Derivation of the Epsilon-Delta Relationship

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The following is based on a homework problem in Reference 1.

The Kronecker delta function is

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{1}$$

The permutation function epsilon can be expressed as the determinant of a matrix filled with Kronecker delta terms.

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix} \tag{2}$$

Now compute $\varepsilon_{inm}\varepsilon_{jpq}$.

Note that the determinant of a matrix is equal to the determinant of its transposed matrix so that

$$|A||B| = \left|A^T\right||B| = \left|A^T B\right| \tag{3}$$

where A and B are both 3x3 matrices.

Thus

$$\varepsilon_{inm}\varepsilon_{jpq} = \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \end{bmatrix} \begin{bmatrix} \delta_{j1} & \delta_{p1} & \delta_{q1} \\ \delta_{j2} & \delta_{p2} & \delta_{q2} \\ \delta_{j3} & \delta_{p3} & \delta_{q3} \end{bmatrix} \right| \tag{4}$$

$\varepsilon_{inm}\varepsilon_{jpq}$

$$= \begin{vmatrix} \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3} & \delta_{i1}\delta_{p1} + \delta_{i2}\delta_{p2} + \delta_{i3}\delta_{p3} & \delta_{i1}\delta_{q1} + \delta_{i2}\delta_{q2} + \delta_{i3}\delta_{q3} \\ \delta_{n1}\delta_{j1} + \delta_{n2}\delta_{j2} + \delta_{n3}\delta_{j3} & \delta_{n1}\delta_{p1} + \delta_{n2}\delta_{p2} + \delta_{n3}\delta_{p3} & \delta_{n1}\delta_{q1} + \delta_{n2}\delta_{q2} + \delta_{n3}\delta_{q3} \\ \delta_{m1}\delta_{j1} + \delta_{m2}\delta_{j2} + \delta_{m3}\delta_{j3} & \delta_{m1}\delta_{p1} + \delta_{m2}\delta_{p2} + \delta_{m3}\delta_{p3} & \delta_{m1}\delta_{q1} + \delta_{m2}\delta_{q2} + \delta_{m3}\delta_{q3} \end{vmatrix} \tag{5}$$

$$\varepsilon_{inm}\varepsilon_{jpq} = \begin{vmatrix} \delta_{ij} & \delta_{ip} & \delta_{iq} \\ \delta_{nj} & \delta_{np} & \delta_{nq} \\ \delta_{mj} & \delta_{mp} & \delta_{mq} \end{vmatrix} \tag{6}$$

$$\begin{aligned} \varepsilon_{inm}\varepsilon_{jpq} = \; & \delta_{ij}\delta_{np}\delta_{mq} + \delta_{ip}\delta_{nq}\delta_{mj} + \delta_{iq}\delta_{nj}\delta_{mp} \\ & - \delta_{ij}\delta_{mp}\delta_{nq} - \delta_{ip}\delta_{nj}\delta_{mq} - \delta_{iq}\delta_{np}\delta_{mj} \end{aligned} \tag{7}$$

Now let j=i.

$$\begin{aligned} \varepsilon_{inm}\varepsilon_{ipq} = \; & \delta_{ii}\delta_{np}\delta_{mq} + \delta_{ip}\delta_{nq}\delta_{mi} + \delta_{iq}\delta_{ni}\delta_{mp} \\ & - \delta_{ii}\delta_{mp}\delta_{nq} - \delta_{ip}\delta_{ni}\delta_{mq} - \delta_{iq}\delta_{np}\delta_{mi} \end{aligned} \tag{8}$$

Note that

$$\delta_{ii} = 3 \tag{9}$$

By substitution,

$$\begin{aligned} \varepsilon_{inm}\varepsilon_{ipq} = \; & 3\delta_{np}\delta_{mq} + \delta_{mp}\delta_{nq} + \delta_{nq}\delta_{mp} \\ & - 3\delta_{mp}\delta_{nq} - \delta_{np}\delta_{mq} - \delta_{mq}\delta_{np} \end{aligned} \tag{10}$$

$$\varepsilon_{inm}\varepsilon_{ipq} = \delta_{np}\delta_{mq} - \delta_{mp}\delta_{nq} \tag{11}$$

Now let p = n.

$$\varepsilon_{inm}\varepsilon_{inq} = \delta_{nn}\delta_{mq} - \delta_{mn}\delta_{nq} \tag{12}$$

$$\varepsilon_{inm}\varepsilon_{inq} = 3\delta_{mq} - \delta_{mq} \tag{13}$$

$$\varepsilon_{inm}\varepsilon_{inq} = 2\delta_{mq} \tag{14}$$

Now let q=m, since $\delta_{mq} = 0$ for $m \neq q$.

$$\varepsilon_{inm}\varepsilon_{inm} = 2\delta_{mm} = 6 \tag{15}$$

Reference

1. T. J. Chung, General Continuum Mechanics, Cambridge, New York, 2007.