Position Vectors
Revision B

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## Position Vectors

Let $\mathbf{r} = x_i \, \mathbf{i}_i$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

$$\nabla \bullet \mathbf{r} = \frac{\partial}{\partial x_i} \mathbf{i}_i \bullet x_j \mathbf{i}_j = \frac{\partial}{\partial x_i} x_j \delta_{ij} = x_{i,i} = 3$$

$$\nabla \text{ x } \mathbf{r} = \frac{\partial}{\partial x_i} \mathbf{i}_i \text{ x } \left( x_j \mathbf{i}_j \right) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} x_j \, \mathbf{i}_k = \varepsilon_{ijk} x_{j,i} \, \mathbf{i}_k = 0$$

$$\nabla \mathbf{r} = x_{i,j} \, \mathbf{i}_i \otimes \mathbf{i}_j = \begin{bmatrix} x_{i,i} & x_{i,j} & x_{i,k} \\ x_{j,i} & x_{j,j} & x_{j,k} \\ x_{k,i} & x_{k,j} & x_{k,k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## APPENDIX A

### Sample Problems

2. [20] Consider a *three*-dimensional scalar field $\phi(\mathbf{r}) = R^m$, with the **position vector** given by $\mathbf{r} = x_i \mathbf{i}_i$, $i = 1,2,3$ and $R \equiv \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x_i x_i}$ is the length of vector $\mathbf{r}$ ( i.e., the distance to the origin $O$) and $m$ is a constant.

Determine the following expressions (your answers need to be functions of distance $R$, position vector $\mathbf{r}$, and the constant $m$ ):

a. $\mathbf{v} = \nabla\phi$;
b. $\nabla \times (\nabla\phi)$;
c. $\nabla^2\phi$;
d. $\nabla\mathbf{v} = \nabla(\nabla\phi)$, the gradient of vector $\mathbf{v} = \nabla\phi$.

a)

$$\mathbf{v} = \nabla\phi = \nabla R^m = \mathbf{i}_i \frac{\partial}{\partial x_i}\left(R^m\right) = m R^{m-1} \frac{\partial R}{\partial x_i} \mathbf{i}_i$$

$$\frac{\partial R}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{x_i x_i} = \frac{1}{2}\frac{2x_i}{\sqrt{x_i x_i}} = \frac{x_i}{\sqrt{x_i x_i}} = \frac{x_i}{R}$$

$$\mathbf{v} = m R^{m-1} \frac{x_i}{R} \mathbf{i}_i = m R^{m-2}\, \mathbf{r}$$

b)

$$\nabla x(\nabla\phi) = \nabla x\left(\phi_{,j}\, \mathbf{i}_j\right)$$

$$\nabla x(\nabla\phi) = \nabla x\left(\phi_{,j}\, \mathbf{i}_j\right)$$

$$\nabla x(\nabla\phi) = \frac{\partial}{\partial x_i}\left(\phi_{,j}\right)\varepsilon_{ijk}\mathbf{i}_k$$

$$\nabla x(\nabla\phi) = \left(\phi_{,ji}\right)\varepsilon_{ijk}\mathbf{i}_k = 0$$

c)

$$\nabla^2\phi = \nabla \bullet \nabla\phi$$
$$= \mathbf{i}_j \frac{\partial}{\partial x_j} \bullet \left(m R^{m-2}\, \mathbf{r}\right)$$
$$= \mathbf{i}_j \frac{\partial}{\partial x_j} \bullet \left(m R^{m-2}\, x_i \mathbf{i}_i\right)$$
$$= \frac{\partial}{\partial x_j} \bullet \left(m R^{m-2}\, x_i\right)\delta_{ij}$$
$$= \frac{\partial}{\partial x_i} \left(m R^{m-2}\, x_i\right)$$
$$= x_i \frac{\partial}{\partial x_i}\left(m R^{m-2}\right) + m R^{m-2} \frac{\partial xi}{\partial x_i}$$
$$= x_i\, m(m-2)R^{m-3} \frac{\partial R}{\partial x_i} + 3 m R^{m-2}$$
$$= x_i\, m(m-2)R^{m-3} \frac{x_i}{R} + 3 m R^{m-2}$$
$$= m(m-2)R^{m-3} \frac{x_i x_i}{R} + 3 m R^{m-2}$$
$$= m(m-2)R^{m-3} \frac{R^2}{R} + 3 m R^{m-2}$$
$$= m(m-2)R^{m-3}\, R + 3 m R^{m-2}$$
$$= m(m-2)R^{m-2} + 3 m R^{m-2}$$
$$= \left[(m-2)+3\right] mR^{m-2}$$
$$= \left[m+1\right] mR^{m-2}$$

d)

$$\nabla\mathbf{v} = \nabla(\nabla\phi)$$

$$\nabla\mathbf{v} = \nabla\left(m R^{m-2}\,\mathbf{r}\right)$$

$$\nabla\mathbf{v} = \nabla\left(m R^{m-2}\right)\otimes\mathbf{r} + m R^{m-2}\,\nabla\mathbf{r}$$

$$\nabla\mathbf{r} = \mathbf{I}$$

$$\nabla\mathbf{v} = \mathbf{i}_i\,\frac{\partial}{\partial x_i}\left(m R^{m-2}\right)\otimes\mathbf{r} + m R^{m-2}\,\mathbf{I}$$

$$\nabla\mathbf{v} = \left(m(m-2)R^{m-3}\,\frac{\partial R}{\partial x_i}\,\mathbf{i}_i\right)\otimes\mathbf{r} + m R^{m-2}\,\mathbf{I}$$

$$\nabla\mathbf{v} = \left(m(m-2)R^{m-3}\,\frac{x_i}{R}\,\mathbf{i}_i\right)\otimes\mathbf{r} + m R^{m-2}\,\mathbf{I}$$

$$\nabla\mathbf{v} = \left(m(m-2)R^{m-3}\,\frac{\mathbf{r}}{R}\right)\otimes\mathbf{r} + m R^{m-2}\,\mathbf{I}$$

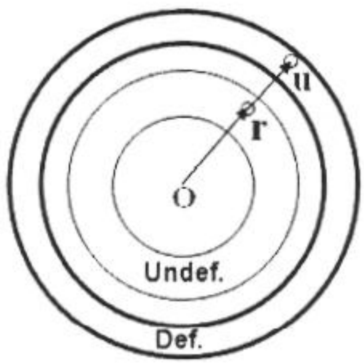
$$\nabla\mathbf{v} = \left(m(m-2)R^{m-2}\,\frac{\mathbf{r}}{R}\right)\otimes\frac{\mathbf{r}}{R} + m R^{m-2}\,\mathbf{I}$$

$$\nabla\mathbf{v} = m R^{m-2}\left\{(m-2)\,\frac{\mathbf{r}}{R}\otimes\frac{\mathbf{r}}{R} + \mathbf{I}\right\}$$

APPENDIX B

Sphere Problem

4. [40] The radial expansion of a *hollow* sphere is shown below (think of a balloon being expanded). Suppose the *displacement* field is found (given) as $\mathbf{u}(\mathbf{r},t)=f(r,t)\mathbf{r}$, where $r\equiv\sqrt{\mathbf{r}\cdot\mathbf{r}}$ (i.e. $r=\sqrt{x_i x_i}$ in component form) is the distance to the origin $O$; $f(r,t)$ is some given function of the distance $r$ and time $t$. Note, the displacement is *radially* outward, as indicated.




(a) Construct the deformation map; that is, find the expression for $\mathbf{R}=\mathbf{R}(\mathbf{r},t)$; (Your answer should be in terms of $f(r,t)$, and the **position vector** $\mathbf{r}=x_i\mathbf{i}_i$.)

For convenience you may use $\lambda(r,t)=1+f(r,t)$ in the following:

(b) compute the *deformation gradient* tensor, $\mathbf{F}=F_{ij}\mathbf{i}_i\otimes\mathbf{i}_j$.

(c) compute the *metric tensor* $\mathbf{G}\equiv\mathbf{F}^T\mathbf{F}$;

(d) compute the *finite* and *infinitesimal* strain *tensors*;

(e) compute the volume ratio, $J=\det\mathbf{F}$. Hint: a unit vector along $\mathbf{r}$, i.e., $\mathbf{n}=\mathbf{r}/r$, is a principal direction (eigenvector) of $\mathbf{F}$; express $\mathbf{F}$ in its principal basis.

a) Deformation map

$$R(\underline{r},t)=\underline{r}+u(\underline{r},t)$$

$$R(\underline{r},t)=\underline{r}+f(r,t)\underline{r}$$

$$R(\underline{r},t)=[1+f(r,t)]\underline{r}$$

$$R(\underline{r},t) = \lambda(r,t)\,\underline{r}$$

b) Deformation gradient tensor

$$F = \nabla R(\underline{r},t)$$

$$F = \nabla\{\lambda(r,t)\,\underline{r}\}$$

$$F = \underline{r} \otimes \nabla\{\lambda\} + \lambda\nabla\{\underline{r}\}$$

$$F = \underline{r} \otimes \nabla\{1 + f\} + \lambda\,\underline{\underline{I}}$$

$$F = \underline{r} \otimes \nabla f + \lambda\,\underline{\underline{I}}$$

$$F = \underline{r} \otimes [f'\ \nabla r] + \lambda\,\underline{\underline{I}}$$

$$\nabla r = \frac{\underline{r}}{r}$$

$$F = \underline{r} \otimes \left[f'\ \frac{\underline{r}}{r}\right] + \lambda\,\underline{\underline{I}}$$

$$F = \frac{f'}{r}\,\underline{r} \otimes \underline{r} + \lambda\,\underline{\underline{I}}$$

c) Metric tensor

$$\underline{\underline{G}} = \underline{\underline{F}}^T\,\underline{\underline{F}}$$

$$\underline{\underline{G}} = \left[\frac{f'}{r}\,\underline{r} \otimes \underline{r} + \lambda\,\underline{\underline{I}}\right]\left[\frac{f'}{r}\,\underline{r} \otimes \underline{r} + \lambda\,\underline{\underline{I}}\right]$$

$$\underline{\underline{G}} = \left[\frac{f'}{r}\,\underline{r} \otimes \underline{r}\right]\left[\frac{f'}{r}\,\underline{r} \otimes \underline{r}\right] + 2\lambda\frac{f'}{r}\,\underline{r} \otimes \underline{r} + \lambda^2\,\underline{\underline{I}}$$

$$\underline{\underline{G}} = \left[\frac{f'}{r}\right]^2 [\, \underline{r} \otimes \underline{r} \,][\, \underline{r} \bullet \underline{r} \,] + 2\lambda \frac{f'}{r} \underline{r} \otimes \underline{r} + \lambda^2 \underline{\underline{I}}$$

d) Finite & infinitesimal strain tensors

$$\begin{aligned} \nabla \mathbf{u} &= \nabla(f\, \mathbf{r}) \\ &= f\, \nabla \mathbf{r} + \mathbf{r} \nabla f \\ &= f\, \underline{\underline{I}} + f' \frac{\mathbf{r}}{r} \otimes \mathbf{r} \end{aligned}$$

Finite

$$\mathbf{E}^* = \frac{1}{2}\left[ (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T (\nabla \mathbf{u}) \right]$$

Infinitesmial

$$\mathbf{E} = \frac{1}{2}\left[ (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \right]$$