INTEGRATION OF THE NORMAL DISTRIBUTION CURVE

By Tom Irvine
Email: tomirvine@aol.com

March 13, 1999

_____

Introduction

Many processes have a normal probability distribution. Broadband random vibration is an example.

The purpose of this report is to derive a formula for integrating the normal distribution curve. This effort is needed due to the limitations of statistical tables published in textbooks.

Derivation

From Reference 1, the probability density function $n(x; \mu, \sigma)$ of a normal distribution is given by

$$n(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left[\frac{x - \mu}{\sigma}\right]^2 \right\}, \quad -\infty < x < \infty \tag{1}$$

where
      x is a continuous random variable,
      μ is the mean,
      σ is the standard deviation.

The normal distribution curve has the shape shown in Figure 1.

For this distribution, the probability P that the random variable X has a value between $x = x_1$ and $x = x_2$ is obtained by integrating the area under the probability density curve.

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} n(x; \mu, \sigma)dx \tag{2}$$

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} \exp\left\{ -\frac{1}{2}\left[\frac{x - \mu}{\sigma}\right]^2 \right\}dx \tag{3}$$

Now transform the data via a random variable Z.

$$Z = \frac{X - \mu}{\sigma} \tag{4}$$

$$dZ = \frac{1}{\sigma} dX \tag{5}$$

By substitution,

$$P\left((\sigma z_1 + \mu) < (\sigma Z + \mu) < (\sigma z_2 + \mu)\right) = \frac{1}{\sqrt{2\pi}} \int_{\sigma z_1 + \mu}^{\sigma z_2 + \mu} \exp\left\{-\frac{z^2}{2}\right\} dz \tag{6a}$$

Now assume a zero mean and a unity standard deviation.

$$P\left(z_1 < Z < z_2\right) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} \exp\left\{-\frac{z^2}{2}\right\} dz \tag{6b}$$

Note that the normal distribution is symmetric about the mean. Now consider the case where $z_2 > 0 > z_1$. Further require that each z limit be equidistant from the zero mean. Thus, $z_2 = -z_1$. Equation (6c) can be restated as

$$P\left(-z_2 < Z < z_2\right) = \frac{2}{\sqrt{2\pi}} \int_0^{z_2} \exp\left\{-\frac{z^2}{2}\right\} dz \tag{7}$$

Let

$$q = \frac{z}{\sqrt{2}} \tag{8}$$

$$dq = \frac{dz}{\sqrt{2}} \tag{9}$$

$$P\left(-z_2 < Z < z_2\right) = \frac{2}{\sqrt{\pi}} \int_0^{q_2} \exp\left\{-q^2\right\} dq \tag{10}$$

where

$$q_2 = \frac{z_2}{\sqrt{2}}$$

There are a number of equivalent methods for evaluating the integral in equation (10). One method is by use of the *error function.* The method used here is based on the *incomplete gamma function* $P(a,w)$, as given in Reference 2.

$$P(a, w) \equiv \frac{\gamma(a, w)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^w [\exp(-t)] \left[t^{(a-1)} \right] dt, \qquad a > 0 \tag{11}$$

Note that the gamma function itself is a standard mathematical function defined as

$$\Gamma(x) = \int_0^\infty \left[t^{x-1} \right] \exp(-t) \, dt, \quad x > 0 \tag{12}$$

$$\gamma(a, w) = \int_0^w \left[t^{a-1} \right] \exp(-t) \, dt \tag{13}$$

Now let

$$u^2 = t \tag{14}$$

$$2u \, du = dt \tag{15}$$

By substitution,

$$\gamma(a, w) = \int_0^{\sqrt{w}} [2u] \left[\exp\left(-u^2\right) \right] \left[u^{2(a-1)} \right] du \tag{16}$$

$$\gamma(a, w) = 2 \int_0^{\sqrt{w}} \left[\exp\left(-u^2\right) \right] \left[u^{(2a-1)} \right] du \tag{17}$$

Now let a=1/2.

$$\gamma\left(\frac{1}{2}, w\right) = 2 \int_0^{\sqrt{w}} \exp\left(-u^2\right) du \tag{18}$$

From Reference 2, the $\gamma(a, w)$ function can be represented in series form as

$$\gamma(a, w) = [\exp(-w)] \left[w^a \right] \sum_{n=0}^\infty \left[\frac{\Gamma(a)}{\Gamma(a+1+n)} w^n \right] \tag{19}$$

$$\gamma\left(\frac{1}{2}, w\right) = [\exp(-w)] \left[\sqrt{w} \right] \sum_{n=0}^\infty \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} w^n \right] \tag{20}$$

3

Equating the respective right-hand-sides of equation (18) and (20).

$$2 \int_0^{\sqrt{w}} \exp\left(-u^2\right) du = \left[\exp(-w)\right]\left[\sqrt{w}\right] \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} w^n \right] \tag{21}$$

Divide each side by $\sqrt{\pi}$.

$$\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{w}} \exp\left(-u^2\right) du = \frac{1}{\sqrt{\pi}} \left[\exp(-w)\right]\left[\sqrt{w}\right] \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} w^n \right] \tag{22}$$

Now let
$$y = \sqrt{w} \tag{23}$$

$$\frac{2}{\sqrt{\pi}} \int_0^{y} \exp\left(-u^2\right) du = \frac{1}{\sqrt{\pi}} \left[\exp\left(-y^2\right)\right][y] \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} y^{2n} \right] \tag{24}$$

Recall equation (10) restated here as equation (25).

$$P\left(-z_2 < Z < z_2\right) = \frac{2}{\sqrt{\pi}} \int_0^{q_2} \exp\left\{-q^2\right\} dq \tag{25}$$

The right side of equation (25) matches the left side of equation (24) for $u = q$ and $y = q_2$.

$$P\left(-z_2 < Z < z_2\right) = \frac{1}{\sqrt{\pi}} \left[\exp\left(-q_2^2\right)\right][q_2] \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+n\right)} q_2^{2n} \right], \quad q_2 = \frac{z_2}{\sqrt{2}} \tag{26}$$

Thus, the probability of the random variable Z can be calculated via (26).

Note that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{27}$$

Furthermore, the gamma function follows a recursive relationship, as given in Reference 2.

$$\Gamma(z+1) = z\,\Gamma(z) \tag{28}$$

Equations (27) and (28) allow for a simple computer program implementation of equation (26). Tables of probability values based on this equation are given in Appendix A.

Examples

1. A certain manufacturing process yields a reliability of 6σ with a zero mean. How many defects are expected per one million parts?

The Table in Appendix A shows that the probability of a part falling outside of the absolute value of 6σ is 1.97317518E-09. Thus, the expected number of part failures is 0.002, which is essentially zero.

2. A 1.5σ shift in the mean occurs in the previous manufacturing process. How many defects are expected per one million parts considering this shift?

The limits are thus -7.5σ to $+4.5\sigma$. Note that the values could be interchanged due to symmetry. For practical purposes, there is zero probability of an occurrence less than -7.5σ.

The Table in Appendix A shows that the probability of a part falling outside of the absolute value of 4.5σ is 6.79534625E-06. The probability from $+4.5\sigma$ to infinity is one-half this amount, or 3.397673125E-06. The number of expected defects in a lot of one million is thus 3.4, rounded to two significant digits.

References

1. R. Walpole and R. Myers, Probability and Statistics for Engineers and Scientists, 2nd edition, Macmillan, New York, 1978.
2. William Press, et al, Numerical Recipes The Art of Scientific Computing, Cambridge University Press, Cambridge, UK, 1986.
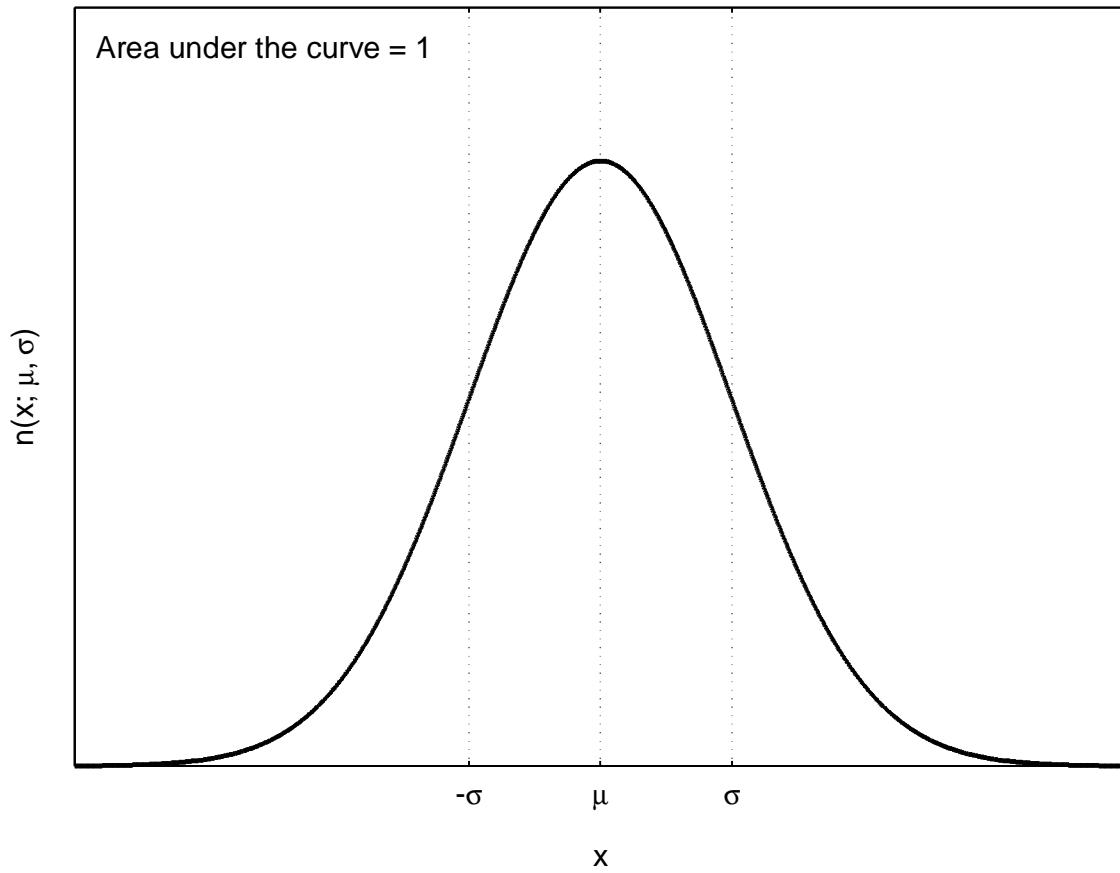
Normal Distribution Curve

Area under the curve = 1

Figure 1.

λ	P[-λ ≤ Z ≤ λ]	P[| Z | > λ]	λ	P[-λ ≤ Z ≤ λ]	P[| Z | > λ]
0.00	0	1	0.47	3.61644982E-01	6.38355018E-01
0.01	7.97871263E-03	9.92021287E-01	0.48	3.68772607E-01	6.31227393E-01
0.02	1.59566274E-02	9.84043373E-01	0.49	3.75866101E-01	6.24133899E-01
0.03	2.39329468E-02	9.76067053E-01	0.50	3.82924923E-01	6.17075077E-01
0.04	3.19068737E-02	9.68093126E-01	0.51	3.89948538E-01	6.10051462E-01
0.05	3.98776117E-02	9.60122388E-01	0.52	3.96936425E-01	6.03063575E-01
0.06	4.78443653E-02	9.52155635E-01	0.53	4.03888069E-01	5.96111931E-01
0.07	5.58063404E-02	9.44193660E-01	0.54	4.10802968E-01	5.89197032E-01
0.08	6.37627440E-02	9.36237256E-01	0.55	4.17680626E-01	5.82319374E-01
0.09	7.17127852E-02	9.28287215E-01	0.56	4.24520562E-01	5.75479438E-01
0.10	7.96556746E-02	9.20344325E-01	0.57	4.31322302E-01	5.68677698E-01
0.11	8.75906251E-02	9.12409375E-01	0.58	4.38085382E-01	5.61914618E-01
0.12	9.55168520E-02	9.04483148E-01	0.59	4.44809350E-01	5.55190650E-01
0.13	1.03433573E-01	8.96566427E-01	0.60	4.51493764E-01	5.48506236E-01
0.14	1.11340010E-01	8.88659990E-01	0.61	4.58138192E-01	5.41861808E-01
0.15	1.19235385E-01	8.80764615E-01	0.62	4.64742213E-01	5.35257787E-01
0.16	1.27118926E-01	8.72881074E-01	0.63	4.71305416E-01	5.28694584E-01
0.17	1.34989863E-01	8.65010137E-01	0.64	4.77827401E-01	5.22172599E-01
0.18	1.42847432E-01	8.57152568E-01	0.65	4.84307778E-01	5.15692222E-01
0.19	1.50690869E-01	8.49309131E-01	0.66	4.90746171E-01	5.09253829E-01
0.20	1.58519419E-01	8.41480581E-01	0.67	4.97142210E-01	5.02857790E-01
0.21	1.66332327E-01	8.33667673E-01	0.68	5.03495539E-01	4.96504461E-01
0.22	1.74128845E-01	8.25871155E-01	0.69	5.09805813E-01	4.90194187E-01
0.23	1.81908230E-01	8.18091770E-01	0.70	5.16072696E-01	4.83927304E-01
0.24	1.89669743E-01	8.10330257E-01	0.71	5.22295864E-01	4.77704136E-01
0.25	1.97412651E-01	8.02587349E-01	0.72	5.28475004E-01	4.71524996E-01
0.26	2.05136226E-01	7.94863774E-01	0.73	5.34609815E-01	4.65390185E-01
0.27	2.12839746E-01	7.87160254E-01	0.74	5.40700006E-01	4.59299994E-01
0.28	2.20522495E-01	7.79477505E-01	0.75	5.46745295E-01	4.53254705E-01
0.29	2.28183762E-01	7.71816238E-01	0.76	5.52745415E-01	4.47254585E-01
0.30	2.35822844E-01	7.64177156E-01	0.77	5.58700107E-01	4.41299893E-01
0.31	2.43439044E-01	7.56560956E-01	0.78	5.64609125E-01	4.35390875E-01
0.32	2.51031669E-01	7.48968331E-01	0.79	5.70472232E-01	4.29527768E-01
0.33	2.58600038E-01	7.41399962E-01	0.80	5.76289203E-01	4.23710797E-01
0.34	2.66143472E-01	7.33856528E-01	0.81	5.82059824E-01	4.17940176E-01
0.35	2.73661302E-01	7.26338698E-01	0.82	5.87783893E-01	4.12216107E-01
0.36	2.81152866E-01	7.18847134E-01	0.83	5.93461216E-01	4.06538784E-01
0.37	2.88617510E-01	7.11382490E-01	0.84	5.99091613E-01	4.00908387E-01
0.38	2.96054585E-01	7.03945415E-01	0.85	6.04674914E-01	3.95325086E-01
0.39	3.03463453E-01	6.96536547E-01	0.86	6.10210957E-01	3.89789043E-01
0.40	3.10843483E-01	6.89156517E-01	0.87	6.15699596E-01	3.84300404E-01
0.41	3.18194052E-01	6.81805948E-01	0.88	6.21140690E-01	3.78859310E-01
0.42	3.25514546E-01	6.74485454E-01	0.89	6.26534114E-01	3.73465886E-01
0.43	3.32804359E-01	6.67195641E-01	0.90	6.31879749E-01	3.68120251E-01
0.44	3.40062893E-01	6.59937107E-01	0.91	6.37177490E-01	3.62822510E-01
0.45	3.47289559E-01	6.52710441E-01	0.92	6.42427241E-01	3.57572759E-01
0.46	3.54483779E-01	6.45516221E-01	0.93	6.47628916E-01	3.52371084E-01

λ	P[-λ ≤ Z ≤ λ]	P[	Z	> λ]	λ	P[-λ ≤ Z ≤ λ]	P[	Z	> λ]
0.94	6.52782439E-01	3.47217561E-01	1.42	8.44392319E-01	1.55607681E-01
0.95	6.57887747E-01	3.42112253E-01	1.43	8.47282981E-01	1.52717019E-01
0.96	6.62944785E-01	3.37055215E-01	1.44	8.50132601E-01	1.49867399E-01
0.97	6.67953508E-01	3.32046492E-01	1.45	8.52941481E-01	1.47058519E-01
0.98	6.72913881E-01	3.27086119E-01	1.46	8.55709926E-01	1.44290074E-01
0.99	6.77825881E-01	3.22174119E-01	1.47	8.58438246E-01	1.41561754E-01
1.00	6.82689492E-01	3.17310508E-01	1.48	8.61126753E-01	1.38873247E-01
1.01	6.87504710E-01	3.12495290E-01	1.49	8.63775764E-01	1.36224236E-01
1.02	6.92271539E-01	3.07728461E-01	1.50	8.66385597E-01	1.33614403E-01
1.03	6.96989994E-01	3.03010006E-01	1.51	8.68956576E-01	1.31043424E-01
1.04	7.01660099E-01	2.98339901E-01	1.52	8.71489024E-01	1.28510976E-01
1.05	7.06281887E-01	2.93718113E-01	1.53	8.73983271E-01	1.26016729E-01
1.06	7.10855401E-01	2.89144599E-01	1.54	8.76439647E-01	1.23560353E-01
1.07	7.15380691E-01	2.84619309E-01	1.55	8.78858484E-01	1.21141516E-01
1.08	7.19857820E-01	2.80142180E-01	1.56	8.81240119E-01	1.18759881E-01
1.09	7.24286856E-01	2.75713144E-01	1.57	8.83584889E-01	1.16415111E-01
1.10	7.28667878E-01	2.71332122E-01	1.58	8.85893134E-01	1.14106866E-01
1.11	7.33000974E-01	2.66999026E-01	1.59	8.88165195E-01	1.11834805E-01
1.12	7.37286238E-01	2.62713762E-01	1.60	8.90401417E-01	1.09598583E-01
1.13	7.41523776E-01	2.58476224E-01	1.61	8.92602144E-01	1.07397856E-01
1.14	7.45713699E-01	2.54286301E-01	1.62	8.94767723E-01	1.05232277E-01
1.15	7.49856129E-01	2.50143871E-01	1.63	8.96898503E-01	1.03101497E-01
1.16	7.53951194E-01	2.46048806E-01	1.64	8.98994833E-01	1.01005167E-01
1.17	7.57999031E-01	2.42000969E-01	1.65	9.01057064E-01	9.89429361E-02
1.18	7.61999785E-01	2.38000215E-01	1.66	9.03085547E-01	9.69144525E-02
1.19	7.65953608E-01	2.34046392E-01	1.67	9.05080636E-01	9.49193636E-02
1.20	7.69860660E-01	2.30139340E-01	1.68	9.07042684E-01	9.29573157E-02
1.21	7.73721107E-01	2.26278893E-01	1.69	9.08972045E-01	9.10279546E-02
1.22	7.77535125E-01	2.22464875E-01	1.70	9.10869074E-01	8.91309255E-02
1.23	7.81302895E-01	2.18697105E-01	1.71	9.12734127E-01	8.72658730E-02
1.24	7.85024606E-01	2.14975394E-01	1.72	9.14567558E-01	8.54324416E-02
1.25	7.88700453E-01	2.11299547E-01	1.73	9.16369725E-01	8.36302752E-02
1.26	7.92330638E-01	2.07669362E-01	1.74	9.18140982E-01	8.18590180E-02
1.27	7.95915370E-01	2.04084630E-01	1.75	9.19881686E-01	8.01183137E-02
1.28	7.99454864E-01	2.00545136E-01	1.76	9.21592193E-01	7.84078066E-02
1.29	8.02949342E-01	1.97050658E-01	1.77	9.23272859E-01	7.67271407E-02
1.30	8.06399031E-01	1.93600969E-01	1.78	9.24924039E-01	7.50759607E-02
1.31	8.09804164E-01	1.90195836E-01	1.79	9.26546089E-01	7.34539114E-02
1.32	8.13164982E-01	1.86835018E-01	1.80	9.28139362E-01	7.18606382E-02
1.33	8.16481729E-01	1.83518271E-01	1.81	9.29704213E-01	7.02957872E-02
1.34	8.19754655E-01	1.80245345E-01	1.82	9.31240995E-01	6.87590049E-02
1.35	8.22984017E-01	1.77015983E-01	1.83	9.32750061E-01	6.72499388E-02
1.36	8.26170076E-01	1.73829924E-01	1.84	9.34231763E-01	6.57682373E-02
1.37	8.29313098E-01	1.70686902E-01	1.85	9.35686450E-01	6.43135496E-02
1.38	8.32413355E-01	1.67586645E-01	1.86	9.37114474E-01	6.28855260E-02
1.39	8.35471123E-01	1.64528877E-01	1.87	9.38516182E-01	6.14838179E-02
1.40	8.38486682E-01	1.61513318E-01	1.88	9.39891922E-01	6.01080779E-02
1.41	8.41460317E-01	1.58539683E-01	1.89	9.41242040E-01	5.87579601E-02

λ	P[-λ ≤ Z ≤ λ]	P[| Z | > λ]	λ	P[-λ ≤ Z ≤ λ]	P[| Z | > λ]
1.90	9.42566880E-01	5.74331196E-02	2.38	9.82687362E-01	1.73126381E-02
1.91	9.43866787E-01	5.61332133E-02	2.39	9.83151627E-01	1.68483728E-02
1.92	9.45142101E-01	5.48578994E-02	2.40	9.83604928E-01	1.63950718E-02
1.93	9.46393162E-01	5.36068378E-02	2.41	9.84047479E-01	1.59525205E-02
1.94	9.47620310E-01	5.23796899E-02	2.42	9.84479493E-01	1.55205071E-02
1.95	9.48823881E-01	5.11761190E-02	2.43	9.84901177E-01	1.50988228E-02
1.96	9.50004210E-01	4.99957903E-02	2.44	9.85312738E-01	1.46872619E-02
1.97	9.51161629E-01	4.88383706E-02	2.45	9.85714379E-01	1.42856215E-02
1.98	9.52296471E-01	4.77035287E-02	2.46	9.86106298E-01	1.38937016E-02
1.99	9.53409064E-01	4.65909355E-02	2.47	9.86488695E-01	1.35113052E-02
2.00	9.54499736E-01	4.55002639E-02	2.48	9.86861762E-01	1.31382383E-02
2.01	9.55568811E-01	4.44311889E-02	2.49	9.87225690E-01	1.27743095E-02
2.02	9.56616612E-01	4.33833875E-02	2.50	9.87580669E-01	1.24193307E-02
2.03	9.57643461E-01	4.23565393E-02	2.51	9.87926884E-01	1.20731162E-02
2.04	9.58649674E-01	4.13503257E-02	2.52	9.88264517E-01	1.17354834E-02
2.05	9.59635569E-01	4.03644308E-02	2.53	9.88593747E-01	1.14062527E-02
2.06	9.60601459E-01	3.93985408E-02	2.54	9.88914753E-01	1.10852469E-02
2.07	9.61547656E-01	3.84523445E-02	2.55	9.89227708E-01	1.07722919E-02
2.08	9.62474467E-01	3.75255329E-02	2.56	9.89532784E-01	1.04672163E-02
2.09	9.63382200E-01	3.66177997E-02	2.57	9.89830149E-01	1.01698515E-02
2.10	9.64271159E-01	3.57288411E-02	2.58	9.90119968E-01	9.88003152E-03
2.11	9.65141644E-01	3.48583559E-02	2.59	9.90402407E-01	9.59759319E-03
2.12	9.65993955E-01	3.40060453E-02	2.60	9.90677624E-01	9.32237605E-03
2.13	9.66828387E-01	3.31716134E-02	2.61	9.90945778E-01	9.05422227E-03
2.14	9.67645233E-01	3.23547667E-02	2.62	9.91207023E-01	8.79297670E-03
2.15	9.68444785E-01	3.15552148E-02	2.63	9.91461513E-01	8.53848682E-03
2.16	9.69227330E-01	3.07726696E-02	2.64	9.91709397E-01	8.29060272E-03
2.17	9.69993154E-01	3.00068459E-02	2.65	9.91950823E-01	8.04917709E-03
2.18	9.70742538E-01	2.92574616E-02	2.66	9.92185935E-01	7.81406515E-03
2.19	9.71475763E-01	2.85242368E-02	2.67	9.92414875E-01	7.58512470E-03
2.20	9.72193105E-01	2.78068950E-02	2.68	9.92637784E-01	7.36221602E-03
2.21	9.72894838E-01	2.71051623E-02	2.69	9.92854798E-01	7.14520190E-03
2.22	9.73581232E-01	2.64187676E-02	2.70	9.93066052E-01	6.93394761E-03
2.23	9.74252557E-01	2.57474429E-02	2.71	9.93271679E-01	6.72832081E-03
2.24	9.74909077E-01	2.50909229E-02	2.72	9.93471808E-01	6.52819163E-03
2.25	9.75551055E-01	2.44489453E-02	2.73	9.93666567E-01	6.33343255E-03
2.26	9.76178749E-01	2.38212508E-02	2.74	9.93856082E-01	6.14391844E-03
2.27	9.76792417E-01	2.32075830E-02	2.75	9.94040474E-01	5.95952647E-03
2.28	9.77392312E-01	2.26076885E-02	2.76	9.94219864E-01	5.78013615E-03
2.29	9.77978683E-01	2.20213166E-02	2.77	9.94394371E-01	5.60562927E-03
2.30	9.78551780E-01	2.14482200E-02	2.78	9.94564110E-01	5.43588985E-03
2.31	9.79111846E-01	2.08881541E-02	2.79	9.94729196E-01	5.27080416E-03
2.32	9.79659123E-01	2.03408773E-02	2.80	9.94889739E-01	5.11026066E-03
2.33	9.80193849E-01	1.98061511E-02	2.81	9.95045850E-01	4.95415000E-03
2.34	9.80716260E-01	1.92837399E-02	2.82	9.95197635E-01	4.80236495E-03
2.35	9.81226589E-01	1.87734111E-02	2.83	9.95345200E-01	4.65480041E-03
2.36	9.81725065E-01	1.82749351E-02	2.84	9.95488647E-01	4.51135338E-03
2.37	9.82211915E-01	1.77880853E-02	2.85	9.95628077E-01	4.37192291E-03

λ	P[-λ ≤ Z ≤ λ]	P[|Z| > λ]	λ	P[-λ ≤ Z ≤ λ]	P[|Z| > λ]
2.86	9.95763590E-01	4.23641008E-03	3.34	9.99162216E-01	8.37783899E-04
2.87	9.95895282E-01	4.10471799E-03	3.35	9.99191884E-01	8.08115604E-04
2.88	9.96023248E-01	3.97675171E-03	3.36	9.99220575E-01	7.79424725E-04
2.89	9.96147582E-01	3.85241826E-03	3.37	9.99248318E-01	7.51681837E-04
2.90	9.96268373E-01	3.73162660E-03	3.38	9.99275142E-01	7.24858298E-04
2.91	9.96385712E-01	3.61428756E-03	3.39	9.99301074E-01	6.98926237E-04
2.92	9.96499686E-01	3.50031386E-03	3.40	9.99326141E-01	6.73858531E-04
2.93	9.96610380E-01	3.38962004E-03	3.41	9.99350371E-01	6.49628795E-04
2.94	9.96717878E-01	3.28212247E-03	3.42	9.99373789E-01	6.26211357E-04
2.95	9.96822261E-01	3.17773929E-03	3.43	9.99396419E-01	6.03581249E-04
2.96	9.96923610E-01	3.07639042E-03	3.44	9.99418286E-01	5.81714187E-04
2.97	9.97022003E-01	2.97799749E-03	3.45	9.99439413E-01	5.60586554E-04
2.98	9.97117516E-01	2.88248383E-03	3.46	9.99459825E-01	5.40175388E-04
2.99	9.97210226E-01	2.78977447E-03	3.47	9.99479542E-01	5.20458365E-04
3.00	9.97300204E-01	2.69979606E-03	3.48	9.99498586E-01	5.01413783E-04
3.01	9.97387523E-01	2.61247690E-03	3.49	9.99516979E-01	4.83020547E-04
3.02	9.97472253E-01	2.52774686E-03	3.50	9.99534742E-01	4.65258158E-04
3.03	9.97554463E-01	2.44553739E-03	3.51	9.99551893E-01	4.48106694E-04
3.04	9.97634219E-01	2.36578149E-03	3.52	9.99568453E-01	4.31546799E-04
3.05	9.97711586E-01	2.28841366E-03	3.53	9.99584440E-01	4.15559667E-04
3.06	9.97786630E-01	2.21336991E-03	3.54	9.99599873E-01	4.00127032E-04
3.07	9.97859412E-01	2.14058771E-03	3.55	9.99614769E-01	3.85231151E-04
3.08	9.97929994E-01	2.07000595E-03	3.56	9.99629145E-01	3.70854794E-04
3.09	9.97998435E-01	2.00156495E-03	3.57	9.99643019E-01	3.56981228E-04
3.10	9.98064794E-01	1.93520643E-03	3.58	9.99656406E-01	3.43594207E-04
3.11	9.98129127E-01	1.87087344E-03	3.59	9.99669322E-01	3.30677961E-04
3.12	9.98191490E-01	1.80851040E-03	3.60	9.99681783E-01	3.18217180E-04
3.13	9.98251937E-01	1.74806303E-03	3.61	9.99693803E-01	3.06197005E-04
3.14	9.98310522E-01	1.68947835E-03	3.62	9.99705397E-01	2.94603016E-04
3.15	9.98367295E-01	1.63270463E-03	3.63	9.99716579E-01	2.83421220E-04
3.16	9.98422309E-01	1.57769139E-03	3.64	9.99727362E-01	2.72638041E-04
3.17	9.98475611E-01	1.52438938E-03	3.65	9.99737760E-01	2.62240309E-04
3.18	9.98527249E-01	1.47275052E-03	3.66	9.99747785E-01	2.52215248E-04
3.19	9.98577272E-01	1.42272794E-03	3.67	9.99757450E-01	2.42550469E-04
3.20	9.98625724E-01	1.37427588E-03	3.68	9.99766766E-01	2.33233954E-04
3.21	9.98672650E-01	1.32734972E-03	3.69	9.99775746E-01	2.24254052E-04
3.22	9.98718094E-01	1.28190597E-03	3.70	9.99784401E-01	2.15599467E-04
3.23	9.98762098E-01	1.23790218E-03	3.71	9.99792741E-01	2.07259247E-04
3.24	9.98804703E-01	1.19529700E-03	3.72	9.99800777E-01	1.99222778E-04
3.25	9.98845950E-01	1.15405008E-03	3.73	9.99808520E-01	1.91479771E-04
3.26	9.98885878E-01	1.11412214E-03	3.74	9.99815980E-01	1.84020255E-04
3.27	9.98924525E-01	1.07547484E-03	3.75	9.99823165E-01	1.76834570E-04
3.28	9.98961929E-01	1.03807087E-03	3.76	9.99830087E-01	1.69913357E-04
3.29	9.98998126E-01	1.00187383E-03	3.77	9.99836752E-01	1.63247547E-04
3.30	9.99033152E-01	9.66848285E-04	3.78	9.99843172E-01	1.56828359E-04
3.31	9.99067040E-01	9.32959712E-04	3.79	9.99849353E-01	1.50647285E-04
3.32	9.99099826E-01	9.00174481E-04	3.80	9.99855304E-01	1.44696088E-04
3.33	9.99131540E-01	8.68459841E-04	3.81	9.99861033E-01	1.38966792E-04

λ	$P[-\lambda \leq Z \leq \lambda]$	$P[\,|Z| > \lambda]$	λ	$P[-\lambda \leq Z \leq \lambda]$	$P[\,|Z| > \lambda]$
3.82	9.99866548E-01	1.33451674E-04	4.30	9.99982920E-01	1.70798109E-05
3.83	9.99871857E-01	1.28143259E-04	4.31	9.99983675E-01	1.63254546E-05
3.84	9.99876966E-01	1.23034310E-04	4.32	9.99984397E-01	1.56029201E-05
3.85	9.99881882E-01	1.18117825E-04	4.33	9.99985089E-01	1.49109342E-05
3.86	9.99886613E-01	1.13387025E-04	4.34	9.99985752E-01	1.42482716E-05
3.87	9.99891165E-01	1.08835353E-04	4.35	9.99986386E-01	1.36137532E-05
3.88	9.99895544E-01	1.04456465E-04	4.36	9.99986994E-01	1.30062444E-05
3.89	9.99899756E-01	1.00244222E-04	4.37	9.99987575E-01	1.24246537E-05
3.90	9.99903807E-01	9.61926880E-05	4.38	9.99988132E-01	1.18679309E-05
3.91	9.99907704E-01	9.22961211E-05	4.39	9.99988665E-01	1.13350661E-05
3.92	9.99911451E-01	8.85489686E-05	4.40	9.99989175E-01	1.08250878E-05
3.93	9.99915054E-01	8.49458616E-05	4.41	9.99989663E-01	1.03370619E-05
3.94	9.99918518E-01	8.14816091E-05	4.42	9.99990130E-01	9.87009013E-06
3.95	9.99921849E-01	7.81511932E-05	4.43	9.99990577E-01	9.42330882E-06
3.96	9.99925050E-01	7.49497634E-05	4.44	9.99991004E-01	8.99588778E-06
3.97	9.99928127E-01	7.18726318E-05	4.45	9.99991413E-01	8.58702894E-06
3.98	9.99931085E-01	6.89152682E-05	4.46	9.99991804E-01	8.19596529E-06
3.99	9.99933927E-01	6.60732953E-05	4.47	9.99992178E-01	7.82195972E-06
4.00	9.99936658E-01	6.33424837E-05	4.48	9.99992536E-01	7.46430392E-06
4.01	9.99939281E-01	6.07187479E-05	4.49	9.99992878E-01	7.12231736E-06
4.02	9.99941802E-01	5.81981414E-05	4.50	9.99993205E-01	6.79534625E-06
4.03	9.99944223E-01	5.57768529E-05	4.51	9.99993517E-01	6.48276257E-06
4.04	9.99946549E-01	5.34512014E-05	4.52	9.99993816E-01	6.18396314E-06
4.05	9.99948782E-01	5.12176329E-05	4.53	9.99994102E-01	5.89836865E-06
4.06	9.99950927E-01	4.90727159E-05	4.54	9.99994375E-01	5.62542282E-06
4.07	9.99952987E-01	4.70131377E-05	4.55	9.99994635E-01	5.36459156E-06
4.08	9.99954964E-01	4.50357008E-05	4.56	9.99994885E-01	5.11536208E-06
4.09	9.99956863E-01	4.31373189E-05	4.57	9.99995123E-01	4.87724215E-06
4.10	9.99958685E-01	4.13150138E-05	4.58	9.99995350E-01	4.64975930E-06
4.11	9.99960434E-01	3.95659117E-05	4.59	9.99995568E-01	4.43246010E-06
4.12	9.99962113E-01	3.78872399E-05	4.60	9.99995775E-01	4.22490940E-06
4.13	9.99963724E-01	3.62763234E-05	4.61	9.99995973E-01	4.02668971E-06
4.14	9.99965269E-01	3.47305821E-05	4.62	9.99996163E-01	3.83740044E-06
4.15	9.99966752E-01	3.32475275E-05	4.63	9.99996343E-01	3.65665733E-06
4.16	9.99968175E-01	3.18247594E-05	4.64	9.99996516E-01	3.48409178E-06
4.17	9.99969540E-01	3.04599639E-05	4.65	9.99996681E-01	3.31935029E-06
4.18	9.99970849E-01	2.91509096E-05	4.66	9.99996838E-01	3.16209384E-06
4.19	9.99972105E-01	2.78954455E-05	4.67	9.99996988E-01	3.01199735E-06
4.20	9.99973309E-01	2.66914980E-05	4.68	9.99997131E-01	2.86874917E-06
4.21	9.99974463E-01	2.55370688E-05	4.69	9.99997268E-01	2.73205049E-06
4.22	9.99975570E-01	2.44302318E-05	4.70	9.99997398E-01	2.60161491E-06
4.23	9.99976631E-01	2.33691312E-05	4.71	9.99997523E-01	2.47716791E-06
4.24	9.99977648E-01	2.23519787E-05	4.72	9.99997642E-01	2.35844643E-06
4.25	9.99978623E-01	2.13770515E-05	4.73	9.99997755E-01	2.24519837E-06
4.26	9.99979557E-01	2.04426904E-05	4.74	9.99997863E-01	2.13718219E-06
4.27	9.99980453E-01	1.95472967E-05	4.75	9.99997966E-01	2.03416649E-06
4.28	9.99981311E-01	1.86893313E-05	4.76	9.99998064E-01	1.93592959E-06
4.29	9.99982133E-01	1.78673118E-05	4.77	9.99998158E-01	1.84225919E-06

λ	$P[-\lambda \leq Z \leq \lambda]$	$P[\,|Z| > \lambda]$	λ	$P[-\lambda \leq Z \leq \lambda]$	$P[\,|Z| > \lambda]$
4.78	9.99998247E-01	1.75295195E-06	5.26	9.99999856E-01	1.44055402E-07
4.79	9.99998332E-01	1.66781313E-06	5.27	9.99999864E-01	1.36423757E-07
4.80	9.99998413E-01	1.58665630E-06	5.28	9.99999871E-01	1.29183888E-07
4.81	9.99998491E-01	1.50930296E-06	5.29	9.99999878E-01	1.22316360E-07
4.82	9.99998564E-01	1.43558221E-06	5.30	9.99999884E-01	1.15802681E-07
4.83	9.99998635E-01	1.36533050E-06	5.31	9.99999890E-01	1.09625236E-07
4.84	9.99998702E-01	1.29839129E-06	5.32	9.99999896E-01	1.03767252E-07
4.85	9.99998765E-01	1.23461474E-06	5.33	9.99999902E-01	9.82127658E-08
4.86	9.99998826E-01	1.17385753E-06	5.34	9.99999907E-01	9.29465822E-08
4.87	9.99998884E-01	1.11598249E-06	5.35	9.99999912E-01	8.79542320E-08
4.88	9.99998939E-01	1.06085841E-06	5.36	9.99999917E-01	8.32219517E-08
4.89	9.99998992E-01	1.00835978E-06	5.37	9.99999921E-01	7.87366431E-08
4.90	9.99999042E-01	9.58366554E-07	5.38	9.99999926E-01	7.44858379E-08
4.91	9.99999089E-01	9.10763929E-07	5.39	9.99999930E-01	7.04576834E-08
4.92	9.99999135E-01	8.65442124E-07	5.40	9.99999933E-01	6.66408971E-08
4.93	9.99999178E-01	8.22296168E-07	5.41	9.99999937E-01	6.30247493E-08
4.94	9.99999219E-01	7.81225708E-07	5.42	9.99999940E-01	5.95990353E-08
4.95	9.99999258E-01	7.42134815E-07	5.43	9.99999944E-01	5.63540533E-08
4.96	9.99999295E-01	7.04931797E-07	5.44	9.99999947E-01	5.32805712E-08
4.97	9.99999330E-01	6.69529017E-07	5.45	9.99999950E-01	5.03698206E-08
4.98	9.99999364E-01	6.35842732E-07	5.46	9.99999952E-01	4.76134577E-08
4.99	9.99999396E-01	6.03792926E-07	5.47	9.99999955E-01	4.50035575E-08
5.00	9.99999427E-01	5.73303145E-07	5.48	9.99999957E-01	4.25325835E-08
5.01	9.99999456E-01	5.44300354E-07	5.49	9.99999960E-01	4.01933735E-08
5.02	9.99999483E-01	5.16714795E-07	5.50	9.99999962E-01	3.79791240E-08
5.03	9.99999510E-01	4.90479835E-07	5.51	9.99999964E-01	3.58833703E-08
5.04	9.99999534E-01	4.65531847E-07	5.52	9.99999966E-01	3.38999655E-08
5.05	9.99999558E-01	4.41810065E-07	5.53	9.99999968E-01	3.20230791E-08
5.06	9.99999581E-01	4.19256477E-07	5.54	9.99999970E-01	3.02471654E-08
5.07	9.99999602E-01	3.97815702E-07	5.55	9.99999971E-01	2.85669606E-08
5.08	9.99999623E-01	3.77434868E-07	5.56	9.99999973E-01	2.69774658E-08
5.09	9.99999642E-01	3.58063516E-07	5.57	9.99999975E-01	2.54739340E-08
5.10	9.99999660E-01	3.39653482E-07	5.58	9.99999976E-01	2.40518585E-08
5.11	9.99999678E-01	3.22158813E-07	5.59	9.99999977E-01	2.27069600E-08
5.12	9.99999694E-01	3.05535657E-07	5.60	9.99999979E-01	2.14351804E-08
5.13	9.99999710E-01	2.89742177E-07	5.61	9.99999980E-01	2.02326618E-08
5.14	9.99999725E-01	2.74738470E-07	5.62	9.99999981E-01	1.90957450E-08
5.15	9.99999740E-01	2.60486459E-07	5.63	9.99999982E-01	1.80209613E-08
5.16	9.99999753E-01	2.46949842E-07	5.64	9.99999983E-01	1.70050161E-08
5.17	9.99999766E-01	2.34093994E-07	5.65	9.99999984E-01	1.60447845E-08
5.18	9.99999778E-01	2.21885900E-07	5.66	9.99999985E-01	1.51372976E-08
5.19	9.99999790E-01	2.10294080E-07	5.67	9.99999986E-01	1.42797527E-08
5.20	9.99999801E-01	1.99288526E-07	5.68	9.99999987E-01	1.34694739E-08
5.21	9.99999811E-01	1.88840632E-07	5.69	9.99999987E-01	1.27039352E-08
5.22	9.99999821E-01	1.78923131E-07	5.70	9.99999988E-01	1.19807441E-08
5.23	9.99999830E-01	1.69510039E-07	5.71	9.99999989E-01	1.12976164E-08
5.24	9.99999839E-01	1.60576597E-07	5.72	9.99999989E-01	1.06524056E-08
5.25	9.99999848E-01	1.52099210E-07	5.73	9.99999990E-01	1.00430655E-08

λ	P[-λ ≤ Z ≤ λ]	P[| Z | > λ]
5.74	9.99999991E-01	9.46765444E-09
5.75	9.99999991E-01	8.92434526E-09
5.76	9.99999992E-01	8.41139436E-09
5.77	9.99999992E-01	7.92715482E-09
5.78	9.99999993E-01	7.47006357E-09
5.79	9.99999993E-01	7.03864433E-09
5.80	9.99999993E-01	6.63149358E-09
5.81	9.99999994E-01	6.24728469E-09
5.82	9.99999994E-01	5.88476290E-09
5.83	9.99999994E-01	5.54273660E-09
5.84	9.99999995E-01	5.22008359E-09
5.85	9.99999995E-01	4.91572960E-09
5.86	9.99999995E-01	4.62867344E-09
5.87	9.99999996E-01	4.35795089E-09
5.88	9.99999996E-01	4.10266543E-09
5.89	9.99999996E-01	3.86195464E-09
5.90	9.99999996E-01	3.63501462E-09
5.91	9.99999997E-01	3.42107731E-09
5.92	9.99999997E-01	3.21941684E-09
5.93	9.99999997E-01	3.02934777E-09
5.94	9.99999997E-01	2.85022195E-09
5.95	9.99999997E-01	2.68142553E-09
5.96	9.99999997E-01	2.52237919E-09
5.97	9.99999998E-01	2.37253495E-09
5.98	9.99999998E-01	2.23137453E-09
5.99	9.99999998E-01	2.09841178E-09
6.00	9.99999998E-01	1.97317518E-09