FREE VIBRATION OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM Revision A

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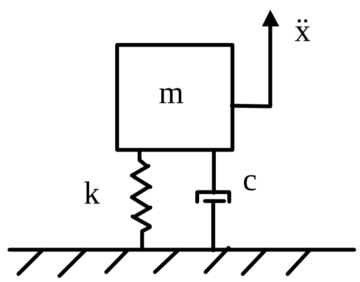
Derivation of the Equation of Motion

Consider a single-degree-of-freedom system.

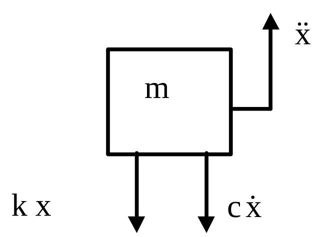


where

m is the mass,
c is the viscous damping coefficient,
k is the stiffness,
x is the absolute displacement of the mass.

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \qquad \text{(A-1)}$$

$$m\ddot{x} = -c\dot{x} - kx \qquad \text{(A-2)}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{A-3}$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \tag{A-4}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n^{\ 2}$$

where

ω_n is the natural frequency in (A-radians/sec),
ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{A-5}$$

Now take the Laplace transform.

$$\mathcal{L}\left\{\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^{\ 2}x\right\} = \mathcal{L}\{0\} \tag{A-6}$$

$$\begin{aligned} &s^2 X(s) - s\,x(0) - \dot{x}(0) \\ &+ 2\xi\omega_n s X(s) - 2\xi\omega_n x(0) \\ &+ \omega_n^{\ 2} X(s) \qquad\qquad = 0 \end{aligned} \tag{A-7}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^{\ 2}\right\}X(s) + \{-1\}\dot{x}(0) + \left\{-s - 2\xi\omega_n\right\}x(0) = 0 \tag{A-8}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^{\ 2}\right\}X(s) = \dot{x}(0) + \left\{s + 2\xi\omega_n\right\}x(0) \tag{A-9}$$

$$X(s) = \left\{\frac{\dot{x}(0) + \left\{s + 2\xi\omega_n\right\}x(0)}{s^2 + 2\xi\omega_n s + \omega_n^{\ 2}}\right\} \tag{A-10}$$

Consider the denominator of equation (A-10),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2 - \left(\xi\omega_n\right)^2 \tag{A-11}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2\left(1 - \xi^2\right) \tag{A-12}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{A-13}$$

Substitute equation (A-13) into (A-12),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_d^2 \tag{A-14}$$

$$X(s) = \left\{\frac{\dot{x}(0) + \left\{s + 2\xi\omega_n\right\}x(0)}{\left(s + \xi\omega_n\right)^2 + \omega_d^2}\right\} \tag{A-15}$$

$$X(s) = \left\{\frac{\left(s + \xi\omega_n\right)x(0)}{\left(s + \xi\omega_n\right)^2 + \omega_d^2}\right\} + \left\{\frac{\dot{x}(0) + \left(\xi\omega_n\right)x(0)}{\left(s + \xi\omega_n\right)^2 + \omega_d^2}\right\} \tag{A-16}$$

$$X(s) = \left\{\frac{\left(s + \xi\omega_n\right)x(0)}{\left(s + \xi\omega_n\right)^2 + \omega_d^2}\right\} + \left\{\frac{\left\{\frac{\dot{x}(0) + \left(\xi\omega_n\right)x(0)}{\omega_d}\right\}\omega_d}{\left(s + \xi\omega_n\right)^2 + \omega_d^2}\right\} \tag{A-17}$$

Oscillatory Motion

Now take the inverse Laplace transform using standard tables. Assume that $\xi < 1$. This case is referred to as oscillatory motion.

The resulting displacement is

$$x(t) = \exp(-\xi\omega_n t)\left\{[x(0)]\cos(\omega_d t) + \left[\frac{\dot{x}(0) + (\xi\omega_n)x(0)}{\omega_d}\right]\sin(\omega_d t)\right\}, \quad \xi < 1$$

(B-1)

An alternate form is

$$x(t) = \left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\{\omega_d[x(0)]\cos(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\}, \quad \xi < 1$$

(B-2)

The velocity is

$$\begin{aligned}\dot{x}(t) = &\left[\frac{-\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\{\omega_d[x(0)]\cos(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\} \\ &+ \exp(-\xi\omega_n t)\{-\omega_d[x(0)]\sin(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\cos(\omega_d t)\}, \quad \xi < 1\end{aligned}$$

(B-3)

$$\begin{aligned}\dot{x}(t) = &\exp(-\xi\omega_n t)\left\{-\xi\omega_n[x(0)]\cos(\omega_d t) + \left[\frac{-\xi\omega_n}{\omega_d}\right][\dot{x}(0) + (\xi\omega_n)x(0)]\sin(\omega_d t)\right\} \\ &+ \exp(-\xi\omega_n t)\{-\omega_d[x(0)]\sin(\omega_d t) + [\dot{x}(0) + (\xi\omega_n)x(0)]\cos(\omega_d t)\}, \quad \xi < 1\end{aligned}$$

(B-4)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[-\omega_d[x(0)] + \left[\frac{-\xi\omega_n}{\omega_d}\right][\dot{x}(0) + (\xi\omega_n)x(0)]\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-5)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\left[-\omega_d + \frac{-\xi^2\ \omega_n{}^2}{\omega_d}\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-6)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_d{}^2 - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-7)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_n{}^2\left(1-\xi^2\right) - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-8)

$$\dot{x}(t) =$$

$$\exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[\left[-\omega_n{}^2 + \xi^2\omega_n{}^2 - \xi^2\omega_n{}^2\right]x(0) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\dot{x}(0)\right]\sin(\omega_d t)\right\},$$

$$\xi < 1$$

(B-9)

$$\dot{x}(t) = \exp(-\xi\omega_n t)\left\{\dot{x}(0)\cos(\omega_d t) + \left[\frac{1}{\omega_d}\right]\left[-\omega_n{}^2\ x(0)\ - \xi\omega_n\ \dot{x}(0)\right]\sin(\omega_d t)\right\}, \quad \xi < 1$$

(B-10)

Critically Damped Motion

Recall,

$$\omega_d = \omega_n \sqrt{1-\xi^2} \tag{C-1}$$

Consider the special case where

$$\xi = 1 \tag{C-2}$$

The damped natural frequency changes to

$$\omega_d = 0 \tag{C-3}$$

This case is referred to as critically damped motion. Substitute equations (C-2) and (C-3) into equation (A-16),

$$X(s) = \left\{ \frac{(s+\omega_n)x(0)}{(s+\omega_n)^2} \right\} + \left\{ \frac{\dot{x}(0)+\omega_n x(0)}{(s+\omega_n)^2} \right\} \tag{C-4}$$

$$X(s) = \left\{ \frac{x(0)}{s+\omega_n} \right\} + \left\{ \frac{\dot{x}(0)+\omega_n x(0)}{(s+\omega_n)^2} \right\} \tag{C-5}$$

The resulting displacement is found via an inverse Laplace transformation.

$$x(t) = \exp(-\omega_n t)\{ [x(0)] + [\dot{x}(0)+\omega_n x(0)]t \}, \quad \xi = 1 \tag{C-6}$$

Non-oscillatory Motion

Now consider the special case where

$$\xi > 1 \tag{D-1}$$

Recall equation (A-10), restated here as equation (D-2).

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\} x(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{D-2}$$

Solve for the roots of the denominator.

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2} \tag{D-3}$$

$$s_{1,2} = \frac{-2\xi\omega_n \pm 2\omega_n \sqrt{\xi^2 - 1}}{2} \tag{D-4}$$

$$s_{1,2} = \omega_n \left[-\xi \pm \sqrt{\xi^2 - 1} \right] \tag{D-5}$$

Note that

$$s_1 - s_2 = \omega_n \left[-\xi + \sqrt{\xi^2 - 1} \right] - \omega_n \left[-\xi - \sqrt{\xi^2 - 1} \right] \tag{D-6}$$

$$s_1 - s_2 = \omega_n \left[-\xi + \sqrt{\xi^2 - 1} \right] + \omega_n \left[\xi + \sqrt{\xi^2 - 1} \right] \tag{D-7}$$

$$s_1 - s_2 = 2\omega_n \sqrt{\xi^2 - 1} \tag{D-8}$$

Equation (D-2) can be rewritten as

$$X(s) = \left\{ \frac{\dot{x}(0) + \{s + 2\xi\omega_n\}x(0)}{[s - s_1][s - s_2]} \right\} \tag{D-9}$$

$$X(s) = \left\{ \frac{x(0)s + [\dot{x}(0) + 2\xi\omega_n x(0)]}{[s - s_1][s - s_2]} \right\} \tag{D-10}$$

Equation (D-10) can be expanded in terms of partial fractions using the following equation from Reference 1.

$$\left\{ \frac{\alpha s + \beta}{(s + \lambda)(s + \sigma)} \right\} = \left\{ \frac{1}{\sigma - \lambda} \right\} \left\{ \left[\frac{\beta - \alpha\lambda}{s + \lambda} \right] + \left[\frac{\alpha\sigma - \beta}{s + \sigma} \right] \right\} \tag{D-11}$$

The expansion is performed in equation (D-12).

$$\left\{ \frac{x(0)s + [\dot{x}(0) + 2\xi\omega_n x(0)]}{[s - s_1][s - s_2]} \right\}$$

$$= \left\{ \frac{1}{-s_2 + s_1} \right\} \left\{ \left[\frac{[\dot{x}(0) + 2\xi\omega_n x(0)] + x(0)s_1}{s - s_1} \right] + \left[\frac{-x(0)s_2 - [\dot{x}(0) + 2\xi\omega_n x(0)]}{s - s_2} \right] \right\}$$

(D-12)

$$X(s) = \left\{ \frac{1}{-s_2 + s_1} \right\} \left\{ \left[\frac{[\dot{x}(0) + 2\xi\omega_n x(0)] + x(0)s_1}{s - s_1} \right] + \left[\frac{-x(0)s_2 - [\dot{x}(0) + 2\xi\omega_n x(0)]}{s - s_2} \right] \right\}$$

(D-13)

Take the inverse Laplace transform.

$$x(t) = \left\{ \frac{1}{-s_2 + s_1} \right\} \{ A \exp(s_1 t) + B \exp(s_2 t) \}$$

where

$$A = [\dot{x}(0) + 2\xi\omega_n x(0)] + x(0) s_1$$
$$B = -x(0) s_2 - [\dot{x}(0) + 2\xi\omega_n x(0)]$$

(D-14)

Apply the appropriate terms to equation (D-14).

$$x(t) = \left\{ \frac{1}{2\omega_n \sqrt{\xi^2 - 1}} \right\} \left\{ A \exp\left[\omega_n \left[-\xi + \sqrt{\xi^2 - 1} \right] t \right] + B \exp\left[\omega_n \left[-\xi - \sqrt{\xi^2 - 1} \right] t \right] \right\}$$

where

$$A = [\dot{x}(0) + 2\xi\omega_n x(0)] + x(0)\omega_n \left[-\xi + \sqrt{\xi^2 - 1} \right]$$
$$B = -x(0)\omega_n \left[-\xi - \sqrt{\xi^2 - 1} \right] - [\dot{x}(0) + 2\xi\omega_n x(0)]$$

(D-15)

Simplify

$$x(t) = \left\{ \frac{1}{2\omega_n \sqrt{\xi^2 - 1}} \right\} \left\{ A \exp\left[\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t \right] + B \exp\left[\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t \right] \right\}$$

where

$$A = \dot{x}(0) + 2\xi\omega_n x(0) - \xi x(0)\omega_n + x(0)\omega_n \sqrt{\xi^2 - 1}$$
$$B = \xi x(0)\omega_n + x(0)\omega_n \sqrt{\xi^2 - 1} - \dot{x}(0) - 2\xi\omega_n x(0)$$

(D-16)

Simplify again,

$$x(t) = \left\{ \frac{1}{2\omega_n \sqrt{\xi^2 - 1}} \right\} \left\{ A \exp\left[\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t \right] + B \exp\left[\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t \right] \right\}$$

where

$$A = \dot{x}(0) + \omega_n x(0) \left[\xi + \sqrt{\xi^2 - 1} \right]$$

$$B = -\dot{x}(0) + \omega_n x(0) \left[-\xi + \sqrt{\xi^2 - 1} \right]$$

(D-17)

Reference

1. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata.com Publications, 1999.