MATERIAL DERIVATIVE

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Introduction

The rate of change of a quantity such as temperature or velocity of a material particle is known as the material derivative.

The material coordinates X are referenced to some fixed point 0. They remain with the particle.

The spatial coordinates x are occupied by different particles at different times.

Lagrangian or Material Description

Consider θ in a material description,

$$\theta = \theta\left(X_1, X_2, X_3, t\right) \tag{1}$$

The material derivative is

$$\frac{D\theta}{Dt} = \left.\frac{\partial\theta}{\partial t}\right|_{X_i \text{ fixed}} \tag{2}$$

The velocity of particle **X** is

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt} = \left.\frac{\partial\mathbf{x}}{\partial t}\right|_{X_i \text{ fixed}} \tag{3}$$

The acceleration of a particle in a given velocity field is

$$\mathbf{a} = \left.\frac{\partial\mathbf{v}}{\partial t}\right|_{X_i \text{ fixed}} \tag{4}$$

Eulerian or Spatial Description

Consider θ in a spatial description,

$$\theta = \theta(x_1, x_2, x_3, t) \tag{5}$$

The material derivative is

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \mathbf{v} \bullet \nabla \theta \tag{6}$$

This is also known as the substantial derivative.

The velocity field in spatial coordinates is

$$\mathbf{v} = v_1(x_1, x_2, x_3, t)\mathbf{i}_1 + v_2(x_1, x_2, x_3, t)\mathbf{i}_2 + v_3(x_1, x_2, x_3, t)\mathbf{i}_3 \tag{7}$$

The acceleration of a particle in a given velocity field is

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \bullet \nabla \mathbf{v} \tag{8}$$