PARTIAL FRACTIONS IN SHOCK AND VIBRATION ANALYSIS
Revision K

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Introduction

Transforming a fraction into the sum of partial fractions is an intermediate step in the solution of certain shock and vibration problems. The purpose of this tutorial is to summarize some common cases.

Term	Appendix
$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\}$	A
$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{\alpha s+\beta}{s^2+2\xi\omega_n s+\omega_n^2}\right\}$	B
$\frac{\alpha s+\beta}{(s+\lambda)(s+\sigma)}$, $\lambda \neq \sigma$	C
$\left[\frac{1}{\rho^2-j2\xi\rho-1}\right]\left[\frac{1}{\rho^2+j2\xi\rho-1}\right]$	D
$\left[\frac{1+j2\xi\rho}{\left(1-\rho^2\right)+j2\xi\rho}\right]\left[\frac{1-j2\xi\rho}{\left(1-\rho^2\right)-j2\xi\rho}\right]$	E
$\frac{s^2}{(s+a)(s+b)}$, $a \neq b$	F
$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$, $a \neq b,\ a \neq c,\ a \neq d, b \neq c, b \neq d, c \neq d$	G

Term	Appendix
$\frac{s}{(s+a)^2}$	H
$\frac{1}{\left(s^2+\alpha^2\right)(s+\beta)^2}$	I
$\frac{1}{s^2}\left\{\frac{1}{s^2+\omega^2}\right\}$	J

Note that

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\xi\omega_n)^2 + \omega_n^2 - \xi^2\omega_n{}^2 \tag{1}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\xi\omega_n)^2 + \omega_n^2\left[1-\xi^2\right] \tag{2}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\xi\omega_n)^2 + \omega_d^{\,2} \tag{3}$$

where

$$\omega_d = \omega_n\sqrt{1-\xi^2} \tag{4}$$

Thus,

$$\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{(s+\xi\omega_n)^2 + \omega_d^{\,2}} \tag{5}$$

The inverse Laplace transform f(t) of equation (5) is

$$f(t) = \frac{1}{\omega_d} \exp(-\xi\omega_n t)\sin(\omega_d t) \tag{6}$$

Furthermore, consider the Laplace transform

$$\hat{F}(s) = \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \tag{7}$$

The inverse Laplace transform $\hat{f}(t)$ is

$$\hat{f}(t) = \exp(-\xi\omega_n t)\cos(\omega_d t) \tag{8}$$

APPENDIX A

Example 1

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} = \left\{\frac{\rho}{s}\right\}+\left\{\frac{\sigma s+\phi}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \quad \text{(A-1)}$$

$$1=\left[s^2+2\xi\omega_n s+\omega_n^2\right]\rho+\left[\sigma s+\phi\right]s \quad \text{(A-2)}$$

$$1=\left[\rho s^2+2\xi\omega_n\rho s+\rho\omega_n^2\right]+\left[\sigma s^2+\phi s\right] \quad \text{(A-3)}$$

$$1=(\rho+\sigma)s^2+(2\xi\omega_n\rho+\phi)s+\rho\omega_n^2 \quad \text{(A-4)}$$

Equation (A-4) implies three separate equations.

$$(\rho+\sigma)=0 \quad \text{(A-5)}$$

$$(2\xi\omega_n\rho+\phi)=0 \quad \text{(A-6)}$$

$$\rho\omega_n^2=1 \quad \text{(A-7)}$$

Equation (A-7) yields

$$\rho=\frac{1}{\omega_n^2} \quad \text{(A-8)}$$

$$\sigma=-\rho \quad \text{(A-9)}$$

$$\sigma = -\frac{1}{\omega_n^2} \quad \text{(A-10)}$$

$$\phi=-2\xi\omega_n\rho \quad \text{(A-11)}$$

$$\phi = -2\xi\omega_n \left[\frac{1}{\omega_n^2} \right] \quad \text{(A-12)}$$

$$\phi = \quad -\frac{2\xi}{\omega_n} \quad \text{(A-13)}$$

Equation (A-1) can thus be expressed as

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} = \left\{\frac{\frac{1}{\omega_n^2}}{s}\right\} + \left\{\frac{-\left[\frac{1}{\omega_n^2}\right]s - \frac{2\xi}{\omega_n}}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \quad \text{(A-14)}$$

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} = \frac{1}{\omega_n^2}\left\{\frac{1}{s}\right\} - \frac{1}{\omega_n^2}\left\{\frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} \quad \text{(A-15)}$$

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} = \frac{1}{\omega_n^2}\left\{\frac{1}{s}\right\} - \frac{1}{\omega_n^2}\left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \quad \text{(A-16)}$$

The following form is a more convenient format prior to taking the inverse Laplace transformation,

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} = \frac{1}{\omega_n^2}\left\{\frac{1}{s}\right\} - \frac{1}{\omega_n^2}\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\} - \frac{1}{\omega_n^2}\left\{\frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

(A-17)

$$\left\{\frac{1}{s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} = \frac{1}{\omega_n^2}\left\{\frac{1}{s}\right\} - \left(\frac{1}{\omega_n^2}\right)\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\} - \left(\frac{1}{\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\left\{\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

(A-18)

The inverse Laplace transform is

$$f(t) = \frac{1}{\omega_n^2}u(t) - \frac{1}{\omega_n^2}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right],\ t \ge 0 \qquad \text{(A-19)}$$

where u(t) is the unit step function.

APPENDIX B

Example 2

Equation (B-1) effectively represents a number of cases since α and β may each be set equal to zero.

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{\alpha s+\beta}{s^2+2\xi\omega_n s+\omega_n^2}\right\} = \left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{s^2+2\xi\omega_n s+\omega_n^2}\right\} \tag{B-1}$$

Multiply through by the common denominator.

$$\alpha s+\beta = \{\lambda s+\rho\}\left\{s^2+2\xi\omega_n s+\omega_n^2\right\}+\{\sigma s+\phi\}\left\{s^2+\omega^2\right\} \tag{B-2}$$

$$\begin{aligned}\alpha s+\beta &= \lambda s^3+(\rho+2\xi\omega_n\lambda)s^2+\left(2\xi\omega_n\rho+\lambda\,\omega_n^2\right)s+\left(\rho\omega_n^2\right)\\ &\quad+\sigma s^3+\phi s^2+\sigma\omega^2 s+\phi\omega^2\end{aligned} \tag{B-3}$$

$$\begin{aligned}\alpha s+\beta &= \\ &[\lambda+\sigma]s^3\\ &+[\rho+2\xi\omega_n\lambda+\phi]s^2\\ &+\left[2\xi\omega_n\rho+\lambda\omega_n^2+\sigma\omega^2\right]s\\ &+\left[\rho\omega_n^2+\phi\omega^2\right]\end{aligned} \tag{B-4}$$

Equation (B-4) implies four separate equations.

$$\lambda+\sigma=0 \tag{B-5}$$

$$\rho+2\xi\omega_n\lambda+\phi=0 \tag{B-6}$$

$$2\xi\omega_n\rho + \lambda\omega_n^2 + \sigma\omega^2 = \alpha \tag{B-7}$$

$$\rho\,\omega_n^2 + \phi\,\omega^2 = \beta \tag{B-8}$$

Equations (B-5) through (B-8) can be assembled into matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2\xi\omega_n & 1 & 0 & 1 \\ \omega_n^2 & 2\xi\omega_n & \omega^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix} \tag{B-9}$$

Gaussian elimination is used to simplify the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2\xi\omega_n & 1 \\ 0 & 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix} \tag{B-10}$$

Equation (B-10) can be reduced to a 3 x 3 matrix.

$$\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix} \tag{B-11}$$

Complete the solution using Cramer's rule.

$$\det \begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \omega^2\left(\omega^2 - \omega_n^2\right) + \omega^2\left(2\xi\omega_n\right)^2 - \omega_n^2\left(\omega^2 - \omega_n^2\right) \tag{B-12}$$

$$\det\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2 \tag{B-13}$$

$$\rho = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}\det\begin{bmatrix} 0 & -2\xi\omega_n & 1 \\ \alpha & \omega^2-\omega_n^2 & 0 \\ \beta & 0 & \omega^2 \end{bmatrix} \tag{B-14}$$

$$\rho = \frac{2\xi\,\omega^2\omega_n\,\alpha - \left(\omega^2-\omega_n^2\right)\beta}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2} \tag{B-15}$$

$$\sigma = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}\det\begin{bmatrix} 1 & 0 & 1 \\ 2\xi\omega_n & \alpha & 0 \\ \omega_n^2 & \beta & \omega^2 \end{bmatrix} \tag{B-16}$$

$$\sigma = \frac{\omega^2\alpha + 2\xi\omega_n\beta - \omega_n^2\alpha}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2} \tag{B-17a}$$

$$\sigma = \frac{\left(\omega^2-\omega_n^2\right)\alpha + 2\xi\omega_n\,\beta}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2} \tag{B-17b}$$

Recall equation (B-5).

$$\lambda = -\sigma \tag{B-18}$$

$$\lambda = -\frac{\left(\omega^2-\omega_n^2\right)\alpha + 2\xi\omega_n\beta}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2} \tag{B-19}$$

$$\phi = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2}\det\begin{bmatrix} 1 & -2\xi\omega_n & 0 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & \alpha \\ \omega_n^2 & 0 & \beta \end{bmatrix} \tag{B-20}$$

$$\phi = \frac{\left(\omega^2-\omega_n^2\right)\beta + \left(2\xi\omega_n\right)^2\beta - 2\xi\omega_n^3\alpha}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2} \tag{B-21a}$$

$$\phi = \frac{\left[\left(\omega^2-\omega_n^2\right) + \left(2\xi\omega_n\right)^2\right]\beta - 2\xi\omega_n^3\alpha}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2} \tag{B-21b}$$

The coefficients are summarized in equation (B-22).

$$\begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2}\begin{bmatrix} -\left(\omega^2-\omega_n^2\right)\alpha - 2\xi\omega_n\beta \\ 2\xi\omega^2\omega_n\alpha - \left(\omega^2-\omega_n^2\right)\beta \\ \left(\omega^2-\omega_n^2\right)\alpha + 2\xi\omega_n\beta \\ -2\xi\omega_n^3\alpha + \left[\left(\omega^2-\omega_n^2\right) + \left(2\xi\omega_n\right)^2\right]\beta \end{bmatrix}$$

(B-22a)

$$\begin{bmatrix}\lambda\\ \rho\\ \sigma\\ \phi\end{bmatrix} = \frac{1}{\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2}\left\{\begin{bmatrix}-\left(\omega^2-\omega_n^2\right)\\ 2\xi\omega^2\omega_n\\ \left(\omega^2-\omega_n^2\right)\\ -2\xi\omega_n^3\end{bmatrix}\alpha+\begin{bmatrix}-2\xi\omega_n\\ -\left(\omega^2-\omega_n^2\right)\\ 2\xi\omega_n\\ \left[\left(\omega^2-\omega_n^2\right)+\left(2\xi\omega_n\right)^2\right]\end{bmatrix}\beta\right\}$$

(B-22b)

Equation (B-1) can thus be rewritten as

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{\alpha s+\beta}{s^2+2\xi\omega_n s+\omega_n^2}\right\} =$$

$$\frac{\left[-\left(\omega^2-\omega_n^2\right)\alpha-2\xi\omega_n\beta\right]s+\left[2\xi\omega^2\omega_n\alpha-\left(\omega^2-\omega_n^2\right)\beta\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[s^2+\omega^2\right]}$$

$$+\frac{\left[\left(\omega^2-\omega_n^2\right)\alpha+2\xi\omega_n\beta\right]s+\left[-2\xi\omega_n^3\alpha+\left[\left(\omega^2-\omega_n^2\right)+\left(2\xi\omega_n\right)^2\right]\beta\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

(B-23)

An alternate form is

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{\alpha s+\beta}{s^2+2\xi\omega_n s+\omega_n^2}\right\} =$$

$$\alpha\frac{\left[-\left(\omega^2-\omega_n^2\right)\right]s+\left[2\xi\omega^2\omega_n\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]\left[s^2+\omega^2\right]}$$

$$+\beta\frac{\left[-2\xi\omega_n\right]s+\left[-\left(\omega^2-\omega_n^2\right)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]\left[s^2+\omega^2\right]}$$

$$+\alpha\frac{\left[\left(\omega^2-\omega_n^2\right)\right]s+\left[-2\xi\omega_n^3\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

$$+\beta\frac{\left[2\xi\omega_n\right]s+\left[\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

(B-24)

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{\alpha s+\beta}{s^2+2\xi\omega_n s+\omega_n^2}\right\} =$$

$$\alpha\frac{\left[-\left(\omega^2-\omega_n^2\right)\right]s+\left[2\xi\omega^2\omega_n\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[s^2+\omega^2\right]}$$

$$+\beta\frac{\left[-2\xi\omega_n\right]s+\left[-\left(\omega^2-\omega_n^2\right)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[s^2+\omega^2\right]}$$

$$+\alpha\frac{\left[\left(\omega^2-\omega_n^2\right)\right]s+\left[-2\xi\omega_n^3\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[\left(s+\xi\omega_n\right)^2+\omega_d^2\right]}$$

$$+\beta\frac{\left[2\xi\omega_n\right]s+\left[\left(\omega^2-\omega_n^2\right)+\left(2\xi\omega_n\right)^2\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\omega\omega_n\right)^2\right]\left[\left(s+\xi\omega_n\right)^2+\omega_d^2\right]}$$

(B-25)

The inverse Laplace transforms for the four terms on the left-hand-side are found as follows

$$F_1(s) = \alpha \frac{\left[-\left(\omega^2 - \omega_n^2\right)\right]s + \left[2\xi\omega^2\omega_n\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]\left[s^2 + \omega^2\right]} \tag{B-26}$$

$$f_1(t) = \frac{\alpha}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left\{\left[-\left(\omega^2 - \omega_n^2\right)\right]\cos(\omega t) + \left[2\xi\omega\omega_n\right]\sin(\omega t)\right\} \tag{B-27}$$

$$F_2(s) = +\beta \frac{\left[-2\xi\omega_n\right]s + \left[-\left(\omega^2 - \omega_n^2\right)\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]\left[s^2 + \omega^2\right]} \tag{B-28}$$

$$f_2(t) = \frac{\beta}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left\{\left[-2\xi\omega_n\right]\cos(\omega t) + \left[-\frac{1}{\omega}\left(\omega^2 - \omega_n^2\right)\right]\sin(\omega t)\right\} \tag{B-29}$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2\right)\right]s + \left[-2\xi\omega_n^3\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]\left[(s+\xi\omega_n)^2 + \omega_d^2\right]} \tag{B-30}$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2\right)\right]s}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]\left[(s+\xi\omega_n)^2 + \omega_d^2\right]}$$

$$+\alpha \frac{\left[-2\xi\omega_n^3\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]\left[(s+\xi\omega_n)^2 + \omega_d^2\right]} \tag{B-31}$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2\right)\right]s}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]\left[(s+\xi\omega_n)^2 + \omega_d^2\right]}$$

$$+\alpha\left(\frac{-2\xi\omega_n^3}{\omega_d}\right)\frac{\omega_d}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\omega_n)^2\right]\left[(s+\xi\omega_n)^2 + \omega_d^2\right]} \tag{B-32}$$

$$f_3(t) = \frac{\alpha \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\left(\omega^2 - \omega_n^2\right)\cos(\omega_d t) + \left(\frac{-2\xi\omega_n^3}{\omega_d}\right)\sin(\omega_d t)\right]$$

(B-33)

$$F_4(s) = +\beta \frac{[2\xi\omega_n]s + \left[\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]\left[(s + \xi\omega_n)^2 + \omega_d^2\right]} \qquad \text{(B-34)}$$

$$F_4(s) = +\beta[2\xi\omega_n] \frac{s + \frac{1}{[2\xi\omega_n]}\left[\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]\left[(s + \xi\omega_n)^2 + \omega_d^2\right]} \qquad \text{(B-35)}$$

$$f_4(t) = \frac{\beta[2\xi\omega_n]\exp(-\xi\omega_n t)}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\cos(\omega_d t)+\left[\frac{\frac{1}{[2\xi\omega_n]}\left[\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2\right]-\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \qquad \text{(B-36)}$$

$$f_4(t) = \frac{\beta\exp(-\xi\omega_n t)}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\cos(\omega_d t)+\left[\frac{\left[\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2\right]-\xi\omega_n[2\xi\omega_n]}{\omega_d}\right]\sin(\omega_d t)\right\} \qquad \text{(B-37)}$$

$$f_4(t) = \frac{\beta\exp(-\xi\omega_n t)}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\cos(\omega_d t)+\left[\frac{\left[\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2\right]-2\xi^2\omega_n{}^2}{\omega_d}\right]\sin(\omega_d t)\right\} \qquad \text{(B-38)}$$

$$f_4(t) = \frac{\beta \exp\left(-\xi \omega_n t\right)}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi \omega \omega_n\right)^2\right]} \left\{ 2\xi\omega_n \cos\left(\omega_d t\right) + \left[\frac{\left[\left(\omega^2 - \omega_n^2\right) + 2\left(\xi\omega_n\right)^2\right]}{\omega_d} \right] \sin\left(\omega_d t\right) \right\} \qquad \text{(B-39)}$$

APPENDIX C

Example 3

Equation (C-1) effectively represents a number of cases since α and β may each be set equal to zero.

$$\left\{\frac{\alpha s+\beta}{(s+\lambda)(s+\sigma)}\right\} = \left\{\frac{p}{s+\lambda}\right\}+\left\{\frac{m}{s+\sigma}\right\} \;,\quad \lambda \neq \sigma \tag{C-1}$$

$$\{\alpha s+\beta\} = \left\{\frac{p}{s+\lambda}\right\}(s+\lambda)(s+\sigma)+\left\{\frac{m}{s+\sigma}\right\}(s+\lambda)(s+\sigma) \tag{C-2}$$

$$(\alpha s+\beta) = (p)(s+\sigma)+(m)(s+\lambda) \tag{C-3}$$

$$(\alpha s+\beta) = (ps+p\sigma)+(ms+m\lambda) \tag{C-4}$$

$$(\alpha s+\beta) = (p+m)s+(p\sigma+m\lambda) \tag{C-5}$$

Equation (C-5) implies equations (C-6) and (C-7).

$$\alpha = (p+m) \tag{C-6}$$

$$\beta = (p\sigma+m\lambda) \tag{C-7}$$

Equation (C-6) yields

$$m = \alpha - p \tag{C-8}$$

Substitute equation (C-8) into (C-7).

$$\beta = (p\sigma+(\alpha-p)\lambda) \tag{C-9}$$

$$\beta = (\sigma-\lambda)p+\alpha\lambda \tag{C-10}$$

$$\beta-\alpha\lambda = (\sigma-\lambda)p \tag{C-11}$$

$$(\sigma-\lambda)p=\beta-\alpha\lambda \tag{C-12}$$

$$p=\frac{\beta-\alpha\lambda}{\sigma-\lambda} \tag{C-13}$$

Substitute equation (C-13) into (C-8).

$$m=\alpha-\left(\frac{\beta-\alpha\lambda}{\sigma-\lambda}\right) \tag{C-14}$$

$$m=\frac{\alpha(\sigma-\lambda)-(\beta-\alpha\lambda)}{\sigma-\lambda} \tag{C-15}$$

$$m=\frac{(\alpha\sigma-\alpha\lambda)-(\beta-\alpha\lambda)}{\sigma-\lambda} \tag{C-16}$$

$$m=\frac{\alpha\sigma-\alpha\lambda-\beta+\alpha\lambda}{\sigma-\lambda} \tag{C-17}$$

$$m=\frac{\alpha\sigma-\beta}{\sigma-\lambda} \tag{C-18}$$

Substitute equations (C-18) and (C-13) into (C-1).

$$\left\{\frac{\alpha s+\beta}{(s+\lambda)(s+\sigma)}\right\}=\left\{\frac{1}{\sigma-\lambda}\right\}\left\{\left[\frac{\beta-\alpha\lambda}{s+\lambda}\right]+\left[\frac{\alpha\sigma-\beta}{s+\sigma}\right]\right\} \tag{C-19}$$

The inverse Laplace transform is

$$f(t)=\left\{\frac{1}{\sigma-\lambda}\right\}\left\{\frac{1}{\lambda}[\beta-\alpha\lambda]\sin(\lambda t)+\frac{1}{\sigma}[\alpha\sigma-\beta]\sin(\sigma t)\right\} \tag{C-20}$$

APPENDIX D

Example 4

The transfer function $H(\rho)$ times its complex conjugate is

$$H(\rho)H*(\rho)=\left[\frac{1}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{1}{\rho^2 + j2\xi\rho - 1}\right] \quad \text{(D-1)}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \quad \text{(D-2)}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad \text{(D-3)}$$

$$R1, R2 = j\xi \pm \sqrt{1-\xi^2} \quad \text{(D-4)}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \quad \text{(D-5)}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad \text{(D-6)}$$

$$R3, R4 = -j\xi \pm \sqrt{1-\xi^2} \quad \text{(D-7)}$$

(D-8)

Summary,

$$R1 = +j\xi + \sqrt{1-\xi^2} \quad \text{(D-9)}$$

$$R2 = +j\xi - \sqrt{1-\xi^2} \tag{D-10}$$

$$R3 = -j\xi + \sqrt{1-\xi^2} \tag{D-11}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{D-12}$$

Note

$$R2 = -R1^* \tag{D-13}$$

$$R3 = R1^* \tag{D-14}$$

$$R4 = -R1^* \tag{D-15}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)}\right] \tag{D-16}$$

$$H(\rho)H^*(\rho) = \left[\frac{1}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)}\right] \tag{D-17}$$

$$H(\rho)H^*(\rho) = \left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] \tag{D-18}$$

Expand into partial fractions.

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = \begin{aligned} &+\frac{\alpha}{(\rho - R1)} \\ &+\frac{\beta}{(\rho - R1^*)} \\ &+\frac{\lambda}{(\rho + R1^*)} \\ &+\frac{\sigma}{(\rho + R1)} \end{aligned} \tag{D-19}$$

Multiply through by the denominator on the left-hand side of equation (D-19).

$$1 = \begin{aligned} &+\frac{\alpha}{(\rho - R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+\frac{\beta}{(\rho - R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+\frac{\lambda}{(\rho + R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+\frac{\sigma}{(\rho + R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \end{aligned} \tag{D-20}$$

$$1 = \begin{aligned} &+\alpha\,(\rho - R1^*)(\rho + R1^*)(\rho + R1) \\ &+\beta\,(\rho - R1)(\rho + R1^*)(\rho + R1) \\ &+\lambda\,(\rho - R1)(\rho - R1^*)(\rho + R1) \\ &+\sigma\,(\rho - R1)(\rho - R1^*)(\rho + R1^*) \end{aligned} \tag{D-21}$$

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^2 - R1*^2\right)(\rho + R1) \\
& +\beta\left(\rho^2 + (-R1 + R1*)\rho - R1R1*\right)(\rho + R1) \\
& +\lambda\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1) \\
& +\sigma\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1*)
\end{aligned}
$$

(D-22)

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + (R1 - R1 + R1*)\rho^2 + \left(-R1R1* - R1^2 + R1R1*\right)\rho - R1^2R1*\right) \\
& +\lambda\left(\rho^3 + (R1 - R1 - R1*)\rho^2 + \left(R1R1* - R1^2 - R1R1*\right)\rho + R1^2R1*\right) \\
& +\sigma\left(\rho^3 + (R1* - R1 - R1*)\rho^2 + \left(-R1R1* - R1R1* - R1*^2\right)\rho + R1R1*^2\right)
\end{aligned}
$$

(D-23)

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + R1*\rho^2 - R1^2\rho - R1^2R1*\right) \\
& +\lambda\left(\rho^3 - R1*\rho^2 - R1^2\rho + R1^2R1*\right) \\
& +\sigma\left(\rho^3 - R1\rho^2 - R1*^2\rho + R1R1*^2\right)
\end{aligned}
$$

(D-24)

$$
\begin{aligned}
1 = \quad & +\left[\alpha + \beta + \lambda + \sigma\right]\rho^3 \\
& +\left[R1\alpha + R1*\beta - R1*\lambda - R1\sigma\right]\rho 2 \\
& +\left[-R1*^2\alpha - R1^2\beta - R1^2\lambda - R1*^2\sigma\right]\rho \\
& +\left[-R1R1*^2\alpha - R1^2R1*\beta + R1^2R1*\lambda + R1R1*^2\sigma\right]
\end{aligned}
$$

(D-25)

$$1 = \quad +[\alpha+\beta+\lambda+\sigma]\rho^3$$
$$+[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma]\rho 2$$
$$+\left[-R1^{*2}\,\alpha - R1^2\,\beta - R1^2\lambda - R1^{*2}\,\sigma\right]\rho$$
$$+[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma]R1R1^*$$

(D-26)

Equation (D-26) can be broken up into four separate equations,

$$\alpha+\beta+\lambda+\sigma = 0 \tag{D-27}$$

$$[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma] = 0 \tag{D-28}$$

$$\left[-R1^{*2}\,\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\,\sigma\right] = 0 \tag{D-29}$$

$$[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma]R1R1^* \quad = 1 \tag{D-30}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/R1R1^* \end{bmatrix}$$

(D-31)

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{D-32}$$

$$R1R1^* = \left[+j\xi + \sqrt{1-\xi^2}\right]\left[-j\xi + \sqrt{1-\xi^2}\right] \tag{D-33}$$

$$R1\ R1^* = 1 \tag{D-34}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-35}$$

Multiply the first row by $R1^{*2}$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & -R1^2 + R1^{*2} & -R1^2 + R1^{*2} & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-36}$$

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-37}$$

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-38}$$

Multiply the first row by -R1 and add to the second row. Also multiply the first row by R1* and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1* & -R1* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-39}$$

Multiply the third row by -R1* and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{D-40}$$

The first row equation yields

$$\alpha = -\sigma \tag{D-41}$$

The third row equation yields

$$\beta = -\lambda \tag{D-42}$$

Equation (D-40) thus reduces to

$$\begin{bmatrix} -2R1* & -2R1 \\ +2R1 & +2R1* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{D-43}$$

$$-2R1*\lambda - 2R1\sigma = 0 \tag{D-44}$$

$$-R1*\lambda = R1\sigma \tag{D-45}$$

$$\lambda = \frac{-R1}{R1*}\,\sigma \tag{D-46}$$

$$2R1\lambda\ + 2R1*\sigma = 1 \tag{D-47}$$

$$2R1\left[\frac{-R1}{R1^*}\ \sigma\right] + 2R1^*\sigma = 1 \tag{D-48}$$

$$\left\{R1\left[\frac{-R1}{R1^*}\ \right] + R1^*\right\}\sigma = \frac{1}{2} \tag{D-49}$$

$$\sigma = \frac{1}{2}\frac{1}{\left\{R1\left[\frac{-R1}{R1^*}\ \right] + R1^*\right\}} \tag{D-50}$$

$$\sigma = \frac{1}{2}\frac{R1^*}{[\,-R1^2\ \ + R1^{*2}\ ]} \tag{D-51}$$

$$\lambda = \frac{-R1}{R1^*}\ \sigma \tag{D-52}$$

$$\lambda = -\frac{1}{2}\frac{R1}{[\,-R1^2\ \ + R1^{*2}\ ]} \tag{D-53}$$

Recall,

$$\alpha = -\sigma \tag{D-54}$$

$$\beta = -\lambda \tag{D-55}$$

The complete solution set is thus

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{2}\frac{1}{[\,-R1^2\ \ + R1^{*2}\ ]}\begin{bmatrix} -R1^* \\ R1 \\ -R1 \\ R1^* \end{bmatrix} \tag{D-56}$$

Note that

$$-R1^2 + R1^{*2} = -j\left[4\xi\sqrt{1-\xi^2}\right] \tag{D-57}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{2}\left[\frac{1}{-j\left[4\xi\sqrt{1-\xi^2}\right]}\right]\begin{bmatrix}-\sqrt{1-\xi^2}+j\xi\\ +\sqrt{1-\xi^2}+j\xi\\ -\sqrt{1-\xi^2}-j\xi\\ +\sqrt{1-\xi^2}-j\xi\end{bmatrix} \tag{D-58}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{j}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}-\sqrt{1-\xi^2}+j\xi\\ +\sqrt{1-\xi^2}+j\xi\\ -\sqrt{1-\xi^2}-j\xi\\ +\sqrt{1-\xi^2}-j\xi\end{bmatrix} \tag{D-59}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}-\xi-j\sqrt{1-\xi^2}\\ -\xi+j\sqrt{1-\xi^2}\\ +\xi-j\sqrt{1-\xi^2}\\ +\xi+j\sqrt{1-\xi^2}\end{bmatrix} \tag{D-60}$$

Let

$$\psi = \frac{\xi}{\sqrt{1-\xi^2}} \tag{D-61}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{8\xi}\begin{bmatrix}-\psi - j\\ -\psi + j\\ +\psi - j\\ +\psi + j\end{bmatrix}$$

(D-62)

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \qquad \text{(D-63)}$$

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = + \frac{-\psi - j}{\left(\rho - \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{-\psi + j}{\left(\rho - \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{+\psi - j}{\left(\rho + \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{+\psi + j}{\left(\rho + \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right]$$

(D-64)

APPENDIX E

Example 5

A transfer function times its complex conjugate is

$$H(\rho)H^*(\rho) = \left[\frac{1 + j2\xi\rho}{\left(1-\rho^2\right) + j2\xi\rho}\right]\left[\frac{1 - j2\xi\rho}{\left(1-\rho^2\right) - j2\xi\rho}\right] \quad \text{(E-1)}$$

Rearrange equation (E-1) as

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2\rho^2}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{1}{\rho^2 + j2\xi\rho - 1}\right] \quad \text{(E-2)}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \quad \text{(E-3)}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad \text{(E-4)}$$

$$R1, R2 = j\xi \pm \sqrt{1-\xi^2} \quad \text{(E-5)}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j2\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \quad \text{(E-6)}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad \text{(E-7)}$$

$$R3, R4 = -j\xi \pm \sqrt{1-\xi^2} \quad \text{(E-8)}$$

Summary,

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{E-9}$$

$$R2 = +j\xi - \sqrt{1-\xi^2} \tag{E-10}$$

$$R3 = -j\xi + \sqrt{1-\xi^2} \tag{E-11}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{E-12}$$

Note

$$R2 = -R1^* \tag{E-13}$$

$$R3 = R1^* \tag{E-14}$$

$$R4 = -R1^* \tag{E-15}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1+4\xi^2\rho^2}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)}\right]$$

(E-16)

$$H(\rho)H^*(\rho) = \left[\frac{1+4\xi^2\rho^2}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)}\right] \tag{E-17}$$

By substitution,

$$H(\rho)H^*(\rho) = \left[\frac{1+4\xi^2\rho^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] \tag{E-18}$$

Expand into partial fractions.

$$\left[\frac{1+4\xi^2\rho^2}{(\rho-R1)(\rho+R1^*)(\rho-R1^*)(\rho+R1)}\right]=$$

$$+\frac{\alpha}{(\rho-R1)}$$
$$+\frac{\beta}{(\rho-R1^*)}$$
$$+\frac{\lambda}{(\rho+R1^*)}$$
$$+\frac{\sigma}{(\rho+R1)}$$

(E-19)

Multiply through by the denominator on the left-hand side of equation (E-19).

$$\left[1+4\xi^2\rho^2\right]=$$

$$+\frac{\alpha}{(\rho-R1)}(\rho-R1)(\rho+R1^*)(\rho-R1^*)(\rho+R1)$$
$$+\frac{\beta}{(\rho-R1^*)}(\rho-R1)(\rho+R1^*)(\rho-R1^*)(\rho+R1)$$
$$+\frac{\lambda}{(\rho+R1^*)}(\rho-R1)(\rho+R1^*)(\rho-R1^*)(\rho+R1)$$
$$+\frac{\sigma}{(\rho+R1)}(\rho-R1)(\rho+R1^*)(\rho-R1^*)(\rho+R1)$$

(E-20)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\alpha\,(\rho - R1^*)(\rho + R1^*)(\rho + R1)\\
&+\beta\,(\rho - R1)(\rho + R1^*)(\rho + R1)\\
&+\lambda\,(\rho - R1)(\rho - R1^*)(\rho + R1)\\
&+\sigma\,(\rho - R1)(\rho - R1^*)(\rho + R1^*)
\end{aligned}$$

(E-21)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\alpha\left(\rho^2 - R1^{*2}\right)(\rho + R1)\\
&+\beta\left(\rho^2 + (-R1 + R1^*)\rho - R1R1^*\right)(\rho + R1)\\
&+\lambda\left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1)\\
&+\sigma\left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1^*)
\end{aligned}$$

(E-22)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\alpha\left(\rho^3 + R1\rho^2 - R1^{*2}\rho - R1R1^{*2}\right)\\
&+\beta\left(\rho^3 + (R1 - R1 + R1^*)\rho^2 + \left(-R1R1^* - R1^2 + R1R1^*\right)\rho - R1^2R1^*\right)\\
&+\lambda\left(\rho^3 + (R1 - R1 - R1^*)\rho^2 + \left(R1R1^* - R1^2 - R1R1^*\right)\rho + R1^2R1^*\right)\\
&+\sigma\left(\rho^3 + (R1^* - R1 - R1^*)\rho^2 + \left(-R1R1^* - R1R1^* - R1^{*2}\right)\rho + R1R1^{*2}\right)
\end{aligned}$$

(E-23)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right)\\
&+\beta\left(\rho^3 + R1*\rho^2 - R1^2\rho - R1^2R1*\right)\\
&+\lambda\left(\rho^3 - R1*\rho^2 - R1^2\rho + R1^2R1*\right)\\
&+\sigma\left(\rho^3 - R1\rho^2 - R1*^2\rho + R1R1*^2\right)
\end{aligned}$$

(E-24)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\left[\alpha+\beta+\lambda+\sigma\right]\rho^3\\
&+\left[R1\alpha + R1*\beta - R1*\lambda - R1\sigma\right]\rho 2\\
&+\left[-R1*^2\alpha - R1^2\beta - R1^2\lambda - R1*^2\sigma\right]\rho\\
&+\left[-R1R1*^2\alpha - R1^2R1*\beta + R1^2R1*\lambda + R1R1*^2\sigma\right]
\end{aligned}$$

(E-25)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$\begin{aligned}
&+\left[\alpha+\beta+\lambda+\sigma\right]\rho^3\\
&+\left[R1\alpha + R1*\beta - R1*\lambda - R1\sigma\right]\rho 2\\
&+\left[-R1*^2\alpha - R1^2\beta - R1^2\lambda - R1*^2\sigma\right]\rho\\
&+\left[-R1*\alpha - R1\beta + R1\lambda + R1*\sigma\right]R1R1*
\end{aligned}$$

(E-26)

Equation (E-26) can be broken up into four separate equations.

$$\alpha + \beta + \lambda + \sigma = 0 \tag{E-27}$$

$$\left[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma \right] = 4\xi^2 \tag{E-28}$$

$$\left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma \right] = 0 \tag{E-29}$$

$$\left[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma \right] R1R1^* = 1 \tag{E-30}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1/(R1R1^*) \end{bmatrix} \tag{E-31}$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{E-32}$$

$$R1R1^* = \left[+j\xi + \sqrt{1-\xi^2} \right] \left[-j\xi + \sqrt{1-\xi^2} \right] \tag{E-33}$$

$$R1\ R1^* = 1 \tag{E-34}$$

By substitution,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{E-35}$$

Multiply the first row by $R1*^2$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & -R1^2+R1*^2 & -R1^2+R1*^2 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-36)

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-37)

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-38)

Multiply the first row by -R1 and add to the second row. Also multiply the first row by R1* and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1* & -R1* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-39)

Multiply the third row by -R1* and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{E-40}$$

The first row equation yields

$$\alpha = -\sigma \tag{E-41}$$

The third row equation yields

$$\lambda = -\beta \tag{E-42}$$

Equation (E-40) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 4\xi^2 \\ 1 \end{bmatrix} \tag{E-43}$$

Complete the solution using Cramer's rule.

$$\text{det er min ant} \begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = 4\left[R1^2 - R1^{*2} \right] \tag{E-44}$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{E-45}$$

$$R1^2 = \left[+j\xi + \sqrt{1-\xi^2} \right] \left[+j\xi + \sqrt{1-\xi^2} \right] \tag{E-46}$$

$$R1^2 = -\xi^2 + \left(1-\xi^2\right) + j\left[2\xi \sqrt{1-\xi^2} \right] \tag{E-47}$$

$$R1^2 = \left(1-2\xi^2\right) + j\left[2\xi \sqrt{1-\xi^2} \right] \tag{E-48}$$

$$R1^* = -j\xi + \sqrt{1-\xi^2} \tag{E-49}$$

$$R1*^2 = \left[-j\xi + \sqrt{1-\xi^2}\right]\left[-j\xi + \sqrt{1-\xi^2}\right] \tag{E-50}$$

$$R1*^2 = -\xi^2 + \left(1-\xi^2\right) - j\left[2\xi\sqrt{1-\xi^2}\right] \tag{E-51}$$

$$R1*^2 = \left(1-2\xi^2\right) - j\left[2\xi\sqrt{1-\xi^2}\right] \tag{E-52}$$

Thus,

$$R1^2 - R1*^2 = j\left[4\xi\sqrt{1-\xi^2}\right] \tag{E-53}$$

$$4\left[R1^2 - R1*^2\right] = j\left[16\xi\sqrt{1-\xi^2}\right] \tag{E-54}$$

$$\text{determinant}\begin{bmatrix} -2R1* & -2R1 \\ +2R1 & +2R1* \end{bmatrix}\begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = j\left[16\xi\sqrt{1-\xi^2}\right] \tag{E-55}$$

$$\lambda = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]}\text{determinant}\begin{bmatrix} 4\xi^2 & -2R1 \\ 1 & +2R1* \end{bmatrix} \tag{E-56}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(2R1*\right) + 2R1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{E-57}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(R1*\right) + R1}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-58}$$

Recall,

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{E-59}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(-j\xi+\sqrt{1-\xi^2}\right)+j\xi+\sqrt{1-\xi^2}}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-60}$$

$$\lambda = \frac{\left(1+4\xi^2\right)\left(\sqrt{1-\xi^2}\right)+j\xi\left(1-4\xi^2\right)}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-61}$$

$$\lambda = \frac{+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\left(\sqrt{1-\xi^2}\right)}{\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-62}$$

$$\sigma = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \det er \min ant \begin{bmatrix} -2R1* & 4\xi^2 \\ 2R1 & 1 \end{bmatrix} \tag{E-63}$$

$$\sigma = \frac{\left\lfloor -2R1* - \left(4\xi^2\right)(2R1)\right\rfloor}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{E-64}$$

$$\sigma = \frac{\left\lfloor -R1* - \left(4\xi^2\right)(R1)\right\rfloor}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-65}$$

$$R1 = +j\xi+\sqrt{1-\xi^2} \tag{E-66}$$

$$\sigma = \frac{\left[-\left(-j\xi+\sqrt{1-\xi^2}\right)-\left(4\xi^2\right)\left(+j\xi+\sqrt{1-\xi^2}\right)\right]}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-67}$$

$$\sigma = \frac{\left[\ + j\xi - \sqrt{1-\xi^2} - j4\xi^3 - 4\xi^2\sqrt{1-\xi^2}\right]}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-68}$$

$$\sigma = \frac{\left[\ -\left(1+4\xi^2\right)\sqrt{1-\xi^2} + j\xi\left(1-4\xi^2\right)\right]}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-69}$$

$$\sigma = \frac{\left[\ +\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{E-70}$$

Recall,

$$\alpha = -\sigma \tag{E-71}$$

$$\lambda = -\beta \tag{E-72}$$

The complete solution set is thus

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{8\xi\sqrt{1-\xi^2}} \begin{bmatrix} -\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ -\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \end{bmatrix} \tag{E-73}$$

The partial fraction expansion is thus

$$H(\rho)H^*(\rho) =$$

$$+\frac{\alpha}{(\rho - R1)}$$

$$+\frac{\beta}{(\rho - R1^*)}$$

$$+\frac{\lambda}{(\rho + R1^*)}$$

$$+\frac{\sigma}{(\rho + R1)}$$

(E-74)

APPENDIX F

$$\frac{s^2}{(s+a)(s+b)} = \alpha + \frac{\beta}{s+a} + \frac{\lambda}{s+b} \tag{F-1}$$

$$s^2 = \alpha(s+a)(s+b) + \frac{\beta}{(s+a)}(s+a)(s+b) + \frac{\lambda}{(s+b)}(s+a)(s+b) \tag{F-2}$$

$$s^2 = \alpha\left(s^2 + (a+b)s + ab\right) + \beta(s+b) + \lambda(s+a) \tag{F-3}$$

$$s^2 = \alpha s^2 + \alpha(a+b)s + \alpha ab + \beta s + \beta b + \lambda s + \lambda a \tag{F-4}$$

$$s^2 = \alpha s^2 + [\alpha(a+b) + \beta + \lambda]s + \alpha ab + \beta b + \lambda a \tag{F-5}$$

Equation (F-5) yields three individual equations.

$$\alpha = 1 \tag{F-6}$$

$$[\alpha(a+b) + \beta + \lambda] = 0 \tag{F-7}$$

$$\alpha ab + \beta b + \lambda a = 0 \tag{F-8}$$

$$[(a+b) + \beta + \lambda] = 0 \tag{F-9}$$

$$ab + \beta b + \lambda a = 0 \tag{F-10}$$

$$\beta + \lambda = -(a+b) \tag{F-11}$$

$$\beta b + \lambda a = -ab \tag{F-12}$$

The equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 \\ b & a \end{bmatrix}\begin{bmatrix} \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} -(a+b) \\ -ab \end{bmatrix} \tag{F-13}$$

Apply Cramer's rule.

$$\beta = \frac{1}{a-b}\det\begin{bmatrix} -(a+b) & 1 \\ -ab & a \end{bmatrix}\begin{bmatrix} \beta \\ \lambda \end{bmatrix} \tag{F-14}$$

$$\beta = \frac{-a^2 - ab + ab}{a-b} \tag{F-15}$$

$$\beta = \frac{-a^2}{a-b} \tag{F-16}$$

$$\lambda = \frac{1}{a-b}\det\begin{bmatrix} 1 & -(a+b) \\ b & -ab \end{bmatrix} \tag{F-17}$$

$$\lambda = \frac{-ab + b(a+b)}{a-b} \tag{F-18}$$

$$\lambda = \frac{-ab + ab + b^2}{a-b} \tag{F-19}$$

$$\lambda = \frac{b^2}{a-b} \tag{F-20}$$

The partial fraction expansion is thus

$$\frac{s^2}{(s+a)(s+b)} = 1 + \left[\frac{-a^2}{a-b}\right]\left[\frac{1}{s+a}\right] + \left[\frac{b^2}{a-b}\right]\left[\frac{\lambda}{s+b}\right], \quad a \neq b \tag{F-21}$$

APPENDIX G

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + \left[\frac{-a^2}{a-b}\right]\left[\frac{1}{s+a}\right] + \left[\frac{b^2}{a-b}\right]\left[\frac{1}{s+b}\right]\right\}\left\{1 + \left[\frac{-c^2}{c-d}\right]\left[\frac{1}{s+c}\right] + \left[\frac{d^2}{c-d}\right]\left[\frac{1}{s+d}\right]\right\} \tag{G-1}$$

Let

$$A = \left[\frac{-a^2}{a-b}\right] \tag{G-2}$$

$$B = \left[\frac{b^2}{a-b}\right] \tag{G-3}$$

$$C = \left[\frac{-c^2}{c-d}\right] \tag{G-4}$$

$$D = \left[\frac{d^2}{c-d}\right] \tag{G-5}$$

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + A\left[\frac{1}{s+a}\right] + B\left[\frac{1}{s+b}\right]\right\}\left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$

(G-6)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ A\left[\frac{1}{s+a}\right]\left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ B\left[\frac{1}{s+b}\right]\left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$

(G-7)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ A\left\{\left[\frac{1}{s+a}\right] + C\left[\frac{1}{s+a}\right]\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+a}\right]\left[\frac{1}{s+d}\right]\right\}$$
$$+ B\left\{\left[\frac{1}{s+b}\right] + C\left[\frac{1}{s+b}\right]\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+b}\right]\left[\frac{1}{s+d}\right]\right\}$$

(G-8)

Recall

$$\frac{1}{(s+a)(s+b)} = \left[\frac{1}{a-b}\right]\left[\frac{-1}{s+a} + \frac{1}{s+b}\right], \quad a \neq b \qquad \text{(G-9)}$$

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ A\left\{\left[\frac{1}{s+a}\right] + C\left[\frac{1}{a-c}\right]\left[\frac{-1}{s+a} + \frac{1}{s+c}\right] + D\left[\frac{1}{a-d}\right]\left[\frac{-1}{s+a} + \frac{1}{s+d}\right]\right\}$$
$$+ B\left\{\left[\frac{1}{s+b}\right] + C\left[\frac{1}{b-c}\right]\left[\frac{-1}{s+b} + \frac{1}{s+c}\right] + D\left[\frac{1}{b-d}\right]\left[\frac{-1}{s+b} + \frac{1}{s+d}\right]\right\}$$

(G-10)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ A\left\{\left[\frac{1}{s+a}\right] + C\left[\frac{1}{a-c}\right]\left[\frac{-1}{s+a} + \frac{1}{s+c}\right] + D\left[\frac{1}{a-d}\right]\left[\frac{-1}{s+a} + \frac{1}{s+d}\right]\right\}$$
$$+ B\left\{\left[\frac{1}{s+b}\right] + C\left[\frac{1}{b-c}\right]\left[\frac{-1}{s+b} + \frac{1}{s+c}\right] + D\left[\frac{1}{b-d}\right]\left[\frac{-1}{s+b} + \frac{1}{s+d}\right]\right\}$$

(G-11)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{1 + C\left[\frac{1}{s+c}\right] + D\left[\frac{1}{s+d}\right]\right\}$$
$$+ \left\{\left[\frac{A}{s+a}\right] + \left[\frac{AC}{a-c}\right]\left[\frac{-1}{s+a}\right] + \left[\frac{AC}{a-c}\right]\left[\frac{1}{s+c}\right] + \left[\frac{AD}{a-d}\right]\left[\frac{-1}{s+a}\right] + \left[\frac{AD}{a-d}\right]\left[\frac{1}{s+d}\right]\right\}$$
$$+ \left\{\left[\frac{B}{s+b}\right] + \left[\frac{BC}{b-c}\right]\left[\frac{-1}{s+b}\right] + \left[\frac{BC}{b-c}\right]\left[\frac{1}{s+c}\right] + \left[\frac{BD}{b-d}\right]\left[\frac{-1}{s+b}\right] + \left[\frac{BD}{b-d}\right]\left[\frac{1}{s+d}\right]\right\}$$

(G-12)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$\begin{aligned}
&1 + A\left[1-\left[\frac{C}{a-c}\right]-\left[\frac{D}{a-d}\right]\right]\left[\frac{1}{s+a}\right] \\
&+ B\left[1-\left[\frac{C}{b-c}\right]-\left[\frac{D}{b-d}\right]\right]\left[\frac{1}{s+b}\right] \\
&+ C\left[1+\left[\frac{A}{a-c}\right]+\left[\frac{B}{b-c}\right]\right]\left[\frac{1}{s+c}\right] \\
&+ D\left[1+\left[\frac{A}{a-d}\right]+\left[\frac{B}{b-d}\right]\right]\left[\frac{1}{s+d}\right]
\end{aligned}$$

(G-13)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$\begin{aligned}
&1 + \left[\frac{-a^2}{a-b}\right]\left[1-\left[\frac{-c^2}{(a-c)(c-d)}\right]-\left[\frac{d^2}{(a-d)(c-d)}\right]\right]\left[\frac{1}{s+a}\right] \\
&+ \left[\frac{b^2}{a-b}\right]\left[1-\left[\frac{-c^2}{(b-c)(c-d)}\right]-\left[\frac{d^2}{(c-d)(b-d)}\right]\right]\left[\frac{1}{s+b}\right] \\
&+ \left[\frac{-c^2}{c-d}\right]\left[1+\left[\frac{-a^2}{(a-b)(a-c)}\right]+\left[\frac{b^2}{(a-b)(b-c)}\right]\right]\left[\frac{1}{s+c}\right] \\
&+ \left[\frac{d^2}{c-d}\right]\left[1+\left[\frac{-a^2}{(a-b)(a-d)}\right]+\left[\frac{b^2}{(a-b)(b-d)}\right]\right]\left[\frac{1}{s+d}\right]
\end{aligned}$$

(G-14)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$1+\left[\frac{-a^2}{a-b}\right]\left[1+\left[\frac{c^2}{(a-c)(c-d)}\right]-\left[\frac{d^2}{(a-d)(c-d)}\right]\right]\left[\frac{1}{s+a}\right]$$
$$+\left[\frac{b^2}{a-b}\right]\left[1+\left[\frac{c^2}{(b-c)(c-d)}\right]-\left[\frac{d^2}{(c-d)(b-d)}\right]\right]\left[\frac{1}{s+b}\right]$$
$$+\left[\frac{-c^2}{c-d}\right]\left[1-\left[\frac{a^2}{(a-b)(a-c)}\right]+\left[\frac{b^2}{(a-b)(b-c)}\right]\right]\left[\frac{1}{s+c}\right]$$
$$+\left[\frac{d^2}{c-d}\right]\left[1-\left[\frac{a^2}{(a-b)(a-d)}\right]+\left[\frac{b^2}{(a-b)(b-d)}\right]\right]\left[\frac{1}{s+d}\right]$$

(G-15)

APPENDIX H

$$\frac{s}{(s+a)^2} = \frac{As+B}{(s+a)} + \frac{C}{(s+a)^2} \tag{H-1}$$

$$s = \frac{(As+B)(s+a)^2}{(s+a)} + C \tag{H-2}$$

$$s = (As+B)(s+a) + C \tag{H-3}$$

$$s = \left(As^2 + Bs\right) + a(As+B) + C \tag{H-4}$$

$$s = As^2 + (aA+B)s + aB + C \tag{H-5}$$

$$A=0 \tag{H-6}$$

$$s = Bs + aB + C \tag{H-7}$$

$$B = 1 \tag{H-8}$$

$$C = -a \tag{H-9}$$

$$\frac{s}{(s+a)^2} = \frac{1}{(s+a)} + \frac{-a}{(s+a)^2} \tag{H-10}$$

APPENDIX I

$$\frac{1}{\left(s^2+\alpha^2\right)(s+\beta)^2}=\frac{As+B}{s^2+\alpha^2}+\frac{Cs+D}{(s+\beta)^2} \qquad \text{(I-1)}$$

$$\frac{1}{\left(s^2+\alpha^2\right)\left(s+2\beta+\beta^2\right)}=\frac{As+B}{s^2+\alpha^2}+\frac{Cs+D}{(s+\beta)^2} \qquad \text{(I-2)}$$

The expansion can be performed using the derivation in Appendix B.

$$\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s^2+2\beta s+\beta^2}\right\} =$$

$$+\frac{[-2\beta]s+\left[-\left(\alpha^2-\beta^2\right)\right]}{\left[\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2\right]\left[s^2+\alpha^2\right]}$$

$$+\frac{[2\beta]s+\left[\left(\alpha^2-\beta^2\right)+(2\beta)^2\right]}{\left[\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2\right]\left[s^2+2\beta s+\beta^2\right]}$$

(I-3)

$$\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s^2+2\beta s+\beta^2}\right\} =$$

$$+\frac{[-2\beta]s+\left[-\alpha^2+\beta^2\right]}{\left[\alpha^4-\beta^4\right]\left[s^2+\alpha^2\right]} + \frac{[2\beta]s+\left[\alpha^2+3\beta^2\right]}{\left[\alpha^4-\beta^4\right][s+\beta]^2}$$

(I-4)

$$\left\{\frac{1}{s^2+\alpha^2}\right\}\left\{\frac{1}{s^2+2\beta s+\beta^2}\right\} =$$

$$\frac{2\beta}{(\alpha^4-\beta^4)}\left\{-\frac{s+\left[\alpha^2-\beta^2\right]/[2\beta]}{[s^2+\alpha^2]} + \frac{s+\left[\alpha^2+3\beta^2\right]/[2\beta]}{[s+\beta]^2}\right\}$$

(I-5)

The inverse Fourier transform is

$$f(t)=\frac{1}{\left[\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2\right]}\left\{[-2\beta]\cos(\alpha t)+\left[-\frac{1}{\alpha}\left(\alpha^2-\beta^2\right)\right]\sin(\alpha t)\right\}$$

$$+\frac{\beta\exp(-\beta t)}{\left[\left(\alpha^2-\beta^2\right)^2+(2\alpha\beta)^2\right]}\left\{2\beta\cos(\beta t)+\left[\frac{\left[\left(\alpha^2-\beta^2\right)+2\left(\beta^2\right)\right]}{\beta}\right]\sin(\beta t)\right\}$$

(I-6)

$$f(t) = \frac{1}{[\alpha^4 - \beta^4]}\left\{ [-2\beta]\cos(\alpha t) + \left[-\frac{1}{\alpha}\left(\alpha^2 - \beta^2\right)\right]\sin(\alpha t)\right\}$$

$$+ \frac{\beta \exp(-\beta t)}{[\alpha^4 - \beta^4]}\left\{2\beta \cos(\beta t) + \left[\frac{\alpha^2 + \beta^2}{\beta}\right]\sin(\beta t)\right\}$$

(I-7)

APPENDIX J

$$\frac{1}{s^2}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\alpha s+\rho}{s^2} + \frac{\beta s+\lambda}{s^2+\omega^2} \tag{J-1}$$

$$1 = (\alpha s+\rho)(s^2+\omega^2) + \beta s^3 + \lambda s^2 \tag{J-2}$$

$$1 = \alpha s^3 + \alpha\omega^2 s + \rho s^2 + \rho\omega^2 + \beta s^3 + \lambda s^2 \tag{J-3}$$

$$1 = (\alpha+\beta)s^3 + (\rho+\lambda)s^2 + \alpha\omega^2 s + \rho\omega^2 \tag{J-4}$$

$$(\alpha+\beta) = 0 \tag{J-5}$$

$$(\rho+\lambda) = 0 \tag{J-6}$$

$$\alpha = 0 \tag{J-7}$$

$$\beta = 0 \tag{J-8}$$

$$\rho = 1/\omega^2 \tag{J-9}$$

$$\lambda = -1/\omega^2 \tag{J-10}$$

$$\frac{1}{s^2}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1/\omega^2}{s^2} + \frac{-1/\omega^2}{s^2+\omega^2} \tag{J-11}$$

The inverse Laplace transform is

$$f(t) = \frac{1}{\omega^2}t - \frac{1}{\omega^3}\sin(\omega t) \tag{J-12}$$