THE DISPLACEMENT GRADIENT AND THE LAGRANGIAN STRAIN TENSOR
Revision B

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Displacement Gradient

Suppose a body having a particular configuration at some reference time t_o changes to another configuration at time t, with both rigid body motion and elastic deformation possible.

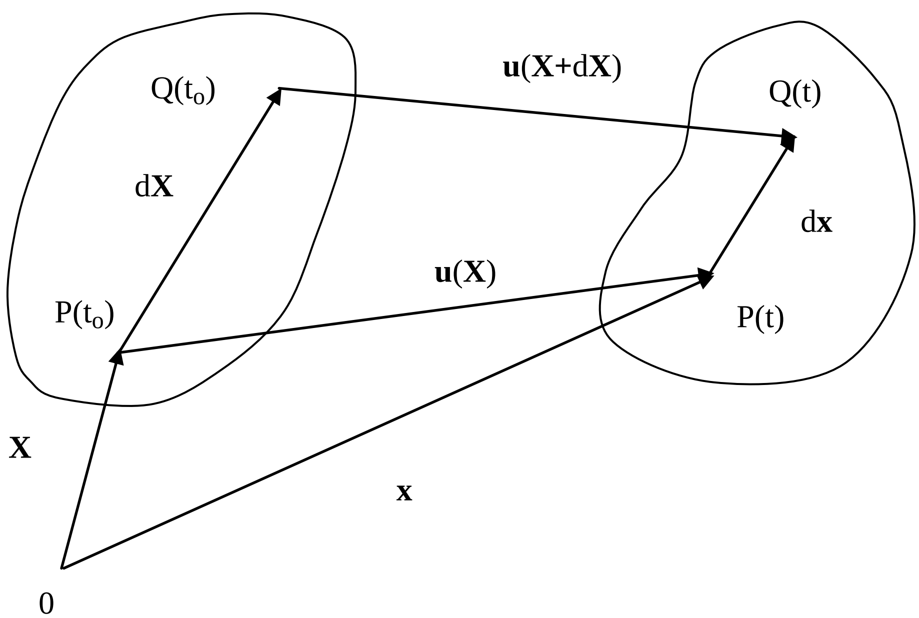

Figure 1.

A typical material point P undergoes a displacement **u** so that it arrives at the position

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t) \tag{1}$$

A neighboring point Q at **X** + d**X** arrives at

$$\mathbf{x} + d\mathbf{x} = \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, \mathbf{t}) \tag{2}$$

Thus

$$d\mathbf{x} = \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, \mathbf{t}) - \mathbf{x} \tag{3}$$

$$d\mathbf{x} = \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, \mathbf{t}) - \left[\mathbf{X} + \mathbf{u}(\mathbf{X}, t)\right] \tag{4}$$

$$d\mathbf{x} = d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, \mathbf{t}) - \mathbf{u}(\mathbf{X}, t) \tag{5}$$

This equation may be rewritten using the gradient of a vector field.

$$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X} \tag{6}$$

$\nabla\mathbf{u}$ is a second order tensor known as the displacement gradient with respect to **X**.

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{7}$$

Note that for rigid body translation $\nabla\mathbf{u} = 0$.

Lagrangian Strain Tensor

Consider two material vectors $d\mathbf{X}^1$ and $d\mathbf{X}^2$ issuing from point P in Figure 1.

Through the motion, $d\mathbf{X}^1$ becomes $d\mathbf{x}^1$ and $d\mathbf{X}^2$ becomes $d\mathbf{x}^2$.

$$d\mathbf{x}^1 = d\mathbf{X}^1 + (\nabla\mathbf{u})d\mathbf{X}^1 \tag{8}$$

$$d\mathbf{x}^2 = d\mathbf{X}^2 + (\nabla\mathbf{u})d\mathbf{X}^2 \tag{9}$$

A measure of the deformation is given by the dot product of $d\mathbf{x}^1$ and $d\mathbf{x}^2$.

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = \left[d\mathbf{X}^1 + (\nabla\mathbf{u})d\mathbf{X}^1\right] \bullet \left[d\mathbf{X}^2 + (\nabla\mathbf{u})d\mathbf{X}^2\right] \tag{10}$$

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = d\mathbf{X}^1 \bullet d\mathbf{X}^2 + d\mathbf{X}^1 \bullet (\nabla\mathbf{u})d\mathbf{X}^2 + d\mathbf{X}^2 \bullet (\nabla\mathbf{u})d\mathbf{X}^1 + \left\{(\nabla\mathbf{u})d\mathbf{X}^1\right\} \bullet \left\{(\nabla\mathbf{u})d\mathbf{X}^2\right\} \tag{11}$$

By the definition of the transpose,

$$d\mathbf{X}^2 \bullet (\nabla\mathbf{u})d\mathbf{X}^1 = d\mathbf{X}^1 \bullet (\nabla\mathbf{u})^T d\mathbf{X}^2 \tag{12}$$

And

$$\left\{(\nabla\mathbf{u})d\mathbf{X}^1\right\} \bullet \left\{(\nabla\mathbf{u})d\mathbf{X}^2\right\} = d\mathbf{X}^1 \bullet (\nabla\mathbf{u})^T (\nabla\mathbf{u})d\mathbf{X}^2 \tag{13}$$

By substitution,

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = d\mathbf{X}^1 \bullet d\mathbf{X}^2 + d\mathbf{X}^1 \bullet (\nabla\mathbf{u})d\mathbf{X}^2 + d\mathbf{X}^1 \bullet (\nabla\mathbf{u})^T d\mathbf{X}^2 + d\mathbf{X}^1 \bullet (\nabla\mathbf{u})^T (\nabla\mathbf{u})d\mathbf{X}^2 \tag{14}$$

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = d\mathbf{X}^1 \bullet d\mathbf{X}^2 + d\mathbf{X}^1 \bullet \left[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T (\nabla\mathbf{u})\right] d\mathbf{X}^2 \tag{15}$$

Define the Lagrangian strain tensor as

$$\mathbf{E}^* = \frac{1}{2}\left[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T(\nabla\mathbf{u}) \right] \tag{16}$$

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = d\mathbf{X}^1 \bullet d\mathbf{X}^2 + 2\, d\mathbf{X}^1 \bullet [E^*] d\mathbf{X}^2 \tag{17}$$

$\mathbf{E}^*$ characterizes the deformation of the neighborhood of the particle P.

Note that for rigid body translation:

$$\nabla\mathbf{u} = 0 \tag{18}$$

$$\mathbf{E}^* = 0 \tag{19}$$

$$d\mathbf{x}^1 \bullet d\mathbf{x}^2 = d\mathbf{X}^1 \bullet d\mathbf{X}^2 \tag{20}$$

For Cartesian coordinates

$$\mathbf{E}^* = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} \right] \tag{21}$$

For small deformations $(\nabla\mathbf{u})^T(\nabla\mathbf{u}) \approx 0$.

$$\mathbf{E} = \frac{1}{2}\left[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T \right] \tag{22}$$

The omission of the * symbol indicates small deformations.

For small deformations in Cartesian coordinates,

$$\mathbf{E} = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right] \tag{23}$$

The infinitesimal strain tensor is

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ & & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \quad \text{, symmetric}$$

(24)

The unit elongation in the x_1 direction is

$$E_{11} = \mathbf{e}_1 \bullet \mathbf{E}\mathbf{e}_1 \tag{25}$$

<u>References</u>

1. Lai, Rubin, Krempl, Introduction to Continuum Mechanics, Revised Edition in SI/Metric Units, Pergamon Unified Engineering Series, Volume 17, New York, 1982

2. Hjelmstad, Fundamentals of Structral Mechanics, Second Edition, Springer, New York, 2005.

3. Chung, General Continuum Mechanics, Cambridge University Press, 2007.

APPENDIX A

Deformation Gradient

$\mathbf{x}$	is the coordinate in the deformed configuration
$\mathbf{z}$	is the coordinate in the undeformed configuration
C	is the curve
$\phi(\mathbf{z})$	is the deformation map

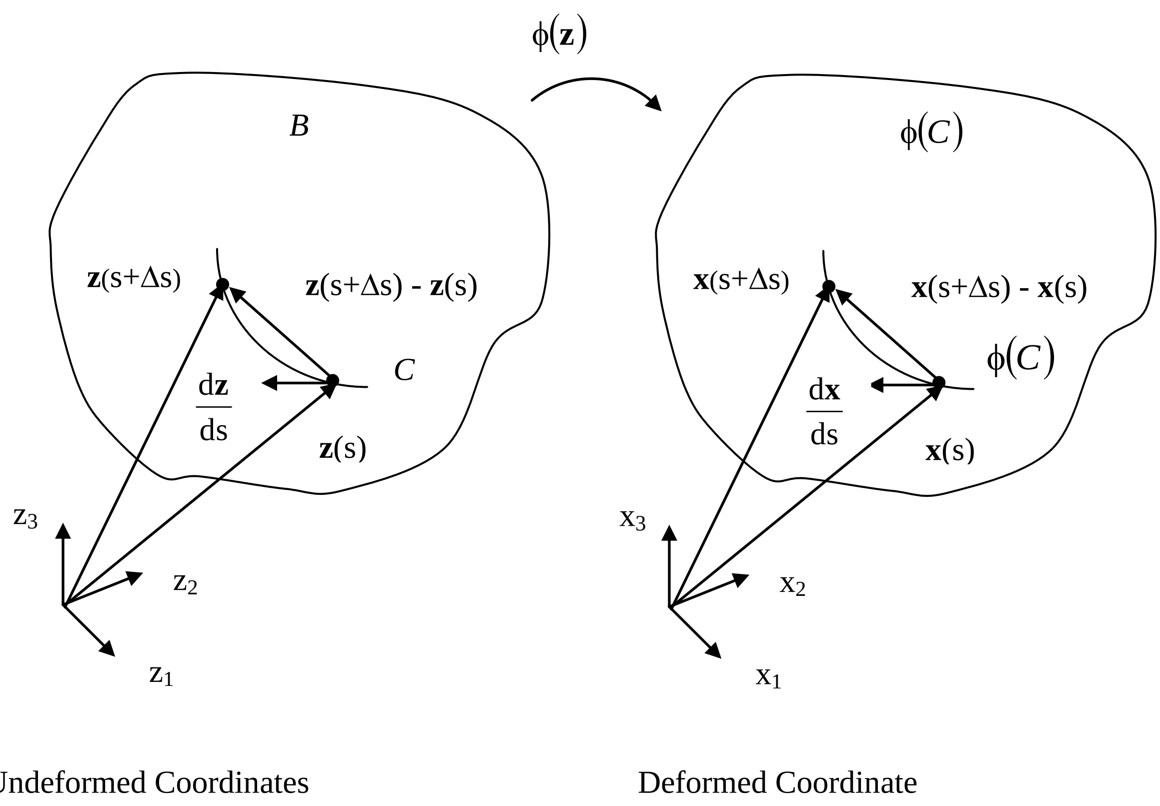

Figure A-1. Measuring the distance between two points on a curve

Noting that the parameterized curve in the deformed configuration is determined by the deformation maps as $\mathbf{x}(\mathbf{s}) = \boldsymbol{\phi}(\mathbf{Z}(s))$, one can apply the chain rule for differentiation to relate the vectors tangent to the curves in the deformed and undeformed configurations. In components, noting that the map can be written

$$x_i(s) = \phi_i\left(z_1(s), z_2(s), z_3(s)\right) \tag{A-1}$$

We can compute the derivative by the chain rule as

$$\frac{d}{ds} x_i(s) = \frac{\partial \phi_i(\mathbf{z})}{\partial z_j} \frac{dz_j(s)}{ds} \tag{A-2}$$

Note that the components $\dfrac{\partial \phi_i(\mathbf{z})}{\partial z_j}$ are simply the components of the tensor $\nabla \boldsymbol{\phi}$.

This tensor plays such an important role in the subsequent developments that we shall give it a special name and symbol.

We call $\mathbf{F}(\mathbf{z})$ the deformation gradient because it characterizes that rate of change of deformation with respect to material coordinates $\mathbf{z}$.

$$\mathbf{F}(\mathbf{z}) \equiv \nabla \boldsymbol{\phi}(\mathbf{z}) \tag{A-3}$$

Thus

$$\frac{d\mathbf{x}}{ds} = \mathbf{F} \frac{d\mathbf{z}}{ds} \tag{A-4}$$

The deformation gradient carries the information about the stretching in the infinitesimal neighborhood of the point $\mathbf{z}$. It also carries information about the rotation of the vector $d\mathbf{z}/ds$.

The deformation gradient F is a tensor with the coordinate representation.

$$\mathbf{F}(\mathbf{z}) \equiv \frac{\partial}{\partial z}\phi_i(\mathbf{z})\left[\mathbf{e}_i \otimes \mathbf{g}_j\right] \tag{A-5}$$

where

$\{\mathbf{e}_i\}$ are the base vectors in the deformed configuration

$\{\mathbf{g}_i\}$ are the base vectors in the undeformed configuration

Strain in Three-dimensional Bodies

The Green deformation tensor **C** is

$$\mathbf{C} \equiv \mathbf{F}^T\mathbf{F} \tag{A-6}$$

The stretch of the line oriented in the direction **n** of the undeformed configuration can then be computed as

$$\lambda^2(\mathbf{n}) = \mathbf{n} \bullet \mathbf{C}\mathbf{n} \tag{A-7}$$

Equation (A-7) holds for any curve with d**z**/ds = **n**.

The Lagrangian strain is the difference between the square of the deformed length and the square of the original length divided by twice the square of the original length.

The strain E in the direction **n** is

$$E(\mathbf{n}) = \frac{1}{2}\left[\lambda^2(\mathbf{n}) - 1\right] \equiv \mathbf{n} \bullet \mathbf{E}\mathbf{n} \tag{A-8}$$

The Lagrangian strain tensor **E** is defined to be have the difference between the Green deformation tensor and the identity tensor **I** as

$$\mathbf{E} \equiv \frac{1}{2}\left[\mathbf{C} - \mathbf{I}\right] \tag{A-9}$$

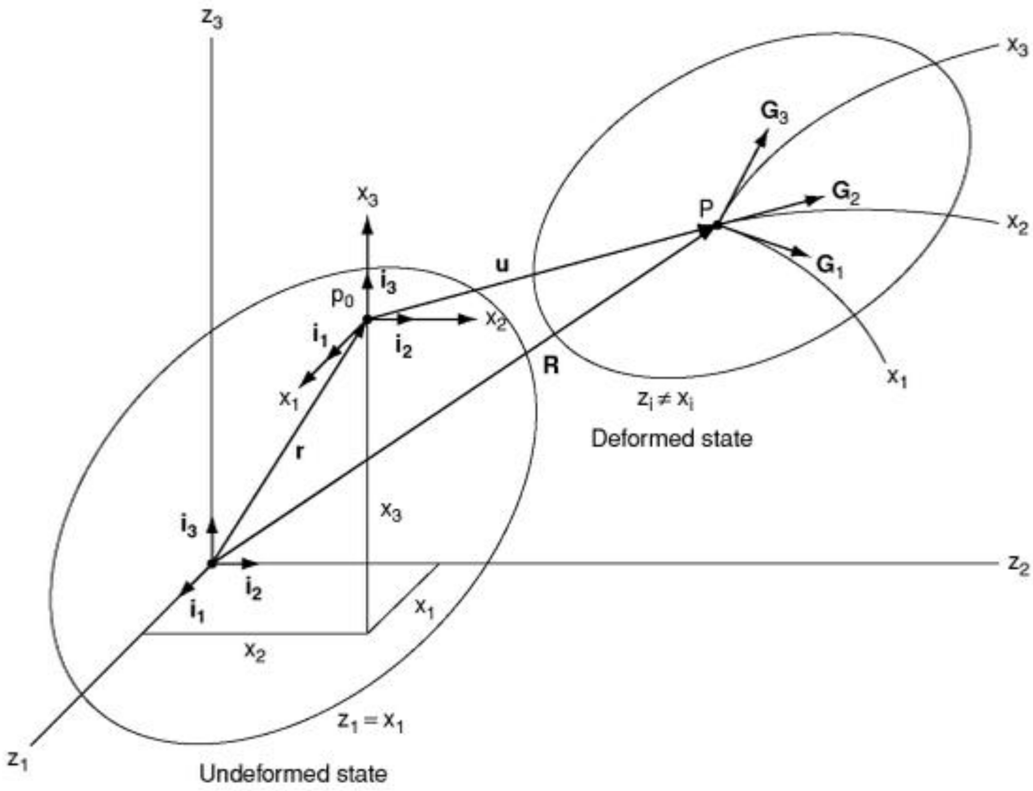

Figure B-1. Lagrangian Coordinates

The undeformed state is defined by rectangular Cartesian coordinates, and the deformed state by arbitrary curvilinear (convected) coordinates.

Strain Tensor in Solids

$$ds^2 - ds_0^2 = \left(G_{ij} - \delta_{ij}\right) dx_i dx_j = 2\gamma_{ij} dx_i dx_j \tag{B-1}$$

The metric tensor G_{ij} is

$$G_{ij} = \mathbf{G}_i \bullet \mathbf{G}_j = \frac{\partial \mathbf{R}}{\partial x_i} \bullet \frac{\partial \mathbf{R}}{\partial x_j} = \frac{\partial z_m}{\partial x_i} \frac{\partial z_m}{\partial x_j} \tag{B-2}$$

where

$\mathbf{G}_i$ is a tangent vector

$\mathbf{R}$ is the position vector to the point *P* in the deformed state at time t=t

Note that

$$\mathbf{G}_i = \left(\delta_{ki} + \frac{\partial u_k}{\partial x_i} \right) \mathbf{i}_k \tag{B-3}$$

$$\mathbf{G}_i = \frac{\partial z_m}{\partial x_i} \mathbf{i}_m \tag{B-4}$$

$$\mathbf{R} = \mathbf{R}(\mathbf{r}, t) \tag{B-5}$$

where

$\mathbf{r}$ is the position vector from the origin of the rectangular Cartesian coordinates to a point P_o at time $t=t_o$ in the undeformed state

$$\mathbf{r} = x_i\, \mathbf{i}_i \qquad \text{(undeformed)} \tag{B-6}$$

$$\mathbf{R} = z_i\, \mathbf{i}_i \qquad \text{(deformed)} \tag{B-7}$$

The displacement vector **u** is

$$\mathbf{u} = \mathbf{R} - \mathbf{r} \tag{B-8}$$

Jacobian

The Jacobian J is

$$J = \left|\frac{\partial z_i}{\partial x_j}\right| = \begin{vmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_2} \\ \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_2} \end{vmatrix} \tag{B-9}$$

$$J = \sqrt{G} = \sqrt{\left| G_{ij} \right|} \tag{B-10}$$

The Jacobian for dilatation is

$$J = \frac{d\Omega}{d\Omega_o} \tag{B-11}$$

APPENDIX C

Finite Strain Tensor

$$\mathbf{H}^* = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \right]$$

$$H_{11}^* = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\frac{\partial u_k}{\partial X_1} \right]$$

$$= \frac{1}{2}\left[2\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_1} \right]$$

$$H_{12}^* = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\frac{\partial u_k}{\partial X_2} \right]$$

$$= \frac{1}{2}\left[\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2} \right]$$

$$H_{13}^* = \frac{1}{2}\left[\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_k}{\partial X_1}\frac{\partial u_k}{\partial X_3} \right]$$

$$= \frac{1}{2}\left[\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3} \right]$$

$$H_{22}^* = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_2} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\frac{\partial u_k}{\partial X_2} \right]$$

$$= \frac{1}{2}\left[2\frac{\partial u_2}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_2} \right]$$

$$H_{23}{}^{*} = \frac{1}{2}\left[\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_k}{\partial X_2}\frac{\partial u_k}{\partial X_3}\right]$$

$$H_{33}{}^{*} = \frac{1}{2}\left[\frac{\partial u_3}{\partial X_3} + \frac{\partial u_3}{\partial X_3} + \frac{\partial u_k}{\partial X_3}\frac{\partial u_k}{\partial X_3}\right]$$

$$= \frac{1}{2}\left[2\frac{\partial u_3}{\partial X_3} + \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_3}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_3}\frac{\partial u_3}{\partial X_3}\right]$$