THE GENERALIZED EIGENVALUE PROBLEM

By Tom Irvine
Email: tomirvine@aol.com

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Introduction

Certain types of acoustic and vibration problems can be represented by the generalized eigenproblem

$$K\ X = \omega^2\ M X \tag{1}$$

where

K is the stiffness matrix,
M is the mass matrix,
ω^2 is the eigenvalue,
ω is the natural frequency,
X is the corresponding eigenvector.

The stiffness and mass matrices have dimension n. Thus, there are n eigenvalues and n corresponding eigenvectors.

Equation (1) can be expressed as

$$\left[K - \omega^2\ M\right] X = 0 \tag{2}$$

The eigenvalues are found by the following equation

$$\text{determinant}\ \left[K - \omega^2\ M\right] = 0 \tag{3}$$

An exact polynomial solution can be obtained for systems with a small dimension. The practical upper limit for an exact solution is n = 4. For the purpose of this tutorial, "exact" means that the roots can be represented in terms of radicals.

Numerical methods are required for larger systems. This tutorial presents the polynomial solution method for systems having dimension $n \leq 4$.

Dimension n = 2

The generalized eigenproblem is shown in terms of its coefficients in equation (A-1).

$$\begin{bmatrix} \left(k_{11} - \omega^2 m_{11}\right) & \left(k_{12} - \omega^2 m_{12}\right) \\ \left(k_{21} - \omega^2 m_{21}\right) & \left(k_{22} - \omega^2 m_{22}\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(A-1)}$$

Set the determinant of the coefficient matrix equal to zero.

$$\left(k_{11} - \omega^2 m_{11}\right)\left(k_{22} - \omega^2 m_{22}\right) - \left(k_{12} - \omega^2 m_{12}\right)\left(k_{21} - \omega^2 m_{21}\right) = 0 \qquad \text{(A-2)}$$

Perform a series of algebraic steps to simplify the polynomial.

$$\begin{aligned} & k_{11}k_{22} - \omega^2 m_{11}k_{22} - \omega^2 m_{22}k_{11} + \omega^4 m_{11}m_{22} \\ & - \left\{ k_{12}k_{21} - \omega^2 m_{12}k_{21} - \omega^2 m_{21}k_{12} + \omega^4 m_{12}m_{21} \right\} = 0 \end{aligned} \qquad \text{(A-3)}$$

$$\begin{aligned} & k_{11}k_{22} - \omega^2 m_{11}k_{22} - \omega^2 m_{22}k_{11} + \omega^4 m_{11}m_{22} \\ & - k_{12}k_{21} + \omega^2 m_{12}k_{21} + \omega^2 m_{21}k_{12} - \omega^4 m_{12}m_{21} = 0 \end{aligned} \qquad \text{(A-4)}$$

The resulting characteristic polynomial is

$$\begin{aligned} & \left[m_{11}m_{22} - m_{12}m_{21} \right] \omega^4 + \left[-m_{11}k_{22} - m_{22}k_{11} + m_{12}k_{21} + m_{21}k_{12} \right] \omega^2 \\ & + \left[k_{11}k_{22} - k_{12}k_{21} \right] = 0 \end{aligned} \qquad \text{(A-5)}$$

The natural frequencies are the roots of equation (A-5). They may be found via the quadratic formula.

$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(A-6)}$$

where

$$a = m_{11}m_{22} - m_{12}m_{21}$$

$$b = -m_{11}k_{22} - m_{22}k_{11} + m_{12}k_{21} + m_{21}k_{12}$$

$$c = k_{11}k_{22} - k_{12}k_{21}$$

Dimension n = 3

The generalized eigenproblem is shown in terms of its coefficients in equation (B-1).

$$\begin{bmatrix} \left(k_{11}-\omega^2 m_{11}\right) & \left(k_{12}-\omega^2 m_{12}\right) & \left(k_{13}-\omega^2 m_{13}\right) \\ \left(k_{21}-\omega^2 m_{21}\right) & \left(k_{22}-\omega^2 m_{22}\right) & \left(k_{23}-\omega^2 m_{23}\right) \\ \left(k_{31}-\omega^2 m_{31}\right) & \left(k_{32}-\omega^2 m_{32}\right) & \left(k_{33}-\omega^2 m_{33}\right) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(B-1)

Again, the eigenvalues are calculated by setting the determinant equal to zero. The resulting characteristic polynomial can be expressed as

$$\alpha\omega^6 + \beta\omega^4 + \eta\omega^2 + \sigma = 0 \qquad \text{(B-2)}$$

where

$$\begin{aligned} \alpha = &-m_{11}m_{22}m_{33} - m_{12}m_{23}m_{31} - m_{13}m_{21}m_{32} \\ &+ m_{11}m_{23}m_{32} + m_{12}m_{21}m_{33} + m_{13}m_{22}m_{31} \end{aligned} \qquad \text{(B-3)}$$

$$\begin{aligned} \beta = \; & \\ &+\left[k_{11}\left(m_{22}m_{33} - m_{23}m_{32}\right) + k_{12}\left(m_{23}m_{31} - m_{21}m_{33}\right) + k_{13}\left(m_{21}m_{32} - m_{22}m_{31}\right)\right] \\ &+\left[k_{21}\left(m_{13}m_{32} - m_{12}m_{33}\right) + k_{22}\left(m_{11}m_{33} - m_{13}m_{31}\right) + k_{23}\left(m_{12}m_{31} - m_{11}m_{32}\right)\right] \\ &+\left[k_{31}\left(m_{12}m_{23} - m_{13}m_{22}\right) + k_{32}\left(m_{13}m_{21} - m_{11}m_{23}\right) + k_{33}\left(m_{11}m_{22} - m_{12}m_{21}\right)\right] \end{aligned}$$

(B-4)

$$\begin{aligned} \eta = \; & \\ &+\left[m_{11}\left(k_{23}k_{32} - k_{22}k_{33}\right) + m_{12}\left(k_{21}k_{33} - k_{23}k_{31}\right) + m_{13}\left(k_{22}k_{31} - k_{21}k_{32}\right)\right] \\ &+\left[m_{21}\left(k_{12}k_{33} - k_{13}k_{32}\right) + m_{22}\left(k_{13}k_{31} - k_{11}k_{33}\right) + m_{23}\left(k_{11}k_{32} - k_{12}k_{31}\right)\right] \\ &+\left[m_{31}\left(k_{13}k_{22} - k_{12}k_{23}\right) + m_{32}\left(k_{11}k_{23} - k_{13}k_{21}\right) + m_{33}\left(k_{12}k_{21} - k_{11}k_{22}\right)\right] \end{aligned}$$

(B-5)

$$\sigma = \left[k_{11}k_{22}k_{33} + k_{12}k_{23}k_{31} + k_{13}k_{32}k_{21} - k_{11}k_{32}k_{23} - k_{12}k_{21}k_{33} - k_{13}k_{22}k_{31}\right]$$

(B-6)

Equation (B-2) can be solved using the methods in References 1 and 2.

Dimension n = 4

The generalized eigenproblem is shown in terms of its coefficients in equation (C-1).

$$\begin{bmatrix} (k_{11}-\omega^2 m_{11}) & (k_{12}-\omega^2 m_{12}) & (k_{13}-\omega^2 m_{13}) & (k_{14}-\omega^2 m_{14}) \\ (k_{21}-\omega^2 m_{21}) & (k_{22}-\omega^2 m_{22}) & (k_{23}-\omega^2 m_{23}) & (k_{24}-\omega^2 m_{24}) \\ (k_{31}-\omega^2 m_{31}) & (k_{32}-\omega^2 m_{32}) & (k_{33}-\omega^2 m_{33}) & (k_{34}-\omega^2 m_{34}) \\ (k_{41}-\omega^2 m_{41}) & (k_{42}-\omega^2 m_{42}) & (k_{43}-\omega^2 m_{43}) & (k_{44}-\omega^2 m_{44}) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(C-1)

Again, the eigenvalues are calculated by setting the determinant equal to zero.

$$+(k_{11}-\omega^2 m_{11}) \begin{vmatrix} (k_{22}-\omega^2 m_{22}) & (k_{23}-\omega^2 m_{23}) & (k_{24}-\omega^2 m_{24}) \\ (k_{32}-\omega^2 m_{32}) & (k_{33}-\omega^2 m_{33}) & (k_{34}-\omega^2 m_{34}) \\ (k_{42}-\omega^2 m_{42}) & (k_{43}-\omega^2 m_{43}) & (k_{44}-\omega^2 m_{44}) \end{vmatrix}$$

$$-(k_{21}-\omega^2 m_{21}) \begin{vmatrix} (k_{12}-\omega^2 m_{12}) & (k_{13}-\omega^2 m_{13}) & (k_{14}-\omega^2 m_{14}) \\ (k_{32}-\omega^2 m_{32}) & (k_{33}-\omega^2 m_{33}) & (k_{34}-\omega^2 m_{34}) \\ (k_{42}-\omega^2 m_{42}) & (k_{43}-\omega^2 m_{43}) & (k_{44}-\omega^2 m_{44}) \end{vmatrix}$$

$$+(k_{31}-\omega^2 m_{31}) \begin{vmatrix} (k_{12}-\omega^2 m_{12}) & (k_{13}-\omega^2 m_{13}) & (k_{14}-\omega^2 m_{14}) \\ (k_{22}-\omega^2 m_{22}) & (k_{23}-\omega^2 m_{23}) & (k_{24}-\omega^2 m_{24}) \\ (k_{42}-\omega^2 m_{42}) & (k_{43}-\omega^2 m_{43}) & (k_{44}-\omega^2 m_{44}) \end{vmatrix}$$

$$-(k_{41}-\omega^2 m_{41}) \begin{vmatrix} (k_{12}-\omega^2 m_{12}) & (k_{13}-\omega^2 m_{13}) & (k_{14}-\omega^2 m_{14}) \\ (k_{22}-\omega^2 m_{22}) & (k_{23}-\omega^2 m_{23}) & (k_{24}-\omega^2 m_{24}) \\ (k_{32}-\omega^2 m_{32}) & (k_{33}-\omega^2 m_{33}) & (k_{34}-\omega^2 m_{34}) \end{vmatrix} = 0$$

(C-2)

After several steps, the equation simplifies to

$$
\begin{aligned}
&\left[\alpha_1 m_{11} - \alpha_2 m_{21} + \alpha_3 m_{31} - \alpha_4 m_{41}\right]\omega^8 \\
&+\left[\left(\alpha_1 k_{11} - \beta_1 m_{11}\right) - \left(\alpha_2 k_{21} - \beta_2 m_{21}\right) + \left(\alpha_3 k_{31} - \beta_3 m_{31}\right) - \left(\alpha_4 k_{41} - \beta_4 m_{41}\right)\right]\omega^6 \\
&+\left[\left(\beta_1 k_{11} - \eta_1 m_{11}\right) - \left(\beta_2 k_{21} - \eta_2 m_{21}\right) + \left(\beta_3 k_{31} - \eta_3 m_{31}\right) - \left(\beta_4 k_{41} - \eta_4 m_{41}\right)\right]\omega^4 \\
&+\left[\left(\eta_1 k_{11} - \sigma_1 m_{11}\right) - \left(\eta_2 k_{21} - \sigma_2 m_{21}\right) + \left(\eta_3 k_{31} - \sigma_3 m_{31}\right) - \left(\eta_4 k_{41} - \sigma_4 m_{41}\right)\right]\omega^2 \\
&+\left[k_{11}\sigma_1 - k_{21}\sigma_1 + k_{31}\sigma_1 - k_{41}\sigma_1\right] = 0
\end{aligned}
$$

(C-3)

The α, β, η, and σ coefficients are calculated using the pattern in the previous section.

For example,

$$
\begin{aligned}
\alpha_1 = &-m_{22}m_{33}m_{44} - m_{23}m_{34}m_{42} - m_{24}m_{32}m_{43} \\
&+ m_{22}m_{34}m_{43} + m_{23}m_{32}m_{44} + m_{24}m_{33}m_{42}
\end{aligned}
$$

(C-4)

$$
\begin{aligned}
\beta_1 = & \\
&+\left[k_{22}\left(m_{33}m_{44} - m_{34}m_{43}\right) + k_{23}\left(m_{34}m_{42} - m_{32}m_{44}\right) + k_{24}\left(m_{32}m_{43} - m_{33}m_{42}\right)\right] \\
&+\left[k_{32}\left(m_{24}m_{43} - m_{23}m_{44}\right) + k_{33}\left(m_{22}m_{44} - m_{24}m_{42}\right) + k_{34}\left(m_{23}m_{42} - m_{22}m_{21}\right)\right] \\
&+\left[k_{42}\left(m_{23}m_{34} - m_{24}m_{33}\right) + k_{21}\left(m_{24}m_{32} - m_{22}m_{34}\right) + k_{44}\left(m_{22}m_{33} - m_{23}m_{32}\right)\right]
\end{aligned}
$$

(C-5)

$$
\begin{aligned}
\eta_1 = & \\
&+\left[m_{22}\left(k_{34}k_{43} - k_{33}k_{44}\right) + m_{23}\left(k_{32}k_{44} - k_{34}k_{42}\right) + m_{24}\left(k_{33}k_{42} - k_{32}k_{43}\right)\right] \\
&+\left[m_{32}\left(k_{23}k_{44} - k_{24}k_{43}\right) + m_{33}\left(k_{24}k_{42} - k_{22}k_{44}\right) + m_{34}\left(k_{22}k_{43} - k_{23}k_{42}\right)\right] \\
&+\left[m_{42}\left(k_{24}k_{33} - k_{23}k_{34}\right) + m_{43}\left(k_{22}k_{34} - k_{24}k_{32}\right) + m_{44}\left(k_{23}k_{32} - k_{22}k_{33}\right)\right]
\end{aligned}
$$

(C-6)

$$
\sigma_1 = \left[k_{22}k_{33}k_{44} + k_{23}k_{34}k_{42} + k_{24}k_{43}k_{32} - k_{22}k_{43}k_{34} - k_{23}k_{32}k_{44} - k_{24}k_{33}k_{42}\right]
$$

(C-7)

Equation (C-3) can be solved by the method in Reference 3.

Theorem

If the K and M matrices are each symmetric with real coefficients, then the eigenvalues are real.

Example

Consider a cantilever beam with mass per length ρ. Assume that the beam has a uniform cross section. Determine the first three natural frequencies.

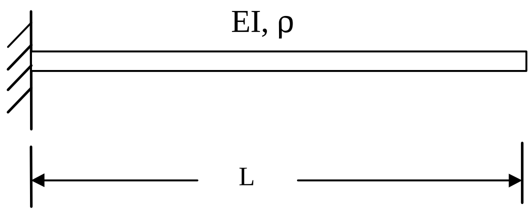

Figure D-1.

The Rayleigh-Ritz method yields the eigen problem shown in Figure D-1. The matrix is shown in upper triangular form due to symmetry.

$$\left|K - \omega^2 M\right| =$$

$$\begin{bmatrix} 3.0440 - 0.2268\ \lambda & -0.5756\ \lambda & -0.2361\ \lambda \\ & 246.57 - 1.9244\ \lambda & -1.0849\ \lambda \\ & & 1902.52 - 1.2454\ \lambda \end{bmatrix} \tag{D-1}$$

where

$$\lambda = \omega^2 \left[\frac{\rho L^4}{EI} \right] \tag{D-2}$$

A complete derivation of equation (D-1) is given in Reference 4.

The resulting eigenvalues are calculated by setting the determinant of the coefficient matrix equal to zero. A characteristic polynomial is formed per the methods previously given in this tutorial. The resulting eigenvalues are shown in Table D-1.

Table D-1. Eigenvalues from the Rayleigh-Ritz Method		
Root Number	λ_n	ω_n
1	12.39	$3.520\sqrt{\frac{EI}{\rho L^4}}$
2	493.6	$22.217\sqrt{\frac{EI}{\rho L^4}}$
3	4526.	$67.276\sqrt{\frac{EI}{\rho L^4}}$

References

1. T. Irvine, Roots of a Cubic Polynomial, Vibrationdata.com Publications, 1999.
2. T. Irvine, Complex Functions and Trigonometric Identities, Vibrationdata.com Publications, 1999.
3. T. Irvine, Roots of a Quardic Polynomial, Vibrationdata.com Publications, 1999.
4. T. Irvine, Rayleigh-Ritz Method for Beams and Rods, Vibrationdata.com Publications, 1999.