LAPLACE'S EQUATION
Revision A

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April 15, 2015

Introduction

Laplace's equation is a second-order partial differential equation operating on a scalar f.

$$\nabla^2 f = 0 \tag{1}$$

It may also be written as the divergence of the gradient of f.

$$\nabla \bullet \nabla f = 0 \tag{2}$$

Laplace's equation is an example of an elliptic partial differential equation. It is also a special case of the Helmholtz and Poisson equations as shown in Appendices A and B, respectively.

The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics; because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Note that Laplace's equation depends on space but not time. Thus it is used for steady-state problems.

In three dimensions, the problem is to find twice-differentiable real-valued functions f, of real variables x, y, and z, such that

Cartesian coordinates

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{3}$$

Cylindrical coordinates

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{4}$$

Spherical coordinates

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial^2 f}{\partial \varphi^2} = 0 \tag{5}$$

<u>Boundary Conditions</u>

Recall Laplace's equation with the amplitude represented by y.

$$\nabla^2 y = 0 \tag{6}$$

Dirichlet boundary conditions specify the solution along the boundary.

$$y(x) = f(x) \qquad \forall\, x \in \partial\Omega \tag{7}$$

where f is a known scalar function defined along the boundary $\partial\Omega$.

In plain English

$\forall$	for all
$\in$	in or belonging to
$\partial\Omega$	volume

Dirichlet boundary conditions specify the solution along the boundary.

$$\frac{\partial}{\partial \mathbf{n}} y(x) = f(x) \qquad \forall\, x \in \partial\Omega \tag{8}$$

where **n** denotes the (typically exterior) normal to the boundary $\partial\Omega$.

Solution

The solutions depend on the boundary conditions and the coordinate system. Examples are shown in the following table.

Coordinate System	Variables	Solution Functions
Cartesian	$X(x)\,Y(y)\,Z(z)$	Exponential functions, circular functions, hyperbolic equations
Cylindrical	$R(r)\,\Theta(\theta)\,Z(z)$	Bessel functions, exponential functions, circular functions
Spherical	$R(r)\,\Theta(\theta)\,\Phi(\phi)$	Legendre polynomial, power, circular functions

APPENDIX A

Helmholtz Equation

The Helmholtz equation is the partial differential equation

$$\nabla^2 A + k^2 A = 0 \tag{A-1}$$

where

∇^2 is the Laplacian

k is the wavenumber

A is the amplitude

In Cartesian coordinates

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + k^2\right] A(x, y, z) = 0 \tag{A-2}$$

The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents the time-independent form of the original equation

APPENDIX B

Possion's Equation

The Possion's equation is

$$\nabla^2 \varphi = f \tag{B-1}$$

where

∇^2	is the Laplacian
φ and f	real or complex-valued functions on a manifold

In Cartesian coordinates

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi(x, y, z) = f(x, y, z) \tag{B-2}$$

APPENDIX C

Position Vector

$$\bar{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \qquad \text{(C-1)}$$

$$\left|\bar{r}\right| = r = \sqrt{x^2 + y^2 + z^2} \qquad \text{(C-2)}$$

Divergence

$$\bar{\nabla} \cdot \bar{r} = 3 \qquad \text{(C-3)}$$

Curl

$$\bar{\nabla} \times \bar{r} = 0 \qquad \text{(C-4)}$$

Gradient

$$\bar{\nabla}\, r = \hat{r} = \frac{\bar{r}}{r} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \qquad \text{unit vector} \qquad \text{(C-5)}$$

Laplacian

$$\bar{\nabla} \cdot \bar{\nabla}\, r = \bar{\nabla}^2\, r = \frac{2}{r} \qquad \text{(C-6)}$$

Laplacian 1/r

$$\bar{\nabla}^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(r) \qquad \text{three-dimensional Dirac delta} \qquad \text{(C-7)}$$