sigma_2 + \sigma_3) - 3\sigma_m{}^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + 2\sigma_m(3\sigma_m) - 3\sigma_m{}^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + 6\sigma_m{}^2 - 3\sigma_m{}^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + 3\sigma_m{}^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + 3\left[\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\right]^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = -\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 + \frac{1}{3}\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 + 2\sigma_1\sigma_2 + 2\sigma_1\sigma_3 + 2\sigma_2\sigma_3\right)$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = \frac{1}{3}\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\right)$$

$$\hat{\mathbf{J}}_{\mathbf{2}} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right] = \frac{1}{2}\operatorname{tr}\hat{\boldsymbol{\sigma}}^2$$

$$\frac{2}{3}\hat{\mathbf{J}}_{\mathbf{2}} = \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right] = \frac{1}{3}\operatorname{tr}\hat{\boldsymbol{\sigma}}^2$$

$$\sqrt{\frac{2}{3}\hat{\mathbf{J}}_{\mathbf{2}}} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2} = \sqrt{\frac{1}{3}\operatorname{tr}\hat{\boldsymbol{\sigma}}^2}$$

$$\tau_o = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$

$$\sigma_{oct} = \sqrt{\frac{2}{3}\hat{J}_2} = \sqrt{\frac{1}{3}\operatorname{tr}\hat{\boldsymbol{\sigma}}^2}$$

(h) specialize your results for uniaxial stress along $\mathbf{p}_1$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

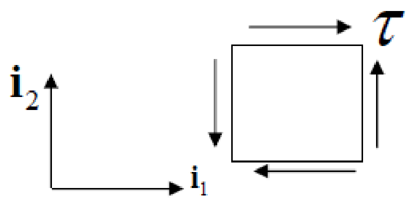
$$\mathbf{t}_{oct} = \frac{\sigma_1}{\sqrt{3}}\mathbf{p}_1$$

$$\sigma_{oct} = \frac{1}{3}\sigma_1$$

$$\tau_{oct} = \frac{1}{3}\sqrt{2(\sigma_1)^2} = \frac{\sqrt{2}}{3}\sigma_1$$

$$\theta = \arccos\left\{\frac{1}{\sqrt{3}}\right\}$$

2. Consider a pure-shear stress state as shown below.



(a) Construct the stress tensor;
(b) Compute the traction vector on planes $\mathbf{n} = \mathbf{i}_1$, and $\mathbf{n} = \mathbf{i}_2$;
(c) Find the principal stresses and principal stress planes;
(d) verify the results given in the class on the orientations and magnitude of the max. shear

stress;
(e) The von-Mises (or equivalent) stress in the material is defined as

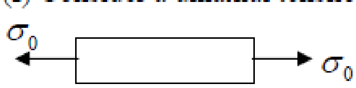
$$\bar{\sigma} = \sigma^{vm} \equiv \sqrt{\frac{3}{2}\hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{\sigma}}} \equiv \sqrt{\frac{3}{2} tr\hat{\boldsymbol{\sigma}}^2} = \sqrt{\frac{3}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij}} = \sqrt{3\hat{J}_2};$$

The von-Mises yield surface are often written as $f = 0$ with the yield function

$$f = \bar{\sigma} - Y$$

where Y is a material property and is called the yield stress of the material.

(f) Consider a uniaxial tensile test of a metal (aluminum alloy) shown below



Show that the yield function under the loading becomes

$$f = |\sigma_0| - Y$$

Hence, the yield stress of the material can be determined by a uniaxial test. Suppose you have found the material yields at $\sigma_0 = 20ksi$.

a) Stress tensor: $\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau_{xy} \\ \tau_{yx} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$

b) Compute the traction vector on planes

$\mathbf{t} = \tau \mathbf{i}_2$ on $\mathbf{n}=\mathbf{i}_1$

$\mathbf{t} = \tau \mathbf{i}_1$ on $\mathbf{n}=\mathbf{i}_2$

c) Find the principal stresses and principal stress planes;

$$\left\{\begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\mathbf{n} = 0$$

$$\det\begin{bmatrix} -\lambda & \tau \\ \tau & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \tau^2 = 0$$

$$\lambda^2 = \tau^2$$

$$\lambda = \pm\tau$$

$$\sigma_1 = -\tau \quad \text{and} \quad \sigma_2 = +\tau$$

$$\begin{bmatrix} \tau & \tau \\ \tau & \tau \end{bmatrix}\mathbf{n}_1 = 0$$

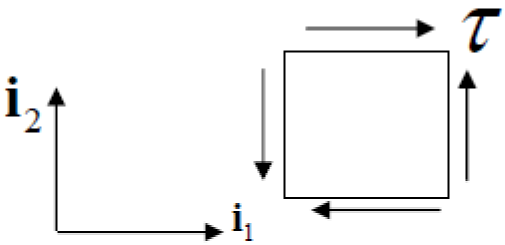
$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

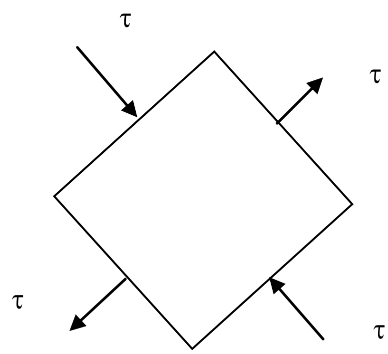
$$\begin{bmatrix} -\tau & \tau \\ \tau & -\tau \end{bmatrix}\mathbf{n}_2 = 0$$

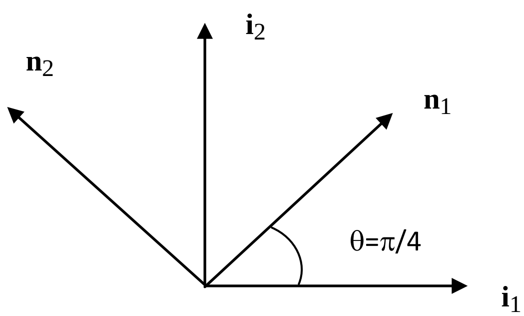
$$\mathbf{n}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tau}{0}$$

$$2\theta_p = \pi/2 \qquad \theta_p = \pi/4$$







d) verify the results given in the class on the orientations and magnitude of the max. shear stress;

(e) The von-Mises (or equivalent) stress in the material is defined as

$$\bar{\sigma} = \sigma^{vm} \equiv \sqrt{\frac{3}{2}\hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{\sigma}}} \equiv \sqrt{\frac{3}{2} tr\hat{\boldsymbol{\sigma}}^2} = \sqrt{\frac{3}{2}\hat{\sigma}_{ij}\hat{\sigma}_{ij}} = \sqrt{3\hat{J}_2}\ ;$$

The von-Mises yield surface are often written as $f = 0$ with the yield function

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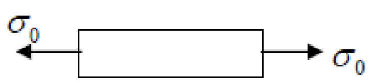
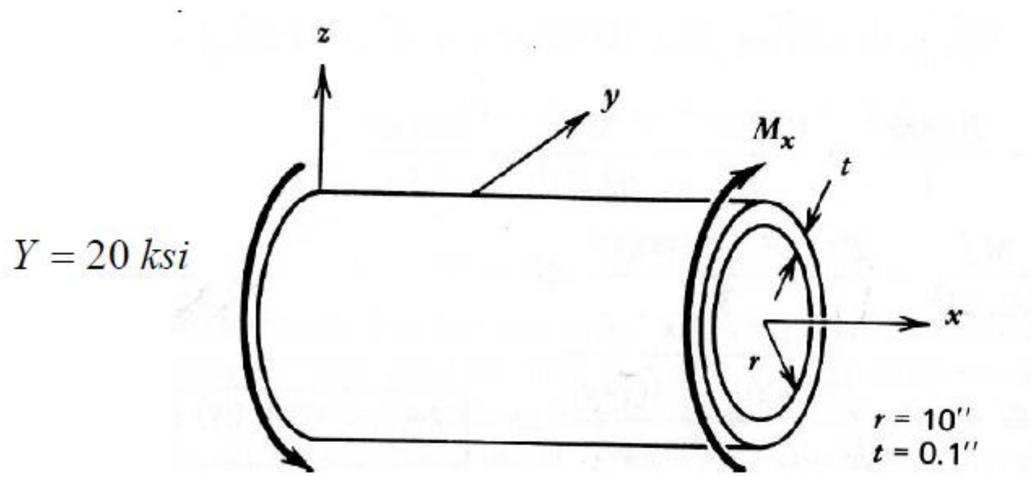


Show that the yield function under the loading becomes

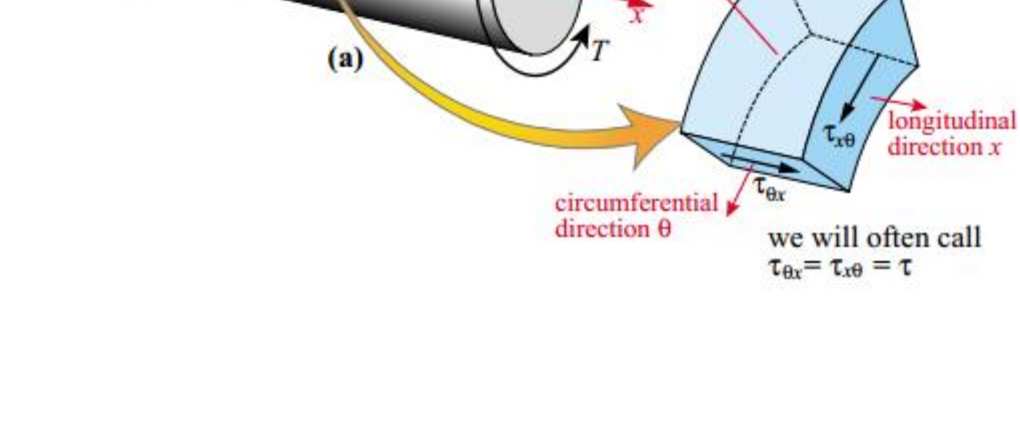
$$f = |\sigma_0| - Y$$

Hence, the yield stress of the material can be determined by a uniaxial test. Suppose you have found the material yields at $\sigma_0 = 20ksi$.

3. The thin-walled pressure vessel with open ends shown below ($r = 10"$, $t = 0.1"$, $Y = 20\,ksi$) is subjected to a torque $M_x = 502,640\ in - lb$.



(a) Assuming the material obeys the von Mises yield criterion, determine if the applied torque will cause yielding in the material.



$$J = \frac{\pi\left(d_2^4 - d_1^4\right)}{32}$$

$$\tau_{x\theta} = \tau_{max}\frac{\rho}{R}, \quad \tau_{max} = \frac{TR}{J}, \quad R = \frac{1}{2}d_2$$

$$\sigma = \begin{bmatrix} 0 & \tau_{x\theta} \\ \tau_{x\theta} & 0 \end{bmatrix}$$

Eigenvalue problem

$$\det\begin{bmatrix} \lambda & \tau_{x\theta} \\ \tau_{x\theta} & \lambda \end{bmatrix} = 0$$

$$\lambda^2 - \tau_{x\theta}{}^2 = 0$$

$$\lambda^2 = \tau_{x\theta}{}^2$$

$$\lambda = \pm\tau_{x\theta}$$

At the outer diameter,

$$\sigma_1 = +\tau_{max}$$

$$\sigma_2 = -\tau_{max}$$

$$\sigma_{vm}{}^2 = \frac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}$$

Results from Matlab script

```
Dimensions (in)
 OD=     20,  ID=   19.8,  thick=    0.1

 Torque=5.026e+05 in-lbf, J=     619 in^4

 R=      10 in

 tau max=    8121 psi
        =   8.121 ksi

 Internal Pressure=       0 psi


 Hoop stress =       0 psi
 open ends

Stress Tensor

A =

   1.0e+03 *

         0    8.1208         0
    8.1208         0         0
         0         0         0

 Invariants
 I1=       0  I2=-6.595e+07  I3=      -0

 Principal Stresses

    8121
       0
   -8121

 Eigenvectors, direction cosines of principal plane
      (column format)


evector =

    0.7071         0    0.7071
    0.7071         0   -0.7071
         0    1.0000         0
```

```
Overall maximum shear stress =      8121 psi
                              =    8.121 ksi

von Mises stress = 1.407e+04 psi
                 =    14.07 ksi
```

14.07 ksi < Y = 20 ksi

Now, ***in addition to*** the applied torque M_x, an internal pressure p is also applied to the vessel.

(b) Determine the internal pressure p_y that will cause yielding using the von Mises yield criterion.

<u>Hoop Stress</u>

The hoop stress can be expressed as:

$\sigma_h = p\ d / 2\ t$

where

σ_h = hoop stress (MPa, psi)

p = internal pressure in the tube or cylinder (MPa, psi)

d = internal diameter of tube or cylinder (mm, in)

t = tube or cylinder wall thickness (mm, in)

<u>Results from Matlab script</u>

```
Dimensions (in)
 OD=      20,   ID=   19.8,   thick=    0.1

 Torque=5.026e+05 in-lbf, J=     619 in^4

 R=       10 in

 tau max=     8121 psi
        =    8.121 ksi
```

```
 Internal Pressure=     142 psi


 Hoop stress = 1.42e+04 psi
 open ends

Stress Tensor

A =

   1.0e+04 *

        0    0.8121         0
   0.8121    1.4200         0
        0         0         0

 Invariants
 I1=1.42e+04  I2=-6.595e+07  I3=      -0

 Principal Stresses

1.789e+04
       0
   -3687

 Eigenvectors, direction cosines of principal plane
      (column format)


evector =

   0.4134         0    0.9106
   0.9106         0   -0.4134
        0    1.0000         0


 Overall maximum shear stress = 1.079e+04 psi
                               =   10.79 ksi

 von Mises stress = 1.999e+04 psi
                  =   19.99 ksi
```

Internal pressure = 142 psi for yielding