APPLICATION OF THE NEWTON-RAPHSON
METHOD TO VIBRATION PROBLEMS
Revision E

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Index

Table 0. Equation Index	
Equation	Location
$\cos(x)\cosh(x) = -1$	Main Text
$\cos(x)\cosh(x) = 1$	Appendix A
$\tan(x) = \tanh(x)$	Appendix B
$-\tan(x) = \tanh(x)$	Appendix C
$J_o(x)$	Appendix D
$J_1(x)$	Appendix E
$J_2(x)$	Appendix F
$\frac{d}{dx} J_o(x)$	Appendix G

Introduction

The Newton-Raphson method is a method for finding the roots of equations. It is particularly useful for transcendental equations, composed of mixed trigonometric and hyperbolic terms. Such equations occur in vibration analysis. An example is the calculation of natural frequencies of continuous structures, such as beams and plates. The purpose of this tutorial is to show how the Newton-Raphson method is applied to vibration problems.

Derivation

The Newton-Raphson method is derived from the Taylor series.

The Taylor series equation is taken from Reference 1. Consider a function f(x) which is continuous and single-valued and has all its derivatives on an interval including x = a.

The Taylor series is defined as

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n \quad (1)$$

where

$$R_n = \frac{f^{(n+1)}(\theta x)}{(n+1)!}(x-a)^{n+1} \quad , \quad 0 < \theta < 1$$

The series represents f(x) for those values of x for which $R_n \rightarrow 0$ as $n \rightarrow \infty$.

Now consider a simplified Taylor series.

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) \quad (2)$$

Solve for x.

$$f(x) = f(a) + (x-a) f'(a) \quad (3)$$

$$f(x) - f(a) = (x-a) f'(a) \quad (4)$$

$$\frac{f(x) - f(a)}{f'(a)} = (x-a) \quad (5)$$

$$\frac{f(x) - f(a)}{f'(a)} + a = x \quad (6)$$

$$x = a + \frac{f(x) - f(a)}{f'(a)} \quad (7)$$

The roots are the values of x for which f(x) = 0. Thus

$$x = a - \frac{f(a)}{f'(a)} \quad (8)$$

Equation (7) is used in an iterative manner to find the roots. This process is demonstrated by an example.

Example

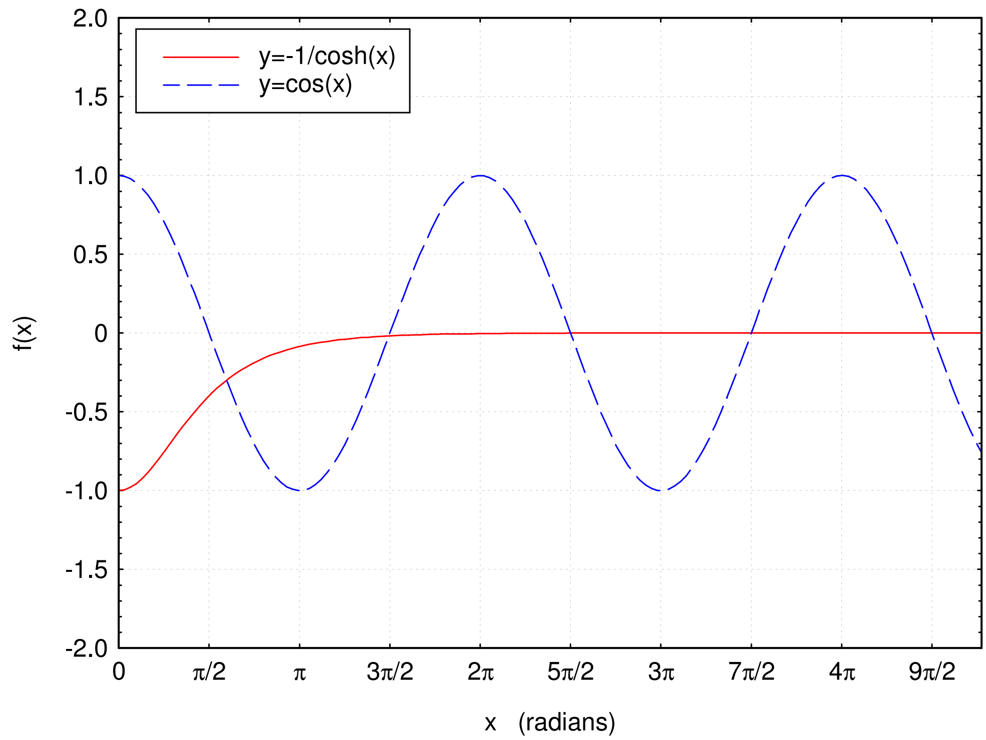

Figure 1.

Consider a cantilever beam undergoing bending vibration. The natural frequencies are governed by the following equation.

$$\cos(x)\cosh(x) = -1 \tag{9a}$$

$$\cos(x) = -1/\cosh(x) \tag{9b}$$

Find the first and second roots.

Equation (9) can be represented as a function.

$$f(x) = \cos(x)\cosh(x) + 1 \tag{10}$$

$$f'(x) = -\sin(x)\cosh(x) + \cos(x)\sinh(x) \tag{11}$$

The Newton-Raphson equation is obtained by substituting equations (10) and (11) into (8).

$$x = a - \frac{\cos(a)\cosh(a) + 1}{-\sin(a)\cosh(a) + \cos(a)\sinh(a)} \tag{12}$$

<u>Graphical Analysis</u>

A graph is a useful tool for obtaining initial estimates of the roots. A graph of equation (9b) is shown in Figure 1, on the previous page.

For reference, note that

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{13}$$

Equation (10) can be rewritten as

$$f(x) = \cos(x)\left[\frac{e^{x} + e^{-x}}{2}\right] + 1 \tag{14}$$

By inspection, an approximate expression for the roots x_n is

$$x_n \approx (2n-1)\pi/2 \text{ , n=1, 2, 3, …..} \tag{15}$$

The subscript n is added to denote that there are multiple roots. Figure 1 confirms that equation (15) gives a reasonable approximation for the first and second roots.

<u>Numerical Analysis</u>

Formula

Estimate the first root as $\pi/2$.

Recall equation (12) which is restated as equation (16).

$$x = a - \frac{\cos(a)\cosh(a) + 1}{-\sin(a)\cosh(a) + \cos(a)\sinh(a)} \tag{16}$$

First Root

The method is carried out by setting a = π/2 in equation (16). An x value is then calculated. Next, a is set equal to x. The process is repeated until x converges to a. The results are shown in Table 1.

Table 1. First Root		
Iteration	a	x
1	π/2	1.969334
2	1.969334	1.881061
3	1.881061	1.875130
4	1.875130	1.875104
5	1.875104	1.875104

The method yields a value of 1.875104 for the first root after five iterations.

Verify by substituting the root into equation (10). Note that the root is in units of radians.

$$\cos(1.875104)\cosh(1.875104) + 1 = 2.842\text{e - }07 \tag{17}$$

The result is approximately equal to zero. Note that the "exact root" may be an irrational number. Only a few decimal places, however, are required for "engineering accuracy." The accuracy of the root is thus verified.

Second Root

Estimate the second root as 3π/2. The results are shown in Table 2.

Table 2. Second Root		
Iteration	a	x
1	3π/2	4.694424
2	4.694424	4.694091
3	4.694091	4.694091

The method rapidly converges to a value of 4.694091 for the second root.

Verify by substituting the root into equation (10). Note that the root is in units of radians.

$$\cos(4.694091)\cosh(4.694091) + 1 = -7.134e-06 \tag{17}$$

The result is approximately equal to zero. The accuracy of the root is verified.

Application to a Cantilever Beam

A cantilever beam is shown in Figure 2.

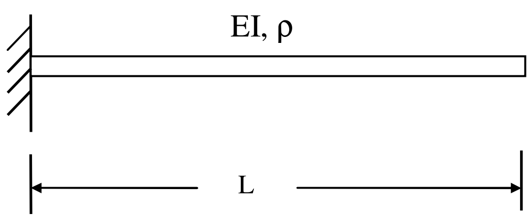

Figure 2.

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)

Again, the natural frequencies are given by equation (9). Let x = $\beta_n L$

Table 3. Roots	
Index	$\beta_n L$
n = 1	1.875104
n = 2	4.694091

$$\omega_n = \beta_n^{\,2}\sqrt{\frac{EI}{\rho}} \tag{19}$$

By substitution,

$$\omega_1 = \left[\frac{1.87510}{L}\right]^2\sqrt{\frac{EI}{\rho}} \tag{20}$$

$$\omega_2 = \left[\frac{4.694091}{L}\right]^2\sqrt{\frac{EI}{\rho}} \tag{21}$$

Note that the natural frequency ω_n is typically expressed in units of (radians/sec).

Further details about the cantilever beam problem are given in Reference 2.

Additional Examples

Additional examples are given in the appendices.

References

1. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.
2. Tom Irvine, Natural Frequencies of Beam Bending Modes Revision K, Vibrationdata, 2004.
3. Tom Irvine, Natural Frequencies of Multispan Beams Revision A, Vibrationdata, 1999.
4. Tom Irvine, Longitudinal Vibration of a Tapered Rod, Vibrationdata, 2003.
5. Tom Irvine, An Introduction to Fluid Slosh, Vibrationdata, 2010.

APPENDIX A

Equation (A-1) is obtained for certain vibration problems. It is the characteristic equation for the following beams:

1. free-free beam
2. clamped-clamped beam
3. clamped-pinned-clamped beam (1 of 2 equations)

$$\cos(x)\cosh(x) = 1 \tag{A-1}$$

An equivalent form is shown in equation (A-2). This equation is graphed in Figure A-1.

$$\cos(x) = 1/\cosh(x) \tag{A-2}$$

The graph in Figure A-1 shows that the roots are given approximately by equation (A-3).

$$x_n \approx \pi\left[\frac{1}{2}+n\right] \quad , \ n=1,2,3, \ldots. \tag{A-3}$$

Return to equation (A-1). The function equals zero if the dependent variable is a root.

$$f(x) = \cos(x)\cosh(x) - 1 \tag{A-4}$$

The derivative is

$$f'(x) = -\sin(x)\cosh(x) + \cos(x)\sinh(x) \tag{A-5}$$

Recall the Newton-Raphson equation.

$$x = a - \frac{f(a)}{f'(a)} \tag{A-6}$$

By substitution,

$$x = a - \frac{\cos(a)\cosh(a)+1}{-\sin(a)\cosh(a)+\cos(a)\sinh(a)} \tag{A-7}$$

The first two roots are calculated as shown in Tables A-1 and A-2, respectively.

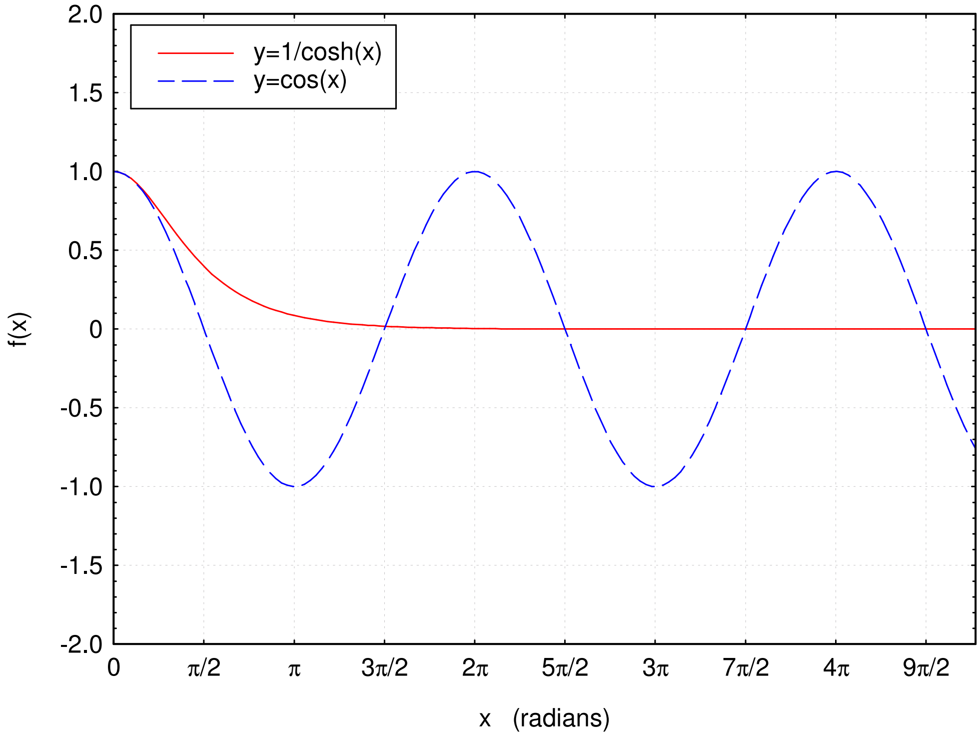

Figure A-1. Graph to Estimate Roots

The roots occur at the intersection of the two curves. Zero is not counted as a root for the purpose of determining the natural frequencies

Table A-1. First Root		
Iteration	a	x
1	4.5	4.80388
2	4.80388	4.73492
3	4.73492	4.73006
4	4.73006	4.73004
5	4.73004	4.73004

Table A-2. Second Root		
Iteration	a	x
1	7.9	7.85527
2	7.85527	7.85321
3	7.85321	7.85320
4	7.85320	7.85320
5	7.85320	7.85320

APPENDIX B

Equation (B-1) is obtained for certain vibration problems. It is the characteristic equation for the following beams:

1. free-pinned beam
2. clamped-pinned beam
3. clamped-pinned-clamped beam (2 of 2 equations)

$$\tan(x) = \tanh(x) \tag{B-1}$$

Equation (B-1) is graphed in Figure B-1. The graph in Figure B-1 shows that the roots are given approximately by equation (B-2).

$$x_n \approx \pi\left(\frac{5}{4}+n\right), \quad n=1, 2, 3, \ldots. \tag{B-2}$$

Return to equation (B-1). The function equals zero if the dependent variable is a root.

$$f(x) = \tan(x) - \tanh(x) \tag{B-3}$$

The derivative is

$$f'(x) = \frac{1}{\cos^2(x)} - \frac{1}{\cosh^2(x)} \tag{B-4}$$

Recall the Newton-Raphson equation.

$$x = a - \frac{f(a)}{f'(a)} \tag{B-5}$$

By substitution,

$$x = a - \frac{\tan(a) - \tanh(a)}{\dfrac{1}{\cos^2(a)} - \dfrac{1}{\cosh^2(a)}} \tag{B-6}$$

Simplifying

$$x = a - \frac{\tan(a)\cosh(a)\cos^2(a) - \tanh(a)\cosh(a)\cos^2(a)}{\cosh^2(a) - \cos^2(a)} \tag{B-7}$$

$$x = a - \frac{\sin(a)\cos(a)\cosh(a) - \sinh(a)\cos^2(a)}{\cosh^2(a) - \cos^2(a)} \tag{B-8}$$

$$x = a - \frac{[\sin(a)\cosh(a) - \sinh(a)\cos(a)]\cos(a)}{\cosh^2(a) - \cos^2(a)} \tag{B-9}$$

The first two roots are calculated as shown in Tables B-1 and B-2, respectively.

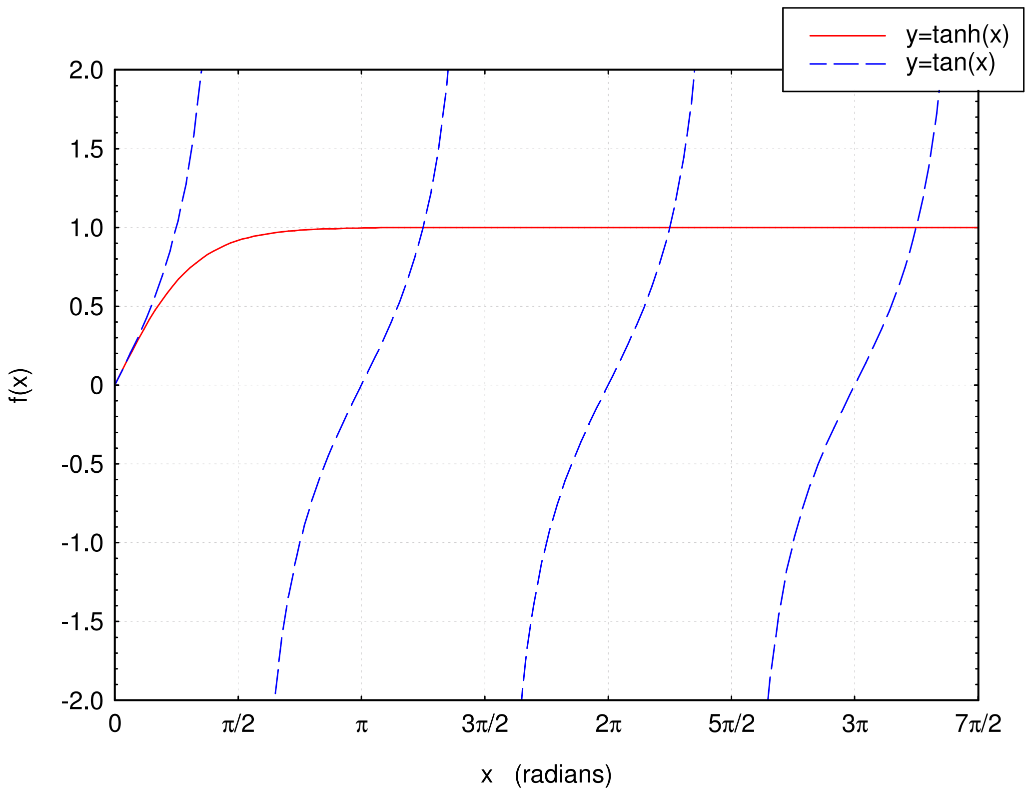

Figure B-1. Graph to Estimate Roots

The roots occur at the intersection of the two curves. Zero is not counted as a root for the purpose of determining the natural frequencies.

Table B-1. First Root		
Iteration	a	x
1	3.9	3.927298
2	3.927298	3.926603
3	3.926603	3.926602
4	3.926602	3.926602

Table B-2. Second Root		
Iteration	a	x
1	7	7.073064
2	7.073064	7.068603
3	7.068603	7.068583
4	7.068583	7.068583

APPENDIX C

Equation (C-1) is obtained for certain vibration problems. It is the characteristic equation for the following beams:

1. free-sliding beam
2. clamped-sliding beam

Equation (C-1) is graphed in Figure C-1.

$$-\tan(x) = \tanh(x) \tag{C-1}$$

The graph in Figure C-1 shows that the roots are given approximately by equation (C-2).

$$x_n \approx \pi\left(\frac{3}{4} + n\right), \quad n=1, 2, 3, \ldots. \tag{C-2}$$

Return to equation (C-1). The function equals zero if the dependent variable is a root.

$$f(x) = \tan(x) + \tanh(x) \tag{C-3}$$

The derivative is

$$f'(x) = \frac{1}{\cos^2(x)} + \frac{1}{\cosh^2(x)} \tag{C-4}$$

Recall the Newton-Raphson equation.

$$x = a - \frac{f(a)}{f'(a)} \tag{C-5}$$

By substitution,

$$x = a - \frac{\tan(a) + \tanh(a)}{\dfrac{1}{\cos^2(a)} + \dfrac{1}{\cosh^2(a)}} \tag{C-6}$$

Simplifying

$$x = a - \frac{\tan(a)\cosh(a)\cos^2(a) + \tanh(a)\cosh(a)\cos^2(a)}{\cosh^2(a) + \cos^2(a)} \tag{C-7}$$

$$x = a - \frac{\sin(a)\cos(a)\cos(a) + \sinh(a)\cos^2(a)}{\cosh^2(a) + \cos^2(a)} \tag{C-8}$$

$$x = a - \frac{[\sin(a)\cosh(a) + \sinh(a)\cos(a)]\cos(a)}{\cosh^2(a) + \cos^2(a)} \tag{C-9}$$

The first two roots are calculated as shown in Tables C-1 and C-2, respectively.

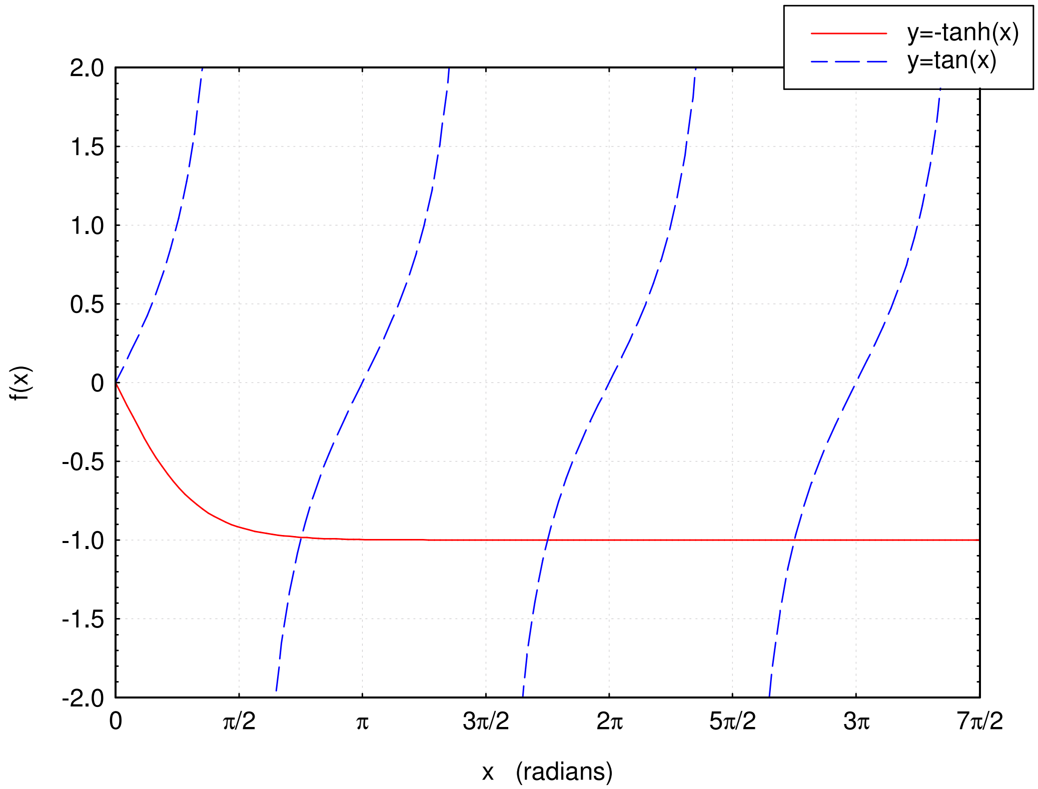

Figure C-1. Graph to Estimate Roots

The roots occur at the intersection of the two curves. Zero is not counted as a root for the purpose of determining the natural frequencies.

Table C-1. First Root		
Iteration	a	x
1	2.4	2.363846
2	2.363846	2.365019
3	2.365019	2.365020
4	2.365020	2.365020

Table C-2. Second Root		
Iteration	a	x
1	5.5	5.497799
2	5.497799	5.497804
3	5.497804	5.497804

APPENDIX D

Certain vibration problems have a solution in terms of zero order Bessel function. An example is the longitudinal vibration of a tapered rod. Another example is the natural frequencies of the acoustic pressure modes inside a cylinder. Note that there are also acoustic modes associated with higher order Bessel functions, which are covered in Appendices E and F.

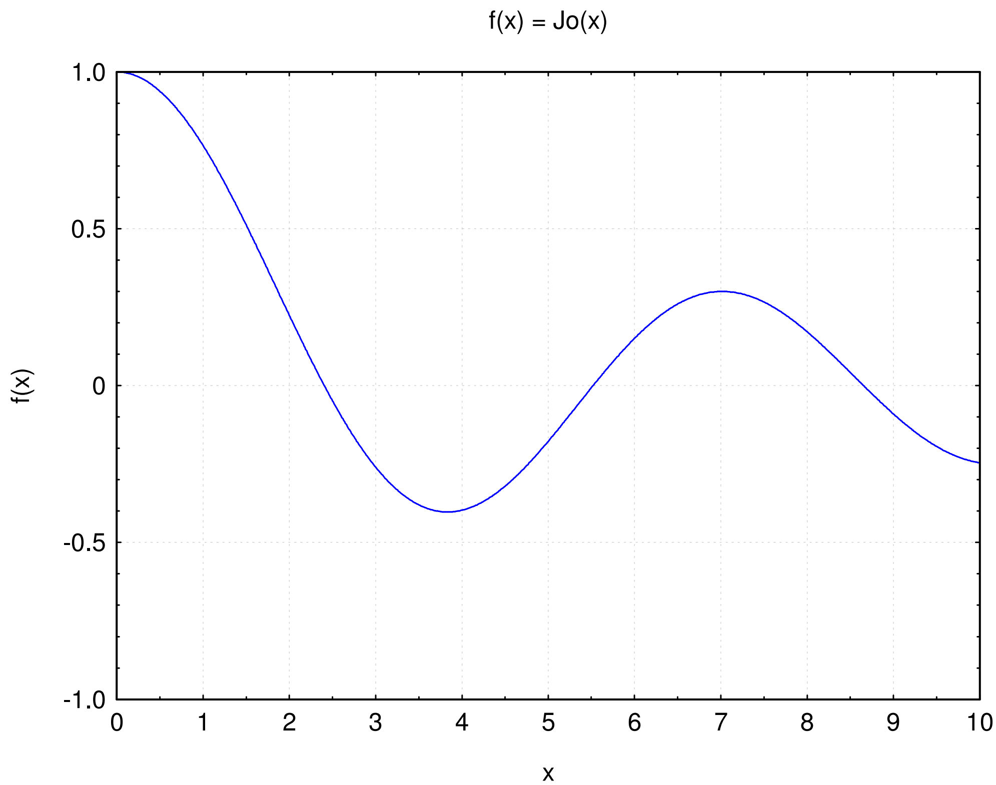

Figure D-1. Graph to Estimate Roots

Find the roots of the $J_o(x)$ function.

$$J_o(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \ldots \tag{D-1}$$

$$J_o(x) = -\frac{(2x)(1/2)^2}{(1!)^2} + \frac{(4x^3)(1/2)^4}{(2!)^2} - \frac{(6x^5)(1/2)^6}{(3!)^2} + \ldots \tag{D-2}$$

$$x = a - \frac{f(a)}{f'(a)} \tag{D-3}$$

$$\frac{d}{dx} J_0(x) = -J_1(x) \tag{D-4}$$

$$x = a - \frac{J_0(a)}{-J_1(a)} \tag{D-5}$$

Table D-1. First Root	
Iteration	Estimate
1	2
2	2.388211
3	2.404770
4	2.404826
5	2.404826

Table D-2. Second Root	
Iteration	Estimate
1	5
2	5.542149
3	5.520030
4	5.520078
5	5.520078

Table D-3. Third Root	
Iteration	Estimate
1	8
2	8.731561
3	8.653220
4	8.653728
5	8.653728

APPENDIX E

Certain vibration problems have a solution in terms of a first order Bessel function.

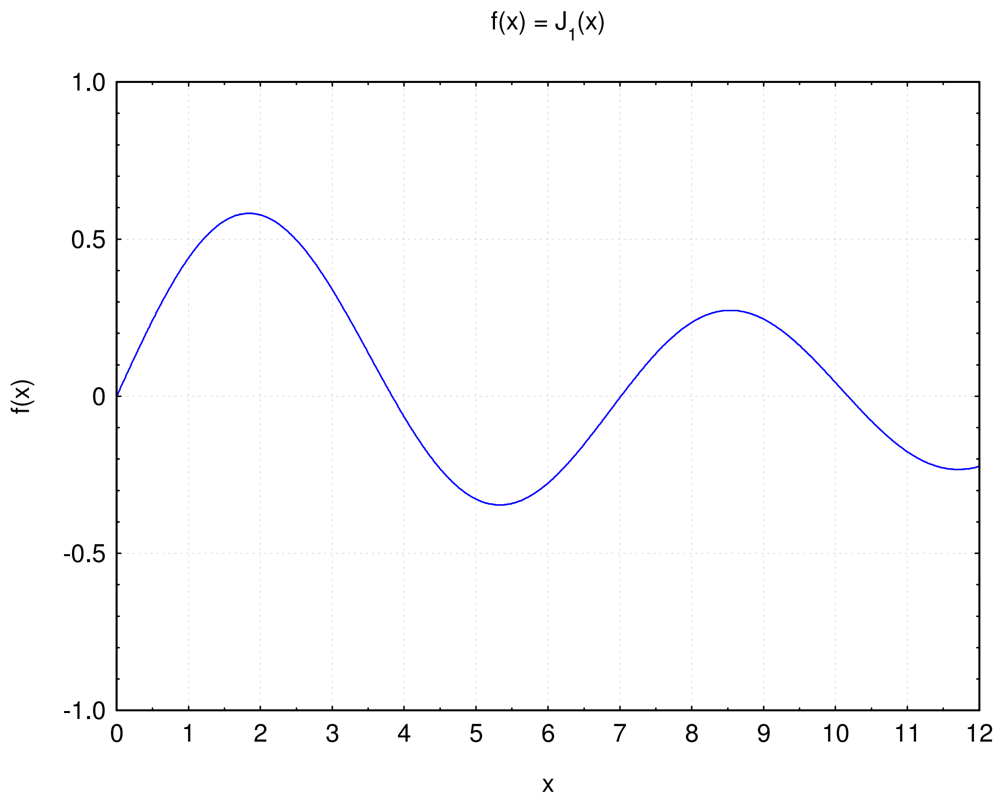

Figure E-1. Graph to Estimate Roots

Find the roots of the $J_1(x)$ function.

$$J_1(x) = 0 \tag{E-1}$$

$$\frac{d}{dx} J_1(x) = -J_2(x) + \frac{1}{x} J_1(x) \tag{E-2}$$

$$x = a - \frac{f(a)}{f'(a)} \tag{E-3}$$

$$x = a - \frac{J_1(a)}{-J_2(a) + \frac{1}{a} J_1(a)} \quad \text{(E-4)}$$

Table E-1. First Root	
Iteration	Estimate
1	3.9
2	3.8309999
3	3.8317059
4	3.8317060
5	3.8317060

Table E-2. Second Root	
Iteration	Estimate
1	7
2	7.0155706
3	7.0155867
4	7.0155867
5	7.0155867

Table E-3. Third Root	
Iteration	Estimate
1	10.2
2	10.173427
3	10.173468
4	10.173468
5	10.173468

APPENDIX F

Certain vibration problems have a solution in terms of a second order Bessel function.

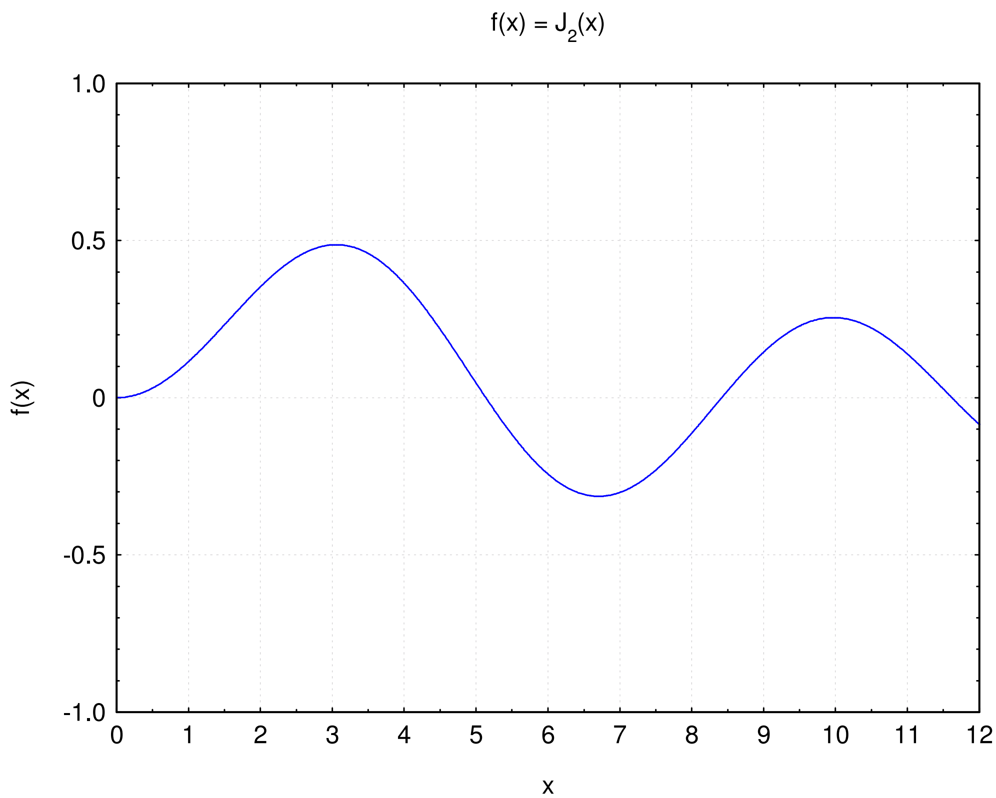

Figure F-1. Graph to Estimate Roots

Find the roots of the $J_2(x)$ function.

$$J_2(x) = 0 \tag{F-1}$$

$$\frac{d}{dx} J_2(x) = -J_3(x) + \frac{1}{x} J_2(x) \tag{F-2}$$

$$x = a - \frac{f(a)}{f'(a)} \tag{F-3}$$

$$x = a - \frac{J_2(a)}{-J_3(a) + \frac{1}{a} J_2(a)} \quad \text{(F-4)}$$

Table F-1. First Root	
Iteration	Estimate
1	5
2	5.1309781
3	5.1356160
4	5.1356223
5	5.1356223

Table F-2. Second Root	
Iteration	Estimate
1	8.5
2	8.4158328
3	8.4172438
4	8.4172441
5	8.4172441

Table F-3. Third Root	
Iteration	Estimate
1	11.5
2	11.618540
3	11.619841
4	11.619841
5	11.619841

APPENDIX G

Certain vibration problems have a solution in terms of the derivative of a zero order Bessel function. An example is the fluid slosh in a cylindrical basin.

Find the roots of the $\frac{d}{dx}J_o(x)$ function.

$$\frac{d}{dx}J_o(x) = 0 \tag{G-1}$$

$$\frac{d}{dx}J_o(x) = -J_1(x) \tag{G-2}$$

$$J_1(x) = 0 \tag{G-3}$$

The roots are thus the same as those in Appendix E, as repeated in Table G-1.

Table G-1. Root Summary

Root	Estimate
1	3.8317060
2	7.0155867
3	10.173468