SIMPLE DROP SHOCK Revision D

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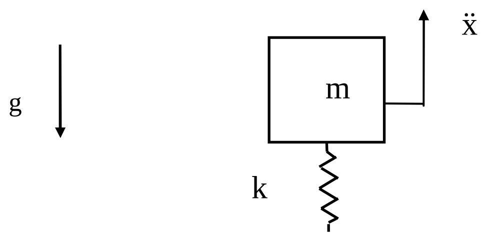
DERIVATION

Consider a single-degree-of-freedom system in a free-fall due to gravity.



Where

m is the mass,
k is the spring stiffness,
x is the absolute displacement of the mass,
g is the gravitational acceleration.

Note that the double-dot denotes acceleration.

Assume

1. The object can be modeled as a single-degree-of-freedom system.
2. The object is dropped from rest.
3. There is no energy dissipation. The collision is perfectly elastic.
4. The object remains attached to the floor via the spring after initial contact.
5. The object freely vibrates at its natural frequency after contact.
6. The system has a linear response.

The initial velocity of the system as it strikes the ground can be found by equating the change in kinetic energy with the change in potential energy.

$$\frac{1}{2} m\dot{x}^2 = mg\Delta h \tag{1}$$

where Δh is the drop height.

Dividing through by (m/2),

$$\dot{x}^2 = 2g\Delta h \tag{2}$$

Thus, the initial velocity when the mass encounters the floor is

$$\dot{x}(0) = \sqrt{2g\Delta h} \tag{3}$$

Furthermore, the initial displacement is taken as zero.

$$x(0) = 0 \tag{4}$$

Assume a displacement equation with constant coefficients a and b.

$$x(t) = a \sin \omega_n t + b \cos \omega_n t \tag{5a}$$

Equation (5a) assumes oscillation at the system's natural frequency. Note that the natural frequency in radians per time is

$$\omega_n = \sqrt{\frac{k}{m}} \tag{5b}$$

The zero displacement initial condition requires b = 0. Thus

$$x(t) = a \sin \omega_n t \tag{6}$$

The velocity is

$$\dot{x}(t) = a\,\omega_n \cos \omega_n t \tag{7}$$

Recall

$$\dot{x}(0) = \sqrt{2g\Delta h} \tag{8}$$

Thus

$$a = \left[\frac{\sqrt{2g\Delta h}}{\omega_n} \right] \tag{9}$$

Substitute equation (9) into (6). The resulting displacement is

$$x(t) = \left[\frac{\sqrt{2g\Delta h}}{\omega_n}\right]\sin\omega_n t \tag{10}$$

The velocity equation is

$$\dot{x}(t) = \sqrt{2g\Delta h}\cos\omega_n t \tag{11}$$

The acceleration equation is

$$\ddot{x}(t) = -\omega_n \sqrt{2g\Delta h}\sin\omega_n t \tag{12}$$

The force transmitted through the spring f(t) is

$$f(t) = kx \tag{13}$$

$$f(t) = k\left[\frac{\sqrt{2g\Delta h}}{\omega_n}\right]\sin\omega_n t \tag{14}$$

Note that

$$k = \omega_n^{\,2}\, m \tag{15}$$

Substitute equation (15) into (14).

$$f(t) = -\omega_n m\sqrt{2g\Delta h}\sin\omega_n t \tag{16}$$

The peak instantaneous power flow is

$$P = \omega_n\, mg\,\Delta h \tag{17}$$

Equation (17) is derived in Appendix A. The total impulse over a half-sine period is

$$I = 2m\sqrt{2g\,\Delta h} \tag{18}$$

Equation (18) is derived in Appendix B.

The shock analysis is only concerned with the maximum values. These are summarized in Table 1.

Table 1. Maximum Absolute Values			
Parameter	Symbol	Maximum	Equivalent Form
Displacement	x	$\frac{\sqrt{2g\Delta h}}{\omega_n}$	$\sqrt{\frac{2mg\Delta h}{k}}$
Velocity	$\dot{x}$	$\sqrt{2g\Delta h}$	$\sqrt{2g\Delta h}$
Acceleration	$\ddot{x}$	$\omega_n \sqrt{2g\Delta h}$	$\sqrt{\frac{2gk\Delta h}{m}}$
Jerk	$\dddot{x}$	$\omega_n^{\,2} \sqrt{2g\Delta h}$	$\left[\frac{k}{m}\right]\sqrt{2g\Delta h}$
Transmitted Force	f	$m\omega_n \sqrt{2g\Delta h}$	$\sqrt{2mgk\Delta h}$
Potential Energy of Spring	PE max	$\frac{k}{\omega_n^{\,2}} g\Delta h$	$mg\Delta h$
Kinetic Energy of Mass	KE max	$\frac{k}{\omega_n^{\,2}} g\Delta h$	$mg\Delta h$
Peak Instantaneous Power Flow	P	$\omega_n mg\Delta h$	$\sqrt{km}\, g\Delta h$
Total Impulse over a Half-sine Period	I	$2m\sqrt{2g\Delta h}$	$2m\sqrt{2g\Delta h}$

The values in Table 1 demonstrate that there are some tradeoffs involved in designing an object with respect to drop shock.

EXAMPLE

Consider a fixed drop height. Also consider that the mass is fixed, but that the spring stiffness is variable. The table values show that lowering the stiffness reduces both the acceleration and the force. A lower stiffness could be achieved by adding isolator mounts or some cushioning material. On the other hand, lowering stiffness also increases displacement. This is acceptable as long as the system remains linear and does not "bottom out."

The displacement limit of the spring is thus a practical constraint.

CONCLUSION

A simple method for modeling drop shock was derived. The derivation was based on a simplified free-vibration model.

A rigorous derivation of the free-vibration equation is given in Appendix C.

APPENDIX A

Again, the mass undergoes free vibration at its natural frequency. The instantaneous power flow is

$$P(t) = f(t)v(t) \tag{A-1}$$

$$P(t) = [m\omega_n \sqrt{2g\Delta h} \;\; \sin \omega_n t][\sqrt{2g\Delta h} \cos \omega_n t] \tag{A-2}$$

$$P(t) = 2mg\Delta h\omega_n \sin \omega_n t \cos \omega_n t \tag{A-3}$$

Apply a trigonometric identity.

$$P(t) = mg\Delta h\omega_n \sin 2\omega_n t \tag{A-4}$$

By inspection, the peak instantaneous power flow occurs at

$$2\omega_n t = \frac{\pi}{2} \tag{A-5}$$

$$t = \frac{\pi}{4\omega_n} \tag{A-6}$$

Substitute (A-5) into (A-4).

$$P_{max} = \omega_n mg\Delta h \tag{A-7}$$

$$P_{max} = \left[\sqrt{\frac{k}{m}}\right] mg\Delta h \tag{A-8}$$

$$P_{max} = g\Delta h\sqrt{km} \tag{A-9}$$

Note that the instantaneous power flow changes direction as the polarity changes with time in equation (A-4).

APPENDIX B

The total impulse over a half-sine duration is

$$I = \int_0^{\pi/\omega_n} [f(t)]dt \tag{B-1}$$

$$I = \int_0^{\pi/\omega_n} [m\omega_n \sqrt{2g\Delta h}\ \sin \omega_n t]dt \tag{B-2}$$

$$I = -m\omega_n \sqrt{2g\Delta h} \left[\frac{1}{\omega_n}\right] \cos \omega_n t \Big|_0^{\pi/\omega_n} \tag{B-3}$$

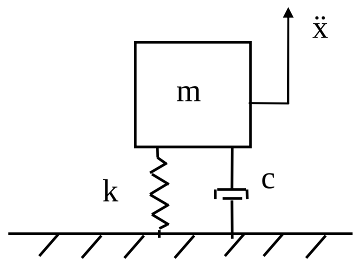
$$I = -m\sqrt{2g\Delta h}[-1-1] \tag{B-4}$$

$$I = 2m\sqrt{2g\Delta h} \tag{B-5}$$

APPENDIX C

FREE VIBRATION DERIVATION

Consider a single-degree-of-freedom system.

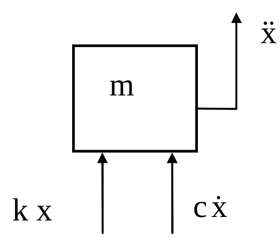


Where

m is the mass,
c is the viscous damping coefficient,
k is the stiffness,
x is the absolute displacement of the mass,
g is the gravitational acceleration.

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{C-1}$$

$$m\ddot{x} = -c\dot{x} - kx \tag{C-2}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{C-3}$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \tag{C-4}$$

By convention,

$$(c\,/\,m) = 2\xi\omega_n$$
$$(k\,/\,m) = \omega_n{}^2$$

where

ω_n is the natural frequency in (radians/sec),
ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n{}^2\,x = 0 \tag{C-5}$$

Now take the Laplace transform.

$$\mathcal{L}\left\{\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n{}^2\,x\right\} = \mathcal{L}\{0\} \tag{C-6}$$

$$\begin{aligned} &s^2\,X(s) - s\,x(0) - \dot{x}(0) \\ &+ 2\xi\omega_n s\,X(s) - 2\xi\omega_n x(0) \\ &+ \omega_n{}^2\,X(s) \qquad\qquad = 0 \end{aligned} \tag{C-7}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}X(s) + \{-1\}\dot{x}(0) + \left\{-s - 2\xi\omega_n\right\}x(0) = 0 \tag{C-8}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}X(s) = \dot{x}(0) + \left\{s + 2\xi\omega_n\right\}x(0) \tag{C-9}$$

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{C-10}$$

Consider the denominator of equation (C-10),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \tag{C-11}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2) \tag{C-12}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{C-13}$$

Substitute equation (C-13) into (C-12),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{C-14}$$

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{C-15}$$

$$X(s) = \left\{ \frac{(s + \xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{C-16}$$

$$X(s) = \left\{ \frac{(s + \xi\omega_n)x(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d} \right\}\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{C-17}$$

Now take the inverse Laplace transform using standard tables. The resulting displacement is

$$x(t) = \exp\left(-\xi\omega_n t\right)\left\{\left[x(0)\right]\cos\left(\omega_d t\right) + \left[\frac{\dot{x}(0) + \left(\xi\omega_n\right)x(0)}{\omega_d}\right]\sin\left(\omega_d t\right)\right\} \tag{C-18}$$

$$x(t) = \left[\frac{1}{\omega_d}\right]\exp\left(-\xi\omega_n t\right)\left\{\omega_d\left[x(0)\right]\cos\left(\omega_d t\right) + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\sin\left(\omega_d t\right)\right\} \tag{C-19}$$

The velocity is

$$\begin{aligned}\dot{x}(t) = &\left[\frac{-\xi\omega_n}{\omega_d}\right]\exp\left(-\xi\omega_n t\right)\left\{\omega_d\left[x(0)\right]\cos\left(\omega_d t\right) + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\sin\left(\omega_d t\right)\right\} \\ &+ \exp\left(-\xi\omega_n t\right)\left\{-\omega_d\left[x(0)\right]\sin\left(\omega_d t\right) + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\cos\left(\omega_d t\right)\right\}\end{aligned} \tag{C-20}$$

$$\begin{aligned}\dot{x}(t) = &\exp\left(-\xi\omega_n t\right)\left\{-\xi\omega_n\left[x(0)\right]\cos\left(\omega_d t\right) + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\left[\frac{-\xi\omega_n}{\omega_d}\right]\sin\left(\omega_d t\right)\right\} \\ &+ \exp\left(-\xi\omega_n t\right)\left\{-\omega_d\left[x(0)\right]\sin\left(\omega_d t\right) + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\cos\left(\omega_d t\right)\right\}\end{aligned} \tag{C-21}$$

$$\begin{aligned}\dot{x}(t) = &\exp\left(-\xi\omega_n t\right)\left\{-\xi\omega_n\left[x(0)\right] + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\right\}\cos\left(\omega_d t\right) \\ &+ \exp\left(-\xi\omega_n t\right)\left\{-\omega_d\left[x(0)\right] + \left[\dot{x}(0) + \left(\xi\omega_n\right)x(0)\right]\left[\frac{-\xi\omega_n}{\omega_d}\right]\right\}\sin\left(\omega_d t\right)\end{aligned} \tag{C-22}$$

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\right\}\cos\left(\omega_d\,t\right)$$
$$+\exp\left(-\xi\,\omega_n\,t\right)\left\{\left[\frac{-\xi\,\omega_n}{\omega_d}\right]\dot{x}(0) - \omega_d\left[x(0)\right] + \left[\left(\xi\,\omega_n\right)\right]\left[\frac{-\xi\,\omega_n}{\omega_d}\right]x(0)\right\}\sin\left(\omega_d\,t\right)$$

(C-23)

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\right\}\cos\left(\omega_d\,t\right)$$
$$+\exp\left(-\xi\,\omega_n\,t\right)\left\{\left[\frac{-\xi\,\omega_n}{\omega_d}\right]\dot{x}(0) + \left\{-\omega_d + \left[\left(\xi\,\omega_n\right)\right]\left[\frac{-\xi\,\omega_n}{\omega_d}\right]\right\}x(0)\right\}\sin\left(\omega_d\,t\right)$$

(C-24)

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\right\}\cos\left(\omega_d\,t\right)$$
$$+\exp\left(-\xi\,\omega_n\,t\right)\left\{\left[\frac{-\xi\,\omega_n}{\omega_d}\right]\dot{x}(0) + \left\{-\omega_d + \left[\frac{-\left(\xi\,\omega_n\right)^2}{\omega_d}\right]\right\}x(0)\right\}\sin\left(\omega_d\,t\right)$$

(C-25)

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\cos\left(\omega_d\,t\right) + \left\{\left[\frac{-\xi\,\omega_n}{\omega_d}\right]\dot{x}(0) + \left\{-\omega_d + \left[\frac{-\left(\xi\,\omega_n\right)^2}{\omega_d}\right]\right\}x(0)\right\}\sin\left(\omega_d\,t\right)\right\}$$

(C-26)

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\cos\left(\omega_d\,t\right) - \frac{1}{\omega_d}\left\{\xi\,\omega_n\dot{x}(0) + \left[\omega_d{}^2 + \left(\xi\,\omega_n\right)^2\right]x(0)\right\}\sin\left(\omega_d\,t\right)\right\}$$

(C-27)

Aside

$$\left[\omega_d^{\,2} + (\xi\,\omega_n)^2\right] = \left(1-\xi^2\right)\omega_n^{\,2} + (\xi\,\omega_n)^2 \tag{C-28}$$

$$\left[\omega_d^{\,2} + (\xi\,\omega_n)^2\right] = \omega_n^{\,2} \tag{C-29}$$

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\cos\left(\omega_d\,t\right) - \frac{1}{\omega_d}\left\{\xi\,\omega_n\dot{x}(0) + \omega_n^{\,2}x(0)\right\}\sin\left(\omega_d\,t\right)\right\}$$

(C-30)

$$\dot{x}(t) = \exp\left(-\xi\,\omega_n\,t\right)\left\{\dot{x}(0)\cos\left(\omega_d\,t\right) - \frac{\omega_n}{\omega_d}\left\{\xi\,\dot{x}(0) + \omega_n x(0)\right\}\sin\left(\omega_d\,t\right)\right\}$$

(C-31)

The acceleration is

$$\begin{aligned}\ddot{x}(t) &= \exp\left(-\xi\,\omega_n\,t\right)\left\{-\xi\,\omega_n\dot{x}(0)\cos\left(\omega_d\,t\right) + \frac{\xi\,\omega_n^{\,2}}{\omega_d}\left\{\xi\,\dot{x}(0) + \omega_n x(0)\right\}\sin\left(\omega_d\,t\right)\right\}\\ &\quad + \exp\left(-\xi\,\omega_n\,t\right)\left\{-\omega_d\dot{x}(0)\sin\left(\omega_d\,t\right) - \omega_n\left\{\xi\,\dot{x}(0) + \omega_n x(0)\right\}\cos\left(\omega_d\,t\right)\right\}\end{aligned}$$

(C-32)

$$\begin{aligned}\ddot{x}(t) &= \exp\left(-\xi\,\omega_n\,t\right)\left\{-\xi\,\omega_n\dot{x}(0)\cos\left(\omega_d\,t\right) - \omega_n\left\{\xi\,\dot{x}(0) + \omega_n x(0)\right\}\cos\left(\omega_d\,t\right)\right\}\\ &\quad + \exp\left(-\xi\,\omega_n\,t\right)\left\{-\omega_d\dot{x}(0)\sin\left(\omega_d\,t\right) + \frac{\xi\,\omega_n^{\,2}}{\omega_d}\left\{\xi\,\dot{x}(0) + \omega_n x(0)\right\}\sin\left(\omega_d\,t\right)\right\}\end{aligned}$$

(C-33)

$$\ddot{x}(t) = \exp\left(-\xi\omega_n t\right)\left\{-\xi\omega_n\dot{x}(0) - \omega_n\left\{\xi\dot{x}(0) + \omega_n x(0)\right\}\right\}\cos\left(\omega_d t\right)$$
$$+\exp\left(-\xi\omega_n t\right)\left\{-\omega_d\dot{x}(0) + \frac{\xi\omega_n^2}{\omega_d}\left\{\xi\dot{x}(0) + \omega_n x(0)\right\}\right\}\sin\left(\omega_d t\right)$$

(C-34)

$$\ddot{x}(t) = \exp\left(-\xi\omega_n t\right)\left\{-\xi\omega_n\dot{x}(0) - \omega_n\xi\dot{x}(0) - \omega_n^2 x(0)\right\}\cos\left(\omega_d t\right)$$
$$+\exp\left(-\xi\omega_n t\right)\left\{-\omega_d\dot{x}(0) + \frac{\xi^2\omega_n^2}{\omega_d}\dot{x}(0) + \frac{\xi\omega_n^3}{\omega_d}x(0)\right\}\sin\left(\omega_d t\right)$$

(C-35)

$$\ddot{x}(t) = \exp\left(-\xi\omega_n t\right)\left\{-2\xi\omega_n\dot{x}(0) - \omega_n^2 x(0)\right\}\cos\left(\omega_d t\right)$$
$$+\exp\left(-\xi\omega_n t\right)\left\{\left[\frac{\xi^2\omega_n^2}{\omega_d} - \omega_d\right]\dot{x}(0) + \frac{\xi\omega_n^3}{\omega_d}x(0)\right\}\sin\left(\omega_d t\right)$$

(C-36)

$$\ddot{x}(t) = \exp\left(-\xi\omega_n t\right)\left\{\omega_n\right\}\left\{-2\xi\dot{x}(0) - \omega_n x(0)\right\}\cos\left(\omega_d t\right)$$
$$+\exp\left(-\xi\omega_n t\right)\left\{\frac{1}{\omega_d}\right\}\left\{\left[\xi^2\omega_n^2 - \omega_d^2\right]\dot{x}(0) + \xi\omega_n^3 x(0)\right\}\sin\left(\omega_d t\right)$$

(C-37)

Furthermore

$$\left[-\omega_d^2 + \left(\xi\omega_n\right)^2\right] = -\left(1-\xi^2\right)\omega_n^2 + \left(\xi\omega_n\right)^2 \quad \text{(C-38)}$$

$$\left[-\omega_d^2 + \left(\xi\omega_n\right)^2\right] = \left(-1+\xi^2\right)\omega_n^2 + \left(\xi\omega_n\right)^2 \quad \text{(C-39)}$$

$$\left[-\omega_d^2 + \left(\xi\omega_n\right)^2\right] = \left(-1+2\xi^2\right)\omega_n^2 \quad \text{(C-40)}$$

By substitution,

$$\ddot{x}(t) = \exp\left(-\xi\,\omega_n\, t\right)\left\{\omega_n\right\}\left\{-2\,\xi\,\dot{x}(0) - \omega_n x(0)\right\}\cos\left(\omega_d\, t\right)$$
$$+\exp\left(-\xi\,\omega_n\, t\right)\left\{\frac{1}{\omega_d}\right\}\left\{\left[\left(-1+2\,\xi^2\right)\omega_n{}^2\right]\dot{x}(0) + \xi\,\omega_n{}^3\, x(0)\right\}\sin\left(\omega_d\, t\right)$$

(C-41)

$$\ddot{x}(t) = \exp\left(-\xi\,\omega_n\, t\right)\left\{\omega_n\right\}\left\{-2\,\xi\,\dot{x}(0) - \omega_n x(0)\right\}\cos\left(\omega_d\, t\right)$$
$$+\exp\left(-\xi\,\omega_n\, t\right)\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{\left[-1+2\,\xi^2\right]\dot{x}(0) + \xi\,\omega_n x(0)\right\}\sin\left(\omega_d\, t\right)$$

(C-42)

$$\ddot{x}(t) =$$
$$\omega_n \exp\left(-\xi\,\omega_n\, t\right)\left\{\left\{-2\,\xi\,\dot{x}(0) - \omega_n x(0)\right\}\cos\left(\omega_d\, t\right) + \left\{\frac{\omega_n}{\omega_d}\right\}\left\{\left[-1+2\,\xi^2\right]\dot{x}(0) + \xi\,\omega_n x(0)\right\}\sin\left(\omega_d\, t\right)\right\}$$

(C-43)