ROOTS OF A CUBIC POLYNOMIAL

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Consider the polynomial

$$y = a\,x^3 + b\,x^2 + c\,x + d \tag{1}$$

where a, b, c, and d are real coefficients.

Find the roots by setting the polynomial equal to zero.

$$a\,x^3 + b\,x^2 + c\,x + d = 0 \tag{2}$$

A procedure for finding the roots is given in Table 1

Table 1. Calculation Steps

Step	Equation	Notes
1	$a\,x^3 + b\,x^2 + c\,x + d = 0$	–
2	$p = \left(\frac{1}{a}\right)\left(-\frac{b^2}{3a} + c\right)$	–
3	$q = \left(\frac{1}{27a^3}\right)\left(2b^3 - 9\,abc + 27a^2 d\right)$	–
4	$\pm\sqrt{q^2 + \frac{4p^3}{27}}$	Two roots are obtained
5	$z = \left[\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}\right]^{1/3}$	A total of six z roots are obtained. The roots consist of three sets of complex conjugate pairs.

Table 1. Calculation Steps (Continued)		
Step	Equation	Notes
6	$w = z - \frac{p}{3z}$	This step yields three pairs of identical w roots. Thus, the number of roots are effectively reduced from six to three in this operation.
7	$x = w - \frac{b}{3a}$	Three x roots are obtained.

Steps 4 and 5 can be performed using the methods in Reference 1.

A derivation of the steps in Table 1 is given in Appendix A. An example is given in Appendix B.

Reference

1. T. Irvine, Complex Functions and Trigonometric Identities, Vibrationdata.com Publications, 1999.

APPENDIX A

<u>Derivation</u>

Again, consider

$$y = a x^3 + b x^2 + c x + d \tag{A-1}$$

Find the roots by setting the polynomial equal to zero.

$$a x^3 + b x^2 + c x + d = 0 \tag{A-2}$$

One solution approach is to use variable transformations. Let

$$x = w - \frac{b}{3a} \tag{A-3}$$

Substitute equation (A-3) into (A-2).

$$a\left(w - \frac{b}{3a}\right)^3 + b\left(w - \frac{b}{3a}\right)^2 + c\left(w - \frac{b}{3a}\right) + d = 0 \tag{A-4}$$

$$a\left(w - \frac{b}{3a}\right)\left(w - \frac{b}{3a}\right)\left(w - \frac{b}{3a}\right) + b\left(w - \frac{b}{3a}\right)\left(w - \frac{b}{3a}\right) + c\left(w - \frac{b}{3a}\right) + d = 0 \tag{A-5}$$

$$a\left(w^2 - \frac{2b}{3a}w + \frac{b^2}{9a^2}\right)\left(w - \frac{b}{3a}\right) + b\left(w^2 - \frac{2b}{3a}w + \frac{b^2}{9a^2}\right) + \left(cw - \frac{bc}{3a}\right) + d = 0 \tag{A-6}$$

$$\left(aw^2 - \frac{2b}{3}w + \frac{b^2}{9a}\right)\left(w - \frac{b}{3a}\right) + \left(bw^2 - \frac{2b^2}{3a}w + \frac{b^3}{9a^2}\right) + \left(cw - \frac{bc}{3a}\right) + d = 0 \tag{A-7}$$

$$\left(aw^2 - \frac{2b}{3}w + \frac{b^2}{9a}\right)w + \left(aw^2 - \frac{2b}{3}w + \frac{b^2}{9a}\right)\left(\frac{-b}{3a}\right) + \left(bw^2 - \frac{2b^2}{3a}w + \frac{b^3}{9a^2}\right) + \left(cw - \frac{bc}{3a}\right) + d = 0 \tag{A-8}$$

$$\left(aw^3 - \frac{2b}{3}w^2 + \frac{b^2}{9a}w\right) + \left(-\frac{b}{3}w^2 + \frac{2b^2}{9a}w - \frac{b^3}{27a^2}\right) + \left(bw^2 - \frac{2b^2}{3a}w + \frac{b^3}{9a^2}\right) + \left(cw - \frac{bc}{3a}\right) + d = 0$$

(A-9)

$$a\,w^3 + \left(-\frac{2b}{3} - \frac{b}{3} + b\right)w^2 + \left(\frac{b^2}{9a} + \frac{2b^2}{9a} - \frac{2b^2}{3a} + c\right)w + \left(-\frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d\right) = 0$$

(A-10)

$$a\,w^3 + \left(\frac{b^2}{9a} + \frac{2b^2}{9a} - \frac{6b^2}{9a} + c\right)w + \left(-\frac{b^3}{27a^2} + \frac{3b^3}{27a^2} - \frac{bc}{3a} + d\right) = 0 \quad \text{(A-11)}$$

$$a\,w^3 + \left(-\frac{b^2}{3a} + c\right)w + \left(\frac{2b^3}{27a^2} - \frac{bc}{3a} + d\right) = 0 \quad \text{(A-12)}$$

Divide through by a.

$$w^3 + \left(\frac{1}{a}\right)\left(-\frac{b^2}{3a} + c\right)w + \left(\frac{1}{a}\right)\left(\frac{2b^3}{27a^2} - \frac{bc}{3a} + d\right) = 0 \quad \text{(A-13)}$$

$$w^3 + \left(\frac{1}{3a^2}\right)\left(-b^2 + 3ac\right)w + \left(\frac{1}{27a^3}\right)\left(2b^3 - 9abc + 27a^2 d\right) = 0 \quad \text{(A-14)}$$

$$w^3 + p\,w + q = 0 \quad \text{(A-15)}$$

where

$$p = \left(\frac{1}{a}\right)\left(-\frac{b^2}{3a} + c\right) \quad \text{(A-16)}$$

and

$$q = \left(\frac{1}{27a^3}\right)\left(2b^3 - 9abc + 27a^2 d\right) \qquad \text{(A-17)}$$

Perform a second transformation. Let

$$w = z - \frac{p}{3z} \qquad \text{(A-18)}$$

$$\left(z - \frac{p}{3z}\right)^3 + p\left(z - \frac{p}{3z}\right) + q = 0 \qquad \text{(A-19)}$$

$$\left(z^2 - \frac{2p}{3} + \frac{p^2}{9z^2}\right)\left(z - \frac{p}{3z}\right) + \left(pz - \frac{p^2}{3z}\right) + q = 0 \qquad \text{(A-20)}$$

$$\left(z^2 - \frac{2p}{3} + \frac{p^2}{9z^2}\right)z + \left(z^2 - \frac{2p}{3} + \frac{p^2}{9z^2}\right)\left(-\frac{p}{3z}\right) + \left(pz - \frac{p^2}{3z}\right) + q = 0 \qquad \text{(A-21)}$$

$$\left(z^3 - \frac{2pz}{3} + \frac{p^2}{9z}\right) + \left(-\frac{pz}{3} + \frac{2p^2}{9z} - \frac{p^3}{27z^3}\right) + \left(pz - \frac{p^2}{3z}\right) + q = 0 \qquad \text{(A-22)}$$

$$z^3 + \left(-\frac{2p}{3} - \frac{p}{3} + p\right)z + q + \left(\frac{p^2}{9} + \frac{2p^2}{9} - \frac{p^2}{3}\right)\frac{1}{z} + \left(-\frac{p^3}{27}\right)\frac{1}{z^3} = 0 \qquad \text{(A-23)}$$

$$z^3 + \left(-\frac{2p}{3} - \frac{p}{3} + \frac{3p}{3}\right)z + q + \left(\frac{p^2}{9} + \frac{2p^2}{9} - \frac{3p^2}{9}\right)\frac{1}{z} + \left(-\frac{p^3}{27}\right)\frac{1}{z^3} = 0 \qquad \text{(A-24)}$$

$$z^3 + q + \left(-\frac{p^3}{27}\right)\frac{1}{z^3} = 0 \qquad \text{(A-25)}$$

Multiply through by z^3.

$$z^6 + q z^3 + \left(-\frac{p^3}{27} \right) = 0 \qquad \text{(A-26)}$$

Solve for z^3 using the quadratic formula.

$$z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \qquad \text{(A-27)}$$

Take the cube root of each side.

$$z = \left[\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \right]^{1/3} \qquad \text{(A-28)}$$

The roots are thus found by using the equations as previously summarized in Table 1.

In certain cases, a cube root of a complex number must be calculated. The formula for obtaining a root of complex number is derived in Reference 1.

APPENDIX B

<u>Example</u>

Determine the roots of

$$x^3 + 5x^2 + 7x + 2 = 0 \qquad \text{(B-1)}$$

Recall the general formula.

$$a\,x^3 + b\,x^2 + c\,x + d = 0 \qquad \text{(B-2)}$$

The coefficients are

$$a=1 \qquad \text{(B-3)}$$

$$b=5 \qquad \text{(B-4)}$$

$$c=7 \qquad \text{(B-5)}$$

$$d=2 \qquad \text{(B-6)}$$

Solve for p.

$$p = \left(\frac{1}{a}\right)\left(-\frac{b^2}{3a} + c\right) \qquad \text{(B-7)}$$

$$p = \left(\frac{1}{1}\right)\left(-\frac{5^2}{3(1)} + 7\right) \qquad \text{(B-8)}$$

$$p= -1.3333 \qquad \text{(B-9)}$$

Solve for q.

$$q = \left(\frac{1}{27a^3}\right)\left(2b^3 - 9\,a\,bc + 27a^2d\right) \qquad \text{(B-10)}$$

$$q = \left(\frac{1}{27(1)^3}\right)\left(2(5)^3 - 9(1)(5)(7) + 27(1)^2(2)\right) \qquad \text{(B-11)}$$

$$q = -0.4074 \qquad \text{(B-12)}$$

Solve for the square root term.

$$\pm\sqrt{q^2 + \frac{4p^3}{27}} = \pm\sqrt{(-0.4074)^2 + \left(\frac{4(-1.3333)^3}{27}\right)} \qquad \text{(B-13)}$$

$$\pm\sqrt{q^2 + \frac{4p^3}{27}} = \pm\sqrt{-0.1852} \qquad \text{(B-14)}$$

$$\pm\sqrt{q^2 + \frac{4p^3}{27}} = \pm\, j\, 0.4303 \qquad \text{(B-15)}$$

Solve for z.

$$z = \left[\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}\right]^{1/3} \qquad \text{(B-16)}$$

$$z = \left[\frac{-(-0.4074) \pm j\, 0.4303}{2}\right]^{1/3} \qquad \text{(B-17)}$$

$$z = \left[\frac{0.4074 \pm j\, 0.4303}{2}\right]^{1/3} \qquad \text{(B-18)}$$

$$z = [0.2037 \pm j\, 0.2152]^{1/3} \qquad \text{(B-19)}$$

Equation (B-19) yields two expressions.

$$z_1 = [\, 0.2037 + j\, 0.2152 \,]^{1/3} \qquad \text{(B-20)}$$

$$z_2 = [\ 0.2037 - j\ 0.2152\ ]^{1/3} \tag{B-22}$$

Each of the two expressions yields three roots for z.

Use the approach given in Reference 1 for finding the cube root of a complex number.

$$z = (\alpha + j\beta)^{1/3} \tag{B-23}$$

$$z_{11} = \left(\alpha^2 + \beta^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3} \arctan\frac{\beta}{\alpha}\right) + j \sin\left(\frac{1}{3} \arctan\frac{\beta}{\alpha}\right)\right\} \tag{B-24}$$

$$\alpha = 0.2037 \tag{B-25}$$

$$\beta = 0.2152 \tag{B-26}$$

$$z_{11} = \left(0.2037^2 + 0.2152^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3} \arctan\frac{0.2152}{0.2037}\right) + j \sin\left(\frac{1}{3} \arctan\frac{0.2152}{0.2037}\right)\right\} \tag{B-27}$$

$$z_{11} = (\ 0.08780)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3} \arctan 1.0562\right) + j \sin\left(\frac{1}{3} \arctan 1.0562\right)\right\} \tag{B-28}$$

$$z_{11} = 0.6667 \left\{\cos\left(\frac{1}{3}(0.8127)\right) + j \sin\left(\frac{1}{3}(0.8127)\right)\right\} \tag{B-29}$$

$$z_{11} = 0.6667 \ \{\cos(0.2709) + j \sin(0.2709)\} \tag{B-30}$$

$$z_{11} = 0.6667 \ \{0.9635 + j\ 0.2670\} \tag{B-31}$$

$$z_{11} = 0.6424 + \ j\ 0.1784 \tag{B-32}$$

Solve for the second root using a transformation.

$$z_{12} = z_{11}\left[\frac{-1 + j\sqrt{3}}{2}\right] \qquad \text{(B-33)}$$

$$z_{12} = [0.6424 + \ j \ 0.1784]\left[\frac{-1 + j\sqrt{3}}{2}\right] \qquad \text{(B-34)}$$

$$z_{12} = -0.4757 + j\,0.4671 \qquad \text{(B-35)}$$

Solve for the third root using a transformation.

$$z_{13} = z_{11}\left[\frac{-1 - j\sqrt{3}}{2}\right] \qquad \text{(B-36)}$$

$$z_{13} = [0.6424 + \ j \ 0.1784]\left[\frac{-1 - j\sqrt{3}}{2}\right] \qquad \text{(B-37)}$$

$$z_{13} = -0.1667 - j\,0.6455 \qquad \text{(B-38)}$$

Recall

$$z_2 = [\ 0.2037 - j\,0.2152\ ]^{\ 1/3} \qquad \text{(B-39)}$$

Use the approach given in Reference 1 for finding the cube root of a complex number.

$$z = (\alpha + j\beta)^{1/3} \qquad \text{(B-40)}$$

$$z_{21} = \left(\alpha^2 + \beta^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3}\arctan\frac{\beta}{\alpha}\right) + j\sin\left(\frac{1}{3}\arctan\frac{\beta}{\alpha}\right)\right\} \qquad \text{(B-41)}$$

$$\alpha = 0.2037 \qquad \text{(B-42)}$$

$$\beta = -0.2152 \qquad \text{(B-43)}$$

$$z_{21} = \left(0.2037^2 + (-0.2152)^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3} \arctan \frac{-0.2152}{0.2037}\right) + j \sin\left(\frac{1}{3} \arctan \frac{-0.2152}{0.2037}\right)\right\}$$

(B-44)

$$z_{21} = (\ 0.08780)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3} \arctan -1.0562\right) + j \sin\left(\frac{1}{3} \arctan -1.0562\right)\right\} \quad \text{(B-45)}$$

$$z_{21} = 0.6667 \left\{\cos\left(\frac{1}{3}(-0.8127)\right) + j \sin\left(\frac{1}{3}(-0.8127)\right)\right\} \quad \text{(B-46)}$$

$$z_{21} = 0.6667 \ \{\cos(-0.2709) + j \sin(-0.2709)\} \quad \text{(B-47)}$$

$$z_{21} = 0.6667 \ \{0.9635 - j\ 0.2670\} \quad \text{(B-48)}$$

$$z_{21} = 0.6424 - j\ 0.1784 \quad \text{(B-49)}$$

Solve for the second root using a transformation.

$$z_{22} = z_{21}\left[\frac{-1 + j\sqrt{3}}{2}\right] \quad \text{(B-50)}$$

$$z_{22} = [0.6424 - j\ 0.1784]\left[\frac{-1 + j\sqrt{3}}{2}\right] \quad \text{(B-51)}$$

$$z_{22} = -0.1667 + j\ 0.6455 \quad \text{(B-52)}$$

Solve for the third root using a transformation.

$$z_{23} = x_{21}\left[\frac{-1 - j\sqrt{3}}{2}\right] \quad \text{(B-53)}$$

$$z_{23} = [0.6424 - j\ 0.1784]\left[\frac{-1 - j\sqrt{3}}{2}\right] \quad \text{(B-54)}$$

$$z_{23} = -0.4757 - j\ 0.4671 \quad \text{(B-55)}$$

The z roots are summarized in Table B-1. The roots represent three sets of complex conjugate pairs.

Table B-1. z roots
$z_{11} = 0.6424 + j\ 0.1784$
$z_{12} = -0.4757 + j\ 0.4671$
$z_{13} = -0.1667 - j\ 0.6455$
$z_{21} = 0.6424 - j\ 0.1784$
$z_{22} = -0.1667 + j\ 0.6455$
$z_{23} = -0.4757 - j\ 0.4671$

Calculate the corresponding w values. Recall step 6 from Table 1.

$$w = z - \frac{p}{3z} \quad \text{(B-56)}$$

Recall

$$p = -1.3333 \quad \text{(B-57)}$$

$$w = z - \frac{p}{3z} \quad \text{(B-58)}$$

$$w_{11} = z_{11} - \frac{p}{3\, z_{11}} \quad \text{(B-59)}$$

$$w_{11} = (0.6424 + \; j \; 0.1784) - \frac{-1.3333}{3\,(0.6424 + \; j \; 0.1784)} \quad \text{(B-60)}$$

$$w_{11} = (0.6424 + \; j \; 0.1784) + \frac{0.4444}{(0.6424 + \; j \; 0.1784)} \quad \text{(B-61)}$$

$$w_{11} = 1.2847 + j\,0 \quad \text{(B-62)}$$

$$w_{11} = 1.2847 \quad \text{(B-63)}$$

$$w_{12} = z_{12} - \frac{p}{3\, z_{12}} \quad \text{(B-64)}$$

$$w_{12} = (-0.4757 + j\,0.4671) - \frac{-1.3333}{3\,(-0.4757 + j\,0.4671)} \quad \text{(B-65)}$$

$$w_{12} = (-0.4757 + j\,0.4671) + \frac{0.4444}{(-0.4757 + j\,0.4671)} \quad \text{(B-66)}$$

$$w_{12} = -0.9513 + j\,0 \quad \text{(B-67)}$$

$$w_{12} = -0.9513 \quad \text{(B-68)}$$

$$w_{13} = z_{13} - \frac{p}{3\, z_{13}} \quad \text{(B-69)}$$

$$w_{13} = (-0.1667 - j\,0.6455) - \frac{-1.3333}{3\,(-0.1667 - j\,0.6455)} \quad \text{(B-70)}$$

$$w_{13} = (-0.1667 - j\,0.6455) + \frac{0.4444}{(-0.1667 - j\,0.6455)} \quad \text{(B-71)}$$

$$w_{13} = -0.3334 + j\,0 \quad \text{(B-72)}$$

$$w_{13} = -0.3334 \quad \text{(B-73)}$$

$$w_{21} = z_{21} - \frac{p}{3\,z_{21}} \qquad \text{(B-74)}$$

$$w_{21} = (0.6424 - \ j\ 0.1784) - \frac{-1.3333}{3\,(0.6424 - \ j\ 0.1784)} \qquad \text{(B-75)}$$

$$w_{21} = (0.6424 - \ j\ 0.1784) + \frac{0.4444}{(0.6424 - \ j\ 0.1784)} \qquad \text{(B-76)}$$

$$w_{21} = 1.2847 + j0 \qquad \text{(B-77)}$$

$$w_{21} = 1.2847 \qquad \text{(B-78)}$$

$$w_{22} = z_{22} - \frac{p}{3\,z_{22}} \qquad \text{(B-79)}$$

$$w_{22} = (-0.1667 + j\,0.6455) - \frac{-1.3333}{3\,(-0.1667 + j\,0.6455)} \qquad \text{(B-80)}$$

$$w_{22} = (-0.1667 + j\,0.6455) + \frac{0.4444}{(-0.1667 + j\,0.6455)} \qquad \text{(B-81)}$$

$$w_{22} = -0.3334 + j\,0 \qquad \text{(B-82)}$$

$$w_{22} = \quad -0.3334 \qquad \text{(B-83)}$$

$$w_{23} = z_{23} - \frac{p}{3\,z_{23}} \qquad \text{(B-84)}$$

$$w_{23} = (-0.4757 - j\,0.4671) - \frac{-1.3333}{3\,(-0.4757 - j\,0.4671)} \qquad \text{(B-85)}$$

$$w_{23} = (-0.4757 - j\,0.4671) + \frac{0.4444}{(-0.4757 - j\,0.4671)} \qquad \text{(B-86)}$$

$$w_{23} = -0.9513 + j\,0 \qquad \text{(B-87)}$$

$$w_{23} = -0.9513 \quad \text{(B-88)}$$

The w roots are summarized in Table B-2.

Table B-2. w roots
$w_{11} = 1.2847$
$w_{12} = -0.9513$
$w_{13} = -0.3334$
$w_{21} = 1.2847$
$w_{22} = -0.3334$
$w_{23} = -0.9513$

The second set of roots can be discarded, since the second set repeats the first set.

Recall

$$x = w - \frac{b}{3a} \quad \text{(B-89)}$$

$$a=1 \quad \text{(B-90)}$$

$$b=5 \quad \text{(B-91)}$$

$$\frac{b}{3a} = \frac{5}{(3)(1)} \quad \text{(B-92)}$$

$$\frac{b}{3a} = 1.6667 \quad \text{(B-93)}$$

$$x_1 = w_{11} - \frac{b}{3a} \quad \text{(B-94)}$$

$$x_1 = 1.2847 - 1.6667 \quad \text{(B-95)}$$

$$x_1 = -0.3820 \quad \text{(B-96)}$$

$$x_2 = w_{12} - \frac{b}{3a} \tag{B-97}$$

$$x_2 = -0.9513 - 1.6667 \tag{B-98}$$

$$x_2 = -2.6180 \tag{B-99}$$

$$x_3 = w_{13} - \frac{b}{3a} \tag{B-100}$$

$$x_3 = -0.3334 - 1.6667 \tag{B-101}$$

$$x_3 = -2.0000 \tag{B-102}$$

The x roots are summarized in Table B-3.

Table B-3. x roots
$x_1 = -0.3820$
$x_2 = -2.6180$
$x_3 = -2.0000$

The roots in Table B-3 satisfy

$$x^3 + 5x^2 + 7x + 2 = 0 \tag{B-103}$$

In this example, the roots were all real. In other examples, two of the roots might form a complex conjugate pair.