THE COSINE INTEGRAL

By Tom Irvine
Email: tomirvine@aol.com

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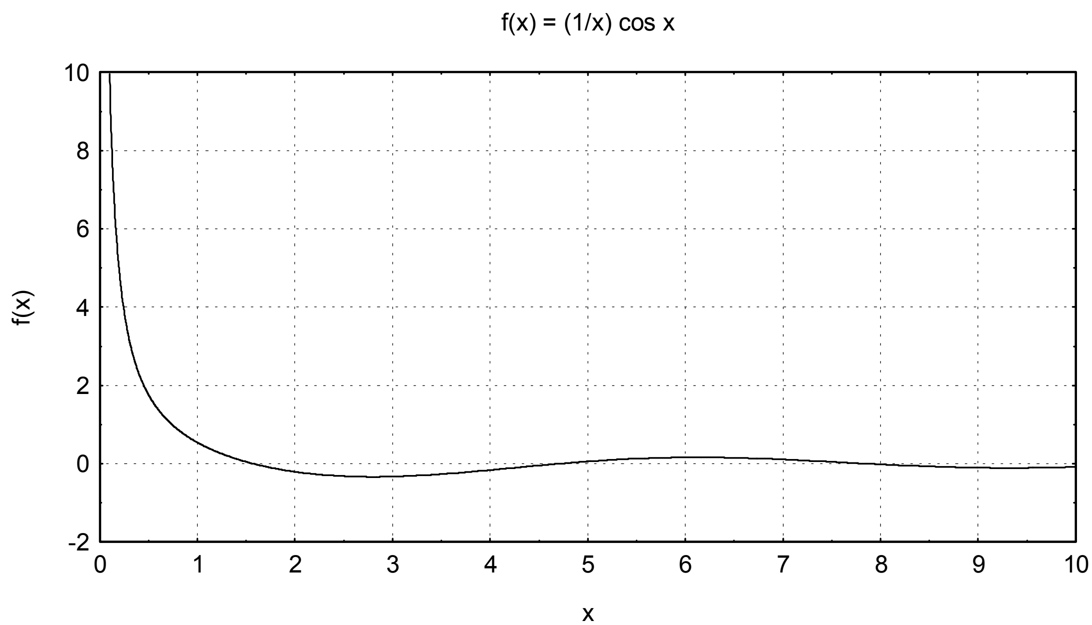

Figure 1.

Integrate the following function.

$$\int_a^b \frac{1}{x} \cos x \, dx \tag{1}$$

The cosine function can be expanded in the following series.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)!} \right) \tag{2}$$

By substitution,

$$\int_a^b \frac{1}{x} \cos x \, dx \quad = \quad \int_a^b \frac{1}{x} \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)!} \right) \right] dx \tag{3}$$

$$\int_a^b \frac{1}{x} \cos x \, dx = \int_a^b \left[\sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k-1}}{(2k)!} \right) \right] dx \tag{4}$$

$$\int_a^b \frac{1}{x} \cos x \, dx \quad = \quad \left[\ln x + \sum_{k=1}^{\infty} (-1)^k \left(\frac{x^{2k}}{(2k)(2k)!} \right) \right] \Bigg|_a^b \tag{5}$$

$$\int_a^b \frac{1}{x} \cos x \, dx \quad = \quad + \left[\ln b + \sum_{k=1}^{\infty} (-1)^k \left(\frac{b^{2k}}{(2k)(2k)!} \right) \right]$$

$$- \left[\ln a + \sum_{k=1}^{\infty} (-1)^k \left(\frac{a^{2k}}{(2k)(2k)!} \right) \right] \tag{6}$$

$$\int_a^b \frac{1}{x}\cos x \, dx = \ln\left(\frac{b}{a}\right) + \sum_{k=1}^{\infty} (-1)^k \left(\frac{b^{2k} - a^{2k}}{(2k)(2k)!} \right) \tag{7}$$

$$\int_a^b \frac{1}{x}\cos x \, dx = \ln\left(\frac{b}{a}\right) + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \left(\frac{b^{2k} - a^{2k}}{(k)(2k)!} \right) \tag{8}$$

Note that each integration limit must be greater than zero due to the natural log term.

For numerical computation, equation (8) is practical only if $b \leq 10$.

Furthermore, the cosine integral is defined in some references as

$$\mathrm{Ci}(x) = -\int_x^{\infty} \frac{1}{u}\cos u \, du \tag{9}$$

The asymptotic expansion of this function is given in Appendix A.

APPENDIX A

An asymptotic solution is given in this section. The result is useful if x >> 1.

$$Ci(x) = -\int_{x}^{\infty} \frac{1}{u} \cos u \, du \tag{A-1}$$

Integrate by parts.

$$Ci(x) = -\int_{x}^{\infty} d\left(\frac{1}{u}\sin u\right) + \int_{x}^{\infty} \sin u \, d\left(\frac{1}{u}\right) \tag{A-2}$$

$$Ci(x) = -\int_{x}^{\infty} d\left(\frac{1}{u}\sin u\right) - \int_{x}^{\infty} \frac{1}{u^2}\sin u \, du \tag{A-3}$$

$$Ci(x) = \quad -\frac{1}{u}\sin u\Big|_{x}^{\infty} \quad - \quad \int_{x}^{\infty} \frac{1}{u^2}\sin u \, du \tag{A-4}$$

$$Ci(x) = \quad \frac{1}{x}\sin x \quad - \quad \int_{x}^{\infty} \frac{1}{u^2}\sin u \, du \tag{A-5}$$

Integrate by parts again.

$$Ci(x) = \quad \frac{1}{x}\sin x \quad + \quad \int_{x}^{\infty} d\left(\frac{1}{u^2}\cos u\right) - \int_{x}^{\infty} \cos u \, d\left(\frac{1}{u^2}\right) \tag{A-6}$$

$$Ci(x) = \quad \frac{1}{x}\sin x \quad + \quad \frac{1}{u^2}\cos u\Big|_{x}^{\infty} + 2\int_{x}^{\infty} \frac{1}{u^3}\cos u \, du \tag{A-7}$$

$$Ci(x) = \quad \frac{1}{x}\sin x \quad - \frac{1}{x^2}\cos x \quad + 2\int_{x}^{\infty} \frac{1}{u^3}\cos u \, du \tag{A-8}$$

Integrate by parts again.

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x + 2\int_x^\infty d\left(\frac{1}{u^3}\sin u\right)du - 2\int_x^\infty \sin u\, d\left(\frac{1}{u^3}\right) \tag{A-9}$$

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x + \frac{2}{u^3}\sin u\Big|_x^\infty + 6\int_x^\infty \frac{1}{u^4}\sin u\, du \tag{A-10}$$

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x - \frac{2}{x^3}\sin x + 6\int_x^\infty \frac{1}{u^4}\sin u\, du \tag{A-11}$$

Integrate by parts again.

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x - \frac{2}{x^3}\sin x - 6\int_x^\infty d\left(\frac{1}{u^4}\cos u\right) + 6\int_x^\infty \cos u\, d\left(\frac{1}{u^4}\right) \tag{A-12}$$

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x - \frac{2}{x^3}\sin x - 6\frac{1}{u^4}\cos u\Big|_x^\infty + 6\int_x^\infty \cos u\, d\left(\frac{1}{u^4}\right) \tag{A-13}$$

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x - \frac{2}{x^3}\sin x + \frac{6}{x^4}\cos x - 24\int_x^\infty \frac{1}{u^5}\cos u\, du \tag{A-14}$$

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1}{x^2}\cos x - \frac{2}{x^3}\sin x + \frac{6}{x^4}\cos x + R \tag{A-15}$$

R is the remainder.

$$R = -24\int_x^{\infty} \frac{1}{u^5}\cos u\,du \tag{A-16}$$

The solution may be restated as

$$\mathrm{Ci}(x) = \frac{1}{x}\sin x - \frac{1!}{x^2}\cos x - \frac{2!}{x^3}\sin x + \frac{3!}{x^4}\cos x + R \tag{A-15}$$

R is the remainder.

$$R = -\,4!\int_x^{\infty} \frac{1}{u^5}\cos u\,du \tag{A-16}$$