CONSERVATION OF MASS

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Introduction

Transform the Lagrangian to Eulerian coordinates to show the conservation of mass for fluids can be derived from the conservation of mass for solids. Include complete details of the algebra involved.

Solids

$$d\Omega = \frac{\partial z_m}{\partial x_1}\frac{\partial z_n}{\partial x_2}\frac{\partial z_p}{\partial x_3}\varepsilon_{mnp}\, d\Omega_o = \left|\frac{\partial z_i}{\partial x_j}\right| d\Omega_o = J\, d\Omega_o = \sqrt{\left|G_{ij}\right|}\, d\Omega_o = \sqrt{G}\, d\Omega_o \tag{1}$$

$$m = \int_{\Omega} \rho\, d\Omega = \int_{\Omega} \rho_o\, d\Omega_o \tag{2}$$

$$\int_{\Omega} \left(\rho J - \rho_o\right) d\Omega_o = 0 \quad \text{in global form} \tag{3}$$

$$\rho J - \rho_o = 0 \quad \text{in local form} \tag{4}$$

Fluids

The Jacobian for dilatation is

$$J = \frac{d\Omega}{d\Omega_o} \tag{5}$$

$$d\Omega_o = \frac{1}{J}\, d\Omega \tag{6}$$

$$\frac{1}{J}\int_{\Omega}\left(\rho J-\rho_o\right)d\Omega=0 \tag{7}$$

Take the material derivative.

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{\partial}{\partial z_i}\frac{\partial z_i}{\partial t}=\frac{\partial}{\partial t}+\frac{\partial}{\partial z_i}V_i \tag{8}$$

$$\frac{D}{Dt}\left\{\frac{1}{J}\int_{\Omega}\left(\rho J-\rho_o\right)d\Omega\right\}=0 \tag{9}$$

$$\frac{D}{Dt}\left\{\int_{\Omega}\left(\rho-\frac{1}{J}\rho_o\right)d\Omega\right\}=0 \tag{10}$$

$$\left\{\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial z_i}\right\}\left\{\frac{1}{J}\int_{\Omega}\left(\rho J-\rho_o\right)d\Omega\right\}=0 \tag{11}$$

$$\int_{\Omega}\left(\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial z_i}\left(\rho V_i\right)\right)d\Omega=0 \quad \text{in global form} \tag{12}$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial z_i}\left(\rho V_i\right)=0 \quad \text{in local form} \tag{13}$$