

PARTIAL FRACTIONS IN SHOCK AND VIBRATION ANALYSIS Revision E

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Introduction

Transforming a fraction into the sum of partial fractions is an intermediate step in the solution of certain shock and vibration problems. The purpose of this tutorial is to summarize some common cases.

Term	Appendix
$\left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\}$	A
$\left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{\alpha s + \beta}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\}$	B
$\frac{\alpha s + \beta}{(s + \lambda)(s + \sigma)}, \quad \lambda \neq \sigma$	C
$\left[\frac{1}{\rho^2 - j2\xi\rho - 1} \right] \left[\frac{1}{\rho^2 + j2\xi\rho - 1} \right]$	D
$\left[\frac{1 + j2\xi\rho}{(1 - \rho^2) + j2\xi\rho} \right] \left[\frac{1 - j2\xi\rho}{(1 - \rho^2) - j2\xi\rho} \right]$	E
$\frac{s^2}{(s + a)(s + b)}, \quad a \neq b$	F
$\frac{s^4}{(s + a)(s + b)(s + c)(s + d)}, \quad a \neq b, a \neq c, a \neq d, b \neq c, b \neq d, c \neq d$	G

Note that

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - \xi^2\omega_n^2 \quad (1)$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 [1 - \xi^2] \quad (2)$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad (3)$$

where

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \quad (4)$$

Thus,

$$\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \quad (5)$$

The inverse Laplace transform $f(t)$ of equation (5) is

$$f(t) = \frac{1}{\omega_d} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad (6)$$

Furthermore, consider the Laplace transform

$$\hat{F}(s) = \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \quad (7)$$

The inverse Laplace transform $\hat{f}(t)$ is

$$\hat{f}(t) = \exp(-\xi\omega_n t)\cos(\omega_d t) \quad (8)$$

APPENDIX A

Example 1

$$\left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \left\{ \frac{\rho}{s} \right\} + \left\{ \frac{\sigma s + \phi}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{A-1})$$

$$1 = \left[s^2 + 2\xi\omega_n s + \omega_n^2 \right] \rho + [\sigma s + \phi] s \quad (\text{A-2})$$

$$1 = \left[\rho s^2 + 2\xi\omega_n \rho s + \rho\omega_n^2 \right] + \left[\sigma s^2 + \phi s \right] \quad (\text{A-3})$$

$$1 = (\rho + \sigma) s^2 + (2\xi\omega_n \rho + \phi) s + \rho\omega_n^2 \quad (\text{A-4})$$

Equation (A-4) implies three separate equations.

$$(\rho + \sigma) = 0 \quad (\text{A-5})$$

$$(2\xi\omega_n \rho + \phi) = 0 \quad (\text{A-6})$$

$$\rho\omega_n^2 = 1 \quad (\text{A-7})$$

Equation (A-7) yields

$$\rho = \frac{1}{\omega_n^2} \quad (\text{A-8})$$

$$\sigma = -\rho \quad (\text{A-9})$$

$$\sigma = -\frac{1}{\omega_n^2} \quad (\text{A-10})$$

$$\phi = -2\xi\omega_n \rho \quad (\text{A-11})$$

$$\phi = -2\xi\omega_n \left[\frac{1}{\omega_n^2} \right] \quad (\text{A-12})$$

$$\phi = -\frac{2\xi}{\omega_n} \quad (\text{A-13})$$

Equation (A-1) can thus be expressed as

$$\left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \left\{ \frac{1}{\omega_n^2} \right\} \left\{ \frac{1}{s} \right\} + \left\{ \frac{-\left[\frac{1}{\omega_n^2} \right] s - \frac{2\xi}{\omega_n}}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{A-14})$$

$$\left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \frac{1}{\omega_n^2} \left\{ \frac{1}{s} \right\} - \frac{1}{\omega_n^2} \left\{ \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{A-15})$$

$$\left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \frac{1}{\omega_n^2} \left\{ \frac{1}{s} \right\} - \frac{1}{\omega_n^2} \left\{ \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (\text{A-16})$$

The following form is a more convenient format prior to taking the inverse Laplace transformation,

$$\begin{aligned} \left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} &= \frac{1}{\omega_n^2} \left\{ \frac{1}{s} \right\} \\ &\quad - \frac{1}{\omega_n^2} \left\{ \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \\ &\quad - \frac{1}{\omega_n^2} \left\{ \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \end{aligned} \tag{A-17}$$

$$\begin{aligned} \left\{ \frac{1}{s} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} &= \frac{1}{\omega_n^2} \left\{ \frac{1}{s} \right\} \\ &\quad - \left(\frac{1}{\omega_n^2} \right) \left\{ \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \\ &\quad - \left(\frac{1}{\omega_n^2} \right) \left(\frac{\xi\omega_n}{\omega_d} \right) \left\{ \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \end{aligned} \tag{A-18}$$

The inverse Laplace transform is

$$f(t) = \frac{1}{\omega_n^2} u(t) - \frac{1}{\omega_n^2} \exp(-\xi\omega_n t) \left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} \sin(\omega_d t) \right], t \geq 0 \tag{A-19}$$

where $u(t)$ is the unit step function.

APPENDIX B

Example 2

Equation (B-1) effectively represents a number of cases since α and β may each be set equal to zero.

$$\left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{\alpha s + \beta}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \left\{ \frac{\lambda s + \rho}{s^2 + \omega^2} \right\} + \left\{ \frac{\sigma s + \phi}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{B-1})$$

Multiply through by the common denominator.

$$\alpha s + \beta = \{\lambda s + \rho\} \{s^2 + 2\xi\omega_n s + \omega_n^2\} + \{\sigma s + \phi\} \{s^2 + \omega^2\} \quad (\text{B-2})$$

$$\begin{aligned} \alpha s + \beta &= \lambda s^3 + (\rho + 2\xi\omega_n \lambda) s^2 + (2\xi\omega_n \rho + \lambda \omega_n^2) s + (\rho \omega_n^2) \\ &\quad + \sigma s^3 + \phi s^2 + \sigma \omega^2 s + \phi \omega^2 \end{aligned} \quad (\text{B-3})$$

$$\begin{aligned} \alpha s + \beta &= \\ &[\lambda + \sigma] s^3 \\ &+ [\rho + 2\xi\omega_n \lambda + \phi] s^2 \\ &+ [2\xi\omega_n \rho + \lambda \omega_n^2 + \sigma \omega^2] s \\ &+ [\rho \omega_n^2 + \phi \omega^2] \end{aligned} \quad (\text{B-4})$$

Equation (B-4) implies four separate equations.

$$\lambda + \sigma = 0 \quad (\text{B-5})$$

$$\rho + 2\xi\omega_n \lambda + \phi = 0 \quad (\text{B-6})$$

$$2\xi\omega_n\rho + \lambda\omega_n^2 + \sigma\omega^2 = \alpha \quad (\text{B-7})$$

$$\rho\omega_n^2 + \phi\omega^2 = \beta \quad (\text{B-8})$$

Equations (B-5) through (B-8) can be assembled into matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2\xi\omega_n & 1 & 0 & 1 \\ \omega_n^2 & 2\xi\omega_n & \omega^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix} \quad (\text{B-9})$$

Gaussian elimination is used to simplify the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2\xi\omega_n & 1 \\ 0 & 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix} \quad (\text{B-10})$$

Equation (B-10) can be reduced to a 3 x 3 matrix.

$$\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix} \quad (\text{B-11})$$

Complete the solution using Cramer's rule.

$$\det \begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \omega^2(\omega^2 - \omega_n^2) + \omega^2(2\xi\omega_n)^2 - \omega_n^2(\omega^2 - \omega_n^2) \quad (\text{B-12})$$

$$\det \begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = (\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \quad (\text{B-13})$$

$$\rho = \frac{1}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \det \begin{bmatrix} 0 & -2\xi\omega_n & 1 \\ \alpha & \omega^2 - \omega_n^2 & 0 \\ \beta & 0 & \omega^2 \end{bmatrix} \quad (\text{B-14})$$

$$\rho = \frac{2\xi\omega^2\omega_n\alpha - (\omega^2 - \omega_n^2)\beta}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-15})$$

$$\sigma = \frac{1}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \det \begin{bmatrix} 1 & 0 & 1 \\ 2\xi\omega_n & \alpha & 0 \\ \omega_n^2 & \beta & \omega^2 \end{bmatrix} \quad (\text{B-16})$$

$$\sigma = \frac{\omega^2\alpha + 2\xi\omega_n\beta - \omega_n^2\alpha}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-17a})$$

$$\sigma = \frac{(\omega^2 - \omega_n^2)\alpha + 2\xi\omega_n\beta}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-17b})$$

Recall equation (B-5).

$$\lambda = -\sigma \quad (\text{B-18})$$

$$\lambda = -\frac{\left(\omega^2 - \omega_n^2\right)\alpha + 2\xi\omega_n\beta}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-19})$$

$$\phi = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \det \begin{bmatrix} 1 & -2\xi\omega_n & 0 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & \alpha \\ \omega_n^2 & 0 & \beta \end{bmatrix} \quad (\text{B-20})$$

$$\phi = \frac{\left(\omega^2 - \omega_n^2\right)\beta + (2\xi\omega_n)^2\beta - 2\xi\omega_n^3\alpha}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-21a})$$

$$\phi = \frac{\left[\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2\right]\beta - 2\xi\omega_n^3\alpha}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \quad (\text{B-21b})$$

The coefficients are summarized in equation (B-22).

$$\begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \begin{bmatrix} -\left(\omega^2 - \omega_n^2\right)\alpha - 2\xi\omega_n\beta \\ 2\xi\omega^2\omega_n\alpha - \left(\omega^2 - \omega_n^2\right)\beta \\ \left(\omega^2 - \omega_n^2\right)\alpha + 2\xi\omega_n\beta \\ -2\xi\omega_n^3\alpha + \left[\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2\right]\beta \end{bmatrix} \quad (\text{B-22a})$$

$$\begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \frac{1}{(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2} \left\{ \begin{bmatrix} -(\omega^2 - \omega_n^2) \\ 2\xi\omega^2\omega_n \\ (\omega^2 - \omega_n^2) \\ -2\xi\omega_n^3 \end{bmatrix} \alpha + \begin{bmatrix} -2\xi\omega_n \\ -(\omega^2 - \omega_n^2) \\ 2\xi\omega_n \\ \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right] \end{bmatrix} \beta \right\}$$

(B-22b)

Equation (B-1) can thus be rewritten as

$$\left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{\alpha s + \beta}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} =$$

$$\frac{\left[-(\omega^2 - \omega_n^2)\alpha - 2\xi\omega_n \beta \right] s + \left[2\xi\omega^2\omega_n \alpha - (\omega^2 - \omega_n^2)\beta \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[s^2 + \omega^2 \right]}$$

$$+ \frac{\left[(\omega^2 - \omega_n^2)\alpha + 2\xi\omega_n \beta \right] s + \left[-2\xi\omega_n^3 \alpha + \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right] \beta \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[s^2 + 2\xi\omega_n s + \omega_n^2 \right]}$$

(B-23)

An alternate form is

$$\begin{aligned}
 \left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{\alpha s + \beta}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = & \\
 \alpha \frac{[-(\omega^2 - \omega_n^2)]s + [2\xi\omega^2\omega_n]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + \omega^2]} & \\
 + \beta \frac{[-2\xi\omega_n]s + [-(\omega^2 - \omega_n^2)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + \omega^2]} & \\
 + \alpha \frac{\left[(\omega^2 - \omega_n^2) \right]s + [-2\xi\omega_n^3]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + 2\xi\omega_n s + \omega_n^2]} & \\
 + \beta \frac{[2\xi\omega_n]s + \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + 2\xi\omega_n s + \omega_n^2]} &
 \end{aligned}$$

(B-24)

$$\begin{aligned}
& \left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{\alpha s + \beta}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \\
& \alpha \frac{\left[-(\omega^2 - \omega_n^2) \right] s + \left[2\xi\omega^2\omega_n \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[s^2 + \omega^2 \right]} \\
& + \beta \frac{\left[-2\xi\omega_n \right] s + \left[-(\omega^2 - \omega_n^2) \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[s^2 + \omega^2 \right]} \\
& + \alpha \frac{\left[(\omega^2 - \omega_n^2) \right] s + \left[-2\xi\omega_n^3 \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \\
& + \beta \frac{\left[2\xi\omega_n \right] s + \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]}
\end{aligned}$$

(B-25)

The inverse Laplace transforms for the four terms on the left-hand-side are found as follows

$$F_1(s) = \alpha \frac{\left[-\left(\omega^2 - \omega_n^2\right) \right] s + \left[2\xi \omega^2 \omega_n \right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi \omega \omega_n)^2 \right] \left[s^2 + \omega^2 \right]} \quad (\text{B-26})$$

$$f_1(t) = \frac{\alpha}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi \omega \omega_n)^2 \right]} \left\{ \left[-\left(\omega^2 - \omega_n^2\right) \right] \cos(\omega t) + \left[2\xi \omega \omega_n \right] \sin(\omega t) \right\} \quad (\text{B-27})$$

$$F_2(s) = +\beta \frac{\left[-2\xi \omega_n \right] s + \left[-\left(\omega^2 - \omega_n^2\right) \right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi \omega \omega_n)^2 \right] \left[s^2 + \omega^2 \right]} \quad (\text{B-28})$$

$$f_2(t) = \frac{\beta}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi \omega \omega_n)^2 \right]} \left\{ \left[-2\xi \omega_n \right] \cos(\omega t) + \left[-\frac{1}{\omega} \left(\omega^2 - \omega_n^2\right) \right] \sin(\omega t) \right\} \quad (\text{B-29})$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2 \right) \right] s + \left[-2\xi\omega_n^3 \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \quad (\text{B-30})$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2 \right) \right] s}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} + \alpha \frac{\left[-2\xi\omega_n^3 \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \quad (\text{B-31})$$

$$F_3(s) = +\alpha \frac{\left[\left(\omega^2 - \omega_n^2 \right) \right] s}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} + \alpha \left(\frac{-2\xi\omega_n^3}{\omega_d} \right) \frac{\omega_d}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]}$$

(B-32)

$$f_3(t) = \frac{\alpha \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\left(\omega^2 - \omega_n^2 \right) \cos(\omega_d t) + \left(\frac{-2\xi\omega_n^3}{\omega_d} \right) \sin(\omega_d t) \right] \quad (\text{B-33})$$

$$F_4(s) = +\beta \frac{[2\xi\omega_n]s + \left[\left(\omega^2 - \omega_n^2 \right) + (2\xi\omega_n)^2 \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \quad (\text{B-34})$$

$$F_4(s) = +\beta \frac{[2\xi\omega_n]s}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} + \beta \frac{\left[\left(\omega^2 - \omega_n^2 \right) + (2\xi\omega_n)^2 \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \quad (\text{B-35})$$

$$F_4(s) = +\beta \frac{[2\xi\omega_n]s}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} + \frac{\beta}{\omega_d} \frac{\left[\left(\omega^2 - \omega_n^2 \right) + (2\xi\omega_n)^2 \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right] \left[(s + \xi\omega_n)^2 + \omega_d^2 \right]} \quad (\text{B-36})$$

$$F_4(s) = \frac{\beta \exp(-\xi \omega_n t)}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi \omega \omega_n)^2 \right]} \left\{ [2\xi \omega_n] \cos(\omega_n t) + \frac{1}{\omega_d} \left[(\omega^2 - \omega_n^2) + (2\xi \omega_n)^2 \right] \sin(\omega_n t) \right\}$$

(B-36)

APPENDIX C

Example 3

Equation (C-1) effectively represents a number of cases since α and β may each be set equal to zero.

$$\left\{ \frac{\alpha s + \beta}{(s + \lambda)(s + \sigma)} \right\} = \left\{ \frac{p}{s + \lambda} \right\} + \left\{ \frac{m}{s + \sigma} \right\}, \quad \lambda \neq \sigma \quad (\text{C-1})$$

$$\{\alpha s + \beta\} = \left\{ \frac{p}{s + \lambda} \right\} (s + \lambda)(s + \sigma) + \left\{ \frac{m}{s + \sigma} \right\} (s + \lambda)(s + \sigma) \quad (\text{C-2})$$

$$(\alpha s + \beta) = (p)(s + \sigma) + (m)(s + \lambda) \quad (\text{C-3})$$

$$(\alpha s + \beta) = (ps + p\sigma) + (ms + m\lambda) \quad (\text{C-4})$$

$$(\alpha s + \beta) = (p + m)s + (p\sigma + m\lambda) \quad (\text{C-5})$$

Equation (C-5) implies equations (C-6) and (C-7).

$$\alpha = (p + m) \quad (\text{C-6})$$

$$\beta = (p\sigma + m\lambda) \quad (\text{C-7})$$

Equation (C-6) yields

$$m = \alpha - p \quad (\text{C-8})$$

Substitute equation (C-8) into (C-7).

$$\beta = (p\sigma + (\alpha - p)\lambda) \quad (\text{C-9})$$

$$\beta = (\sigma - \lambda)p + \alpha\lambda \quad (\text{C-10})$$

$$\beta - \alpha\lambda = (\sigma - \lambda)p \quad (\text{C-11})$$

$$(\sigma - \lambda)p = \beta - \alpha\lambda \quad (\text{C-12})$$

$$p = \frac{\beta - \alpha\lambda}{\sigma - \lambda} \quad (\text{C-13})$$

Substitute equation (C-13) into (C-8).

$$m = \alpha - \left(\frac{\beta - \alpha\lambda}{\sigma - \lambda} \right) \quad (\text{C-14})$$

$$m = \frac{\alpha(\sigma - \lambda) - (\beta - \alpha\lambda)}{\sigma - \lambda} \quad (\text{C-15})$$

$$m = \frac{(\alpha\sigma - \alpha\lambda) - (\beta - \alpha\lambda)}{\sigma - \lambda} \quad (\text{C-16})$$

$$m = \frac{\alpha\sigma - \alpha\lambda - \beta + \alpha\lambda}{\sigma - \lambda} \quad (\text{C-17})$$

$$m = \frac{\alpha\sigma - \beta}{\sigma - \lambda} \quad (\text{C-18})$$

Substitute equations (C-18) and (C-13) into (C-1).

$$\left\{ \frac{\alpha s + \beta}{(s + \lambda)(s + \sigma)} \right\} = \left\{ \frac{1}{\sigma - \lambda} \right\} \left\{ \left[\frac{\beta - \alpha\lambda}{s + \lambda} \right] + \left[\frac{\alpha\sigma - \beta}{s + \sigma} \right] \right\} \quad (\text{C-19})$$

The inverse Laplace transform is

$$f(t) = \left\{ \frac{1}{\sigma - \lambda} \right\} \left\{ \frac{1}{\lambda} [\beta - \alpha\lambda] \sin(\lambda t) + \frac{1}{\sigma} [\alpha\sigma - \beta] \sin(\sigma t) \right\} \quad (\text{C-20})$$

APPENDIX D

Example 4

The transfer function $H(\rho)$ times its complex conjugate is

$$H(\rho)H^*(\rho) = \left[\frac{1}{\rho^2 - j2\xi\rho - 1} \right] \left[\frac{1}{\rho^2 + j2\xi\rho - 1} \right] \quad (D-1)$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \quad (D-2)$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad (D-3)$$

$$R1, R2 = j\xi \pm \sqrt{1 - \xi^2} \quad (D-4)$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \quad (D-5)$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad (D-6)$$

$$R3, R4 = -j\xi \pm \sqrt{1 - \xi^2} \quad (D-7)$$

(D-8)

Summary,

$$R1 = +j\xi + \sqrt{1 - \xi^2} \quad (D-9)$$

$$R2 = +j\xi - \sqrt{1 - \xi^2} \quad (D-10)$$

$$R3 = -j\xi + \sqrt{1 - \xi^2} \quad (D-11)$$

$$R4 = -j\xi - \sqrt{1 - \xi^2} \quad (D-12)$$

Note

$$R2 = -R1^* \quad (D-13)$$

$$R3 = R1^* \quad (D-14)$$

$$R4 = -R1^* \quad (D-15)$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1}{\left(\rho - j\xi - \sqrt{1 - \xi^2}\right)\left(\rho - j\xi + \sqrt{1 - \xi^2}\right)\left(\rho + j\xi - \sqrt{1 - \xi^2}\right)\left(\rho + j\xi + \sqrt{1 - \xi^2}\right)} \right] \quad (D-16)$$

$$H(\rho)H^*(\rho) = \left[\frac{1}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)} \right] \quad (D-17)$$

$$H(\rho)H^*(\rho) = \left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] \quad (D-18)$$

Expand into partial fractions.

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] = \begin{aligned} &+ \frac{\alpha}{(\rho - R1)} \\ &+ \frac{\beta}{(\rho - R1^*)} \\ &+ \frac{\lambda}{(\rho + R1^*)} \\ &+ \frac{\sigma}{(\rho + R1)} \end{aligned} \tag{D-19}$$

Multiply through by the denominator on the left-hand side of equation (D-19).

$$1 = \begin{aligned} &+ \frac{\alpha}{(\rho - R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+ \frac{\beta}{(\rho - R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+ \frac{\lambda}{(\rho + R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ &+ \frac{\sigma}{(\rho + R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \end{aligned} \tag{D-20}$$

$$1 = \begin{aligned} &+ \alpha (\rho - R1^*)(\rho + R1^*)(\rho + R1) \\ &+ \beta (\rho - R1)(\rho + R1^*)(\rho + R1) \\ &+ \lambda (\rho - R1)(\rho - R1^*)(\rho + R1) \\ &+ \sigma (\rho - R1)(\rho - R1^*)(\rho + R1^*) \end{aligned} \tag{D-21}$$

$$\begin{aligned}
1 = & + \alpha \left(\rho^2 - R1^*{}^2 \right) (\rho + R1) \\
& + \beta \left(\rho^2 + (-R1 + R1^*)\rho - R1R1^* \right) (\rho + R1) \\
& + \lambda \left(\rho^2 + (-R1 - R1^*)\rho + R1R1^* \right) (\rho + R1) \\
& + \sigma \left(\rho^2 + (-R1 - R1^*)\rho + R1R1^* \right) (\rho + R1^*)
\end{aligned}
\tag{D-22}$$

$$\begin{aligned}
1 = & + \alpha \left(\rho^3 + R1\rho^2 - R1^*{}^2 \rho - R1R1^*{}^2 \right) \\
& + \beta \left(\rho^3 + (R1 - R1 + R1^*)\rho^2 + \left(-R1R1^* - R1^2 + R1R1^* \right) \rho - R1^2 R1^* \right) \\
& + \lambda \left(\rho^3 + (R1 - R1 - R1^*)\rho^2 + \left(R1R1^* - R1^2 - R1R1^* \right) \rho + R1^2 R1^* \right) \\
& + \sigma \left(\rho^3 + (R1^* - R1 - R1^*)\rho^2 + \left(-R1R1^* - R1R1^* - R1^*{}^2 \right) \rho + R1R1^*{}^2 \right)
\end{aligned}
\tag{D-23}$$

$$\begin{aligned}
1 = & + \alpha \left(\rho^3 + R1\rho^2 - R1^*{}^2 \rho - R1R1^*{}^2 \right) \\
& + \beta \left(\rho^3 + R1^*\rho^2 - R1^2 \rho - R1^2 R1^* \right) \\
& + \lambda \left(\rho^3 - R1^*\rho^2 - R1^2 \rho + R1^2 R1^* \right) \\
& + \sigma \left(\rho^3 - R1\rho^2 - R1^*{}^2 \rho + R1R1^*{}^2 \right)
\end{aligned}
\tag{D-24}$$

$$\begin{aligned}
1 = & + [\alpha + \beta + \lambda + \sigma] \rho^3 \\
& + [R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma] \rho^2 \\
& + [-R1^*{}^2 \alpha - R1^2 \beta - R1^2 \lambda - R1^*{}^2 \sigma] \rho \\
& + [-R1R1^*{}^2 \alpha - R1^2 R1^* \beta + R1^2 R1^* \lambda + R1R1^*{}^2 \sigma]
\end{aligned}
\tag{D-25}$$

$$\begin{aligned}
1 = & +[\alpha + \beta + \lambda + \sigma]\rho^3 \\
& + [R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma]\rho^2 \\
& + \left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma \right]\rho \\
& + \left[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma \right]R1R1^*
\end{aligned} \tag{D-26}$$

Equation (D-26) can be broken up into four separate equations,

$$\alpha + \beta + \lambda + \sigma = 0 \tag{D-27}$$

$$\left[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma \right] = 0 \tag{D-28}$$

$$\left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma \right] = 0 \tag{D-29}$$

$$\left[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma \right]R1R1^* = 1 \tag{D-30}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/R1R1^* \end{bmatrix} \tag{D-31}$$

Recall

$$R1 = +j\xi + \sqrt{1 - \xi^2} \tag{D-32}$$

$$R1R1^* = \left[+j\xi + \sqrt{1 - \xi^2} \right] \left[-j\xi + \sqrt{1 - \xi^2} \right] \tag{D-33}$$

$$R1 R1^* = 1 \quad (D-34)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(D-35)$$

Multiply the first row by $R1^{*2}$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & -R1^2 + R1^{*2} & -R1^2 + R1^{*2} & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(D-36)$$

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(D-37)$$

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(D-38)$$

Multiply the first row by $-R1$ and add to the second row. Also multiply the first row by $R1^*$ and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1^* & -R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (D-39)$$

Multiply the third row by $-R1^*$ and add to the second row. Also, multiply the third row by $R1$ and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (D-40)$$

The first row equation yields

$$\alpha = -\sigma \quad (D-41)$$

The third row equation yields

$$\beta = -\lambda \quad (D-42)$$

Equation (D-40) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (D-43)$$

$$-2R1^*\lambda - 2R1\sigma = 0 \quad (D-44)$$

$$-R1^*\lambda = R1\sigma \quad (D-45)$$

$$\lambda = \frac{-R1}{R1^*} \sigma \quad (D-46)$$

$$2R1\lambda + 2R1^*\sigma = 1 \quad (D-47)$$

$$2R1 \left[\begin{array}{c} -R1 \\ R1^* \end{array} \right] \sigma + 2R1^* \sigma = 1 \quad (D-48)$$

$$\left\{ R1 \left[\begin{array}{c} -R1 \\ R1^* \end{array} \right] + R1^* \right\} \sigma = \frac{1}{2} \quad (D-49)$$

$$\sigma = \frac{1}{2} \frac{1}{\left\{ R1 \left[\begin{array}{c} -R1 \\ R1^* \end{array} \right] + R1^* \right\}} \quad (D-50)$$

$$\sigma = \frac{1}{2} \frac{R1^*}{[-R1^2 + R1^{*2}]} \quad (D-51)$$

$$\lambda = \frac{-R1}{R1^*} \sigma \quad (D-52)$$

$$\lambda = -\frac{1}{2} \frac{R1}{[-R1^2 + R1^{*2}]} \quad (D-53)$$

Recall,

$$\alpha = -\sigma \quad (D-54)$$

$$\beta = -\lambda \quad (D-55)$$

The complete solution set is thus

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{2} \frac{1}{[-R1^2 + R1^{*2}]} \begin{bmatrix} -R1^* \\ R1 \\ -R1 \\ R1^* \end{bmatrix} \quad (D-56)$$

Note that

$$-R1^2 + R1*^2 = -j \left[4\xi \sqrt{1-\xi^2} \right] \quad (D-57)$$

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{2} \left[\frac{1}{-j \left[4\xi \sqrt{1-\xi^2} \right]} \right] \begin{bmatrix} -\sqrt{1-\xi^2} + j\xi \\ +\sqrt{1-\xi^2} + j\xi \\ -\sqrt{1-\xi^2} - j\xi \\ +\sqrt{1-\xi^2} - j\xi \end{bmatrix} \quad (D-58)$$

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{j}{\left[8\xi \sqrt{1-\xi^2} \right]} \begin{bmatrix} -\sqrt{1-\xi^2} + j\xi \\ +\sqrt{1-\xi^2} + j\xi \\ -\sqrt{1-\xi^2} - j\xi \\ +\sqrt{1-\xi^2} - j\xi \end{bmatrix} \quad (D-59)$$

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{\left[8\xi \sqrt{1-\xi^2} \right]} \begin{bmatrix} -\xi - j\sqrt{1-\xi^2} \\ -\xi + j\sqrt{1-\xi^2} \\ +\xi - j\sqrt{1-\xi^2} \\ +\xi + j\sqrt{1-\xi^2} \end{bmatrix} \quad (D-60)$$

Let

$$\psi = \frac{\xi}{\sqrt{1-\xi^2}} \quad (D-61)$$

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{8\xi} \begin{bmatrix} -\psi - j \\ -\psi + j \\ +\psi - j \\ +\psi + j \end{bmatrix} \quad (\text{D-62})$$

Recall

$$R1 = +j\xi + \sqrt{1 - \xi^2} \quad (\text{D-63})$$

$$\begin{aligned} \left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] = & + \frac{-\psi - j}{\left(\rho - \sqrt{1 - \xi^2} - j\xi \right)} \left[\frac{1}{8\xi} \right] \\ & + \frac{-\psi + j}{\left(\rho - \sqrt{1 - \xi^2} + j\xi \right)} \left[\frac{1}{8\xi} \right] \\ & + \frac{+\psi - j}{\left(\rho + \sqrt{1 - \xi^2} - j\xi \right)} \left[\frac{1}{8\xi} \right] \\ & + \frac{+\psi + j}{\left(\rho + \sqrt{1 - \xi^2} + j\xi \right)} \left[\frac{1}{8\xi} \right] \end{aligned} \quad (\text{D-64})$$

APPENDIX E

Example 5

A transfer function times its complex conjugate is

$$H(\rho)H^*(\rho) = \left[\frac{1 + j2\xi\rho}{(1 - \rho^2) + j2\xi\rho} \right] \left[\frac{1 - j2\xi\rho}{(1 - \rho^2) - j2\xi\rho} \right] \quad (\text{E-1})$$

Rearrange equation (E-1) as

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2\rho^2}{\rho^2 - j2\xi\rho - 1} \right] \left[\frac{1}{\rho^2 + j2\xi\rho - 1} \right] \quad (\text{E-2})$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \quad (\text{E-3})$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad (\text{E-4})$$

$$R1, R2 = j\xi \pm \sqrt{1 - \xi^2} \quad (\text{E-5})$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j2\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \quad (\text{E-6})$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \quad (\text{E-7})$$

$$R3, R4 = -j\xi \pm \sqrt{1 - \xi^2} \quad (\text{E-8})$$

Summary,

$$R1 = +j\xi + \sqrt{1 - \xi^2} \quad (E-9)$$

$$R2 = +j\xi - \sqrt{1 - \xi^2} \quad (E-10)$$

$$R3 = -j\xi + \sqrt{1 - \xi^2} \quad (E-11)$$

$$R4 = -j\xi - \sqrt{1 - \xi^2} \quad (E-12)$$

Note

$$R2 = -R1^* \quad (E-13)$$

$$R3 = R1^* \quad (E-14)$$

$$R4 = -R1^* \quad (E-15)$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{\left(\rho - j\xi - \sqrt{1 - \xi^2}\right)\left(\rho - j\xi + \sqrt{1 - \xi^2}\right)\left(\rho + j\xi - \sqrt{1 - \xi^2}\right)\left(\rho + j\xi + \sqrt{1 - \xi^2}\right)} \right] \quad (E-16)$$

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)} \right] \quad (E-17)$$

By substitution,

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] \quad (E-18)$$

Expand into partial fractions.

$$\left[\frac{1 + 4\xi^2 \rho^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] =$$

$$+ \frac{\alpha}{(\rho - R1)}$$

$$+ \frac{\beta}{(\rho - R1^*)}$$

$$+ \frac{\lambda}{(\rho + R1^*)}$$

$$+ \frac{\sigma}{(\rho + R1)}$$

(E-19)

Multiply through by the denominator on the left-hand side of equation (E-19).

$$\left[1 + 4\xi^2 \rho^2 \right] =$$

$$+ \frac{\alpha}{(\rho - R1)} (\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)$$

$$+ \frac{\beta}{(\rho - R1^*)} (\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)$$

$$+ \frac{\lambda}{(\rho + R1^*)} (\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)$$

$$+ \frac{\sigma}{(\rho + R1)} (\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)$$

(E-20)

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + \alpha (\rho - R1^*)(\rho + R1^*)(\rho + R1) \\
& + \beta (\rho - R1)(\rho + R1^*)(\rho + R1) \\
& + \lambda (\rho - R1)(\rho - R1^*)(\rho + R1) \\
& + \sigma (\rho - R1)(\rho - R1^*)(\rho + R1^*)
\end{aligned}
\tag{E-21}$$

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + \alpha \left(\rho^2 - R1^{*2}\right)(\rho + R1) \\
& + \beta \left(\rho^2 + (-R1 + R1^*)\rho - R1R1^*\right)(\rho + R1) \\
& + \lambda \left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1) \\
& + \sigma \left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1^*)
\end{aligned}
\tag{E-22}$$

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + \alpha \left(\rho^3 + R1\rho^2 - R1^{*2}\rho - R1R1^{*2}\right) \\
& + \beta \left(\rho^3 + (R1 - R1 + R1^*)\rho^2 + (-R1R1^* - R1^2 + R1R1^*)\rho - R1^2R1^*\right) \\
& + \lambda \left(\rho^3 + (R1 - R1 - R1^*)\rho^2 + (R1R1^* - R1^2 - R1R1^*)\rho + R1^2R1^*\right) \\
& + \sigma \left(\rho^3 + (R1^* - R1 - R1^*)\rho^2 + (-R1R1^* - R1R1^* - R1^{*2})\rho + R1R1^{*2}\right)
\end{aligned}
\tag{E-23}$$

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + \alpha \left(\rho^3 + R1\rho^2 - R1^{*2} \rho - R1R1^{*2} \right) \\
& + \beta \left(\rho^3 + R1^*\rho^2 - R1^2 \rho - R1^2R1^* \right) \\
& + \lambda \left(\rho^3 - R1^*\rho^2 - R1^2 \rho + R1^2R1^* \right) \\
& + \sigma \left(\rho^3 - R1\rho^2 - R1^{*2} \rho + R1R1^{*2} \right)
\end{aligned}
\tag{E-24}$$

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + [\alpha + \beta + \lambda + \sigma] \rho^3 \\
& + [R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma] \rho^2 \\
& + [-R1^{*2} \alpha - R1^2 \beta - R1^2 \lambda - R1^{*2} \sigma] \rho \\
& + [-R1R1^{*2} \alpha - R1^2R1^*\beta + R1^2R1^*\lambda + R1R1^{*2} \sigma]
\end{aligned}
\tag{E-25}$$

$$\begin{aligned}
\left[1 + 4\xi^2 \rho^2\right] = & \\
& + [\alpha + \beta + \lambda + \sigma] \rho^3 \\
& + [R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma] \rho^2 \\
& + [-R1^{*2} \alpha - R1^2 \beta - R1^2 \lambda - R1^{*2} \sigma] \rho \\
& + [-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma] R1R1^*
\end{aligned}
\tag{E-26}$$

Equation (E-26) can be broken up into four separate equations.

$$\alpha + \beta + \lambda + \sigma = 0 \quad (\text{E-27})$$

$$[R1\alpha + R1*\beta - R1*\lambda - R1\sigma] = 4\xi^2 \quad (\text{E-28})$$

$$[-R1*^2\alpha - R1^2\beta - R1^2\lambda - R1*^2\sigma] = 0 \quad (\text{E-29})$$

$$[-R1*\alpha - R1\beta + R1\lambda + R1*\sigma]R1R1* = 1 \quad (\text{E-30})$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ -R1*^2 & -R1^2 & -R1^2 & -R1*^2 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1/(R1R1*) \end{bmatrix} \quad (\text{E-31})$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \quad (\text{E-32})$$

$$R1R1* = \left[+j\xi + \sqrt{1-\xi^2} \right] \left[-j\xi + \sqrt{1-\xi^2} \right] \quad (\text{E-33})$$

$$R1R1* = 1 \quad (\text{E-34})$$

By substitution,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ -R1*^2 & -R1^2 & -R1^2 & -R1*^2 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \quad (\text{E-35})$$

Multiply the first row by $R1^*2$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & -R1^2 + R1^*2 & -R1^2 + R1^*2 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-36)

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-37)

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-38)

Multiply the first row by -R1 and add to the second row. Also multiply the first row by $R1^*$ and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1^* & -R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix}$$

(E-39)

Multiply the third row by $-R1^*$ and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \quad (\text{E-40})$$

The first row equation yields

$$\alpha = -\sigma \quad (\text{E-41})$$

The third row equation yields

$$\lambda = -\beta \quad (\text{E-42})$$

Equation (E-40) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 4\xi^2 \\ 1 \end{bmatrix} \quad (\text{E-43})$$

Complete the solution using Cramer's rule.

$$\det \text{er min ant} \begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = 4 \begin{bmatrix} R1^2 - R1^{*2} \end{bmatrix} \quad (\text{E-44})$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \quad (\text{E-45})$$

$$R1^2 = \left[+j\xi + \sqrt{1-\xi^2} \right] \left[+j\xi + \sqrt{1-\xi^2} \right] \quad (\text{E-46})$$

$$R1^2 = -\xi^2 + (1-\xi^2) + j \left[2\xi \sqrt{1-\xi^2} \right] \quad (\text{E-47})$$

$$R1^2 = (1-2\xi^2) + j \left[2\xi \sqrt{1-\xi^2} \right] \quad (\text{E-48})$$

$$R1^* = -j\xi + \sqrt{1-\xi^2} \quad (\text{E-49})$$

$$R1^{*2} = \left[-j\xi + \sqrt{1-\xi^2} \right] \left[-j\xi + \sqrt{1-\xi^2} \right] \quad (E-50)$$

$$R1^{*2} = -\xi^2 + (1-\xi^2) - j \left[2\xi \sqrt{1-\xi^2} \right] \quad (E-51)$$

$$R1^{*2} = (1-2\xi^2) - j \left[2\xi \sqrt{1-\xi^2} \right] \quad (E-52)$$

Thus,

$$R1^2 - R1^{*2} = j \left[4\xi \sqrt{1-\xi^2} \right] \quad (E-53)$$

$$4 \left[R1^2 - R1^{*2} \right] = j \left[16\xi \sqrt{1-\xi^2} \right] \quad (E-54)$$

$$\text{determinant} \begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = j \left[16\xi \sqrt{1-\xi^2} \right] \quad (E-55)$$

$$\lambda = \frac{1}{j \left[16\xi \sqrt{1-\xi^2} \right]} \text{determinant} \begin{bmatrix} 4\xi^2 & -2R1 \\ 1 & +2R1^* \end{bmatrix} \quad (E-56)$$

$$\lambda = \frac{(4\xi^2)(2R1^*) + 2R1}{j \left[16\xi \sqrt{1-\xi^2} \right]} \quad (E-57)$$

$$\lambda = \frac{(4\xi^2)(R1^*) + R1}{j \left[8\xi \sqrt{1-\xi^2} \right]} \quad (E-58)$$

Recall,

$$R1 = +j\xi + \sqrt{1-\xi^2} \quad (E-59)$$

$$\lambda = \frac{(4\xi^2)(-j\xi + \sqrt{1-\xi^2}) + j\xi + \sqrt{1-\xi^2}}{j[8\xi\sqrt{1-\xi^2}]} \quad (\text{E-60})$$

$$\lambda = \frac{(1+4\xi^2)(\sqrt{1-\xi^2}) + j\xi(1-4\xi^2)}{j[8\xi\sqrt{1-\xi^2}]} \quad (\text{E-61})$$

$$\lambda = \frac{+\xi(1-4\xi^2) - j(1+4\xi^2)(\sqrt{1-\xi^2})}{[8\xi\sqrt{1-\xi^2}]} \quad (\text{E-62})$$

$$\sigma = \frac{1}{j[16\xi\sqrt{1-\xi^2}]} \det \text{er min ant} \begin{bmatrix} -2R1^* & 4\xi^2 \\ 2R1 & 1 \end{bmatrix} \quad (\text{E-63})$$

$$\sigma = \frac{[-2R1^* - (4\xi^2)](2R1)}{j[16\xi\sqrt{1-\xi^2}]} \quad (\text{E-64})$$

$$\sigma = \frac{[-R1^* - (4\xi^2)](R1)}{j[8\xi\sqrt{1-\xi^2}]} \quad (\text{E-65})$$

$$R1 = +j\xi + \sqrt{1-\xi^2} \quad (\text{E-66})$$

$$\sigma = \frac{\left[-(-j\xi + \sqrt{1-\xi^2}) - (4\xi^2)(+j\xi + \sqrt{1-\xi^2}) \right]}{j[8\xi\sqrt{1-\xi^2}]} \quad (\text{E-67})$$

$$\sigma = \frac{\left[+j\xi - \sqrt{1-\xi^2} - j4\xi^3 - 4\xi^2\sqrt{1-\xi^2} \right]}{j \left[8\xi\sqrt{1-\xi^2} \right]} \quad (\text{E-68})$$

$$\sigma = \frac{\left[-\left(1+4\xi^2\right)\sqrt{1-\xi^2} + j\xi\left(1-4\xi^2\right) \right]}{j \left[8\xi\sqrt{1-\xi^2} \right]} \quad (\text{E-69})$$

$$\sigma = \frac{\left[+\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \right]}{j \left[8\xi\sqrt{1-\xi^2} \right]} \quad (\text{E-70})$$

Recall,

$$\alpha = -\sigma \quad (\text{E-71})$$

$$\lambda = -\beta \quad (\text{E-72})$$

The complete solution set is thus

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{8\xi\sqrt{1-\xi^2}} \begin{bmatrix} -\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ -\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \end{bmatrix} \quad (\text{E-73})$$

The partial fraction expansion is thus

$$H(\rho)H^*(\rho) =$$

$$\begin{aligned} &+ \frac{\alpha}{(\rho - R_1)} \\ &+ \frac{\beta}{(\rho - R_1^*)} \\ &+ \frac{\lambda}{(\rho + R_1^*)} \\ &+ \frac{\sigma}{(\rho + R_1)} \end{aligned}$$

(E-74)

APPENDIX F

$$\frac{s^2}{(s+a)(s+b)} = \alpha + \frac{\beta}{s+a} + \frac{\lambda}{s+b} \quad (\text{F-1})$$

$$s^2 = \alpha(s+a)(s+b) + \frac{\beta}{(s+a)}(s+a)(s+b) + \frac{\lambda}{(s+b)}(s+a)(s+b) \quad (\text{F-2})$$

$$s^2 = \alpha(s^2 + (a+b)s + ab) + \beta(s+b) + \lambda(s+a) \quad (\text{F-3})$$

$$s^2 = \alpha s^2 + \alpha(a+b)s + \alpha ab + \beta s + \beta b + \lambda s + \lambda a \quad (\text{F-4})$$

$$s^2 = \alpha s^2 + [\alpha(a+b) + \beta + \lambda]s + \alpha ab + \beta b + \lambda a \quad (\text{F-5})$$

Equation (F-5) yields three individual equations.

$$\alpha = 1 \quad (\text{F-6})$$

$$[\alpha(a+b) + \beta + \lambda] = 0 \quad (\text{F-7})$$

$$\alpha ab + \beta b + \lambda a = 0 \quad (\text{F-8})$$

$$[(a+b) + \beta + \lambda] = 0 \quad (\text{F-9})$$

$$ab + \beta b + \lambda a = 0 \quad (\text{F-10})$$

$$\beta + \lambda = -(a+b) \quad (\text{F-11})$$

$$\beta b + \lambda a = -ab \quad (\text{F-12})$$

The equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} -(a+b) \\ -ab \end{bmatrix} \quad (\text{F-13})$$

Apply Cramer's rule.

$$\beta = \frac{1}{a-b} \det \begin{bmatrix} -(a+b) & 1 \\ -ab & a \end{bmatrix} \begin{bmatrix} \beta \\ \lambda \end{bmatrix} \quad (\text{F-14})$$

$$\beta = \frac{-a^2 - ab + ab}{a-b} \quad (\text{F-15})$$

$$\beta = \frac{-a^2}{a-b} \quad (\text{F-16})$$

$$\lambda = \frac{1}{a-b} \det \begin{bmatrix} 1 & -(a+b) \\ b & -ab \end{bmatrix} \quad (\text{F-17})$$

$$\lambda = \frac{-ab + b(a+b)}{a-b} \quad (\text{F-18})$$

$$\lambda = \frac{-ab + ab + b^2}{a-b} \quad (\text{F-19})$$

$$\lambda = \frac{b^2}{a-b} \quad (\text{F-20})$$

The partial fraction expansion is thus

$$\frac{s^2}{(s+a)(s+b)} = 1 + \left[\frac{-a^2}{a-b} \right] \left[\frac{1}{s+a} \right] + \left[\frac{b^2}{a-b} \right] \left[\frac{\lambda}{s+b} \right], \quad a \neq b \quad (\text{F-21})$$

APPENDIX G

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{ 1 + \left[\frac{-a^2}{a-b} \right] \left[\frac{1}{s+a} \right] + \left[\frac{b^2}{a-b} \right] \left[\frac{1}{s+b} \right] \right\} \left\{ 1 + \left[\frac{-c^2}{c-d} \right] \left[\frac{1}{s+c} \right] + \left[\frac{d^2}{c-d} \right] \left[\frac{1}{s+d} \right] \right\}$$

(G-1)

Let

$$A = \left[\begin{array}{c} -a^2 \\ a-b \end{array} \right]$$

(G-2)

$$B = \left[\begin{array}{c} b^2 \\ a-b \end{array} \right]$$

(G-3)

$$C = \left[\begin{array}{c} -c^2 \\ c-d \end{array} \right]$$

(G-4)

$$D = \left[\begin{array}{c} d^2 \\ c-d \end{array} \right]$$

(G-5)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{ 1 + A \left[\frac{1}{s+a} \right] + B \left[\frac{1}{s+b} \right] \right\} \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\}$$

(G-6)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\}$$

$$+ A \left[\frac{1}{s+a} \right] \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\}$$

$$+ B \left[\frac{1}{s+b} \right] \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\}$$

(G-7)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)}$$

$$= \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\}$$

$$+ A \left\{ \left[\frac{1}{s+a} \right] + C \left[\frac{1}{s+a} \right] \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+a} \right] \left[\frac{1}{s+d} \right] \right\}$$

$$+ B \left\{ \left[\frac{1}{s+b} \right] + C \left[\frac{1}{s+b} \right] \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+b} \right] \left[\frac{1}{s+d} \right] \right\}$$

(G-8)

Recall

$$\frac{1}{(s+a)(s+b)} = \left[\frac{1}{a-b} \right] \left[\frac{-1}{s+a} + \frac{1}{s+b} \right], \quad a \neq b$$

(G-9)

$$\begin{aligned}
& \frac{s^4}{(s+a)(s+b)(s+c)(s+d)} \\
&= \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\} \\
&+ A \left\{ \left[\frac{1}{s+a} \right] + C \left[\frac{1}{a-c} \right] \left[\frac{-1}{s+a} + \frac{1}{s+c} \right] + D \left[\frac{1}{a-d} \right] \left[\frac{-1}{s+a} + \frac{1}{s+d} \right] \right\} \\
&+ B \left\{ \left[\frac{1}{s+b} \right] + C \left[\frac{1}{b-c} \right] \left[\frac{-1}{s+b} + \frac{1}{s+c} \right] + D \left[\frac{1}{b-d} \right] \left[\frac{-1}{s+b} + \frac{1}{s+d} \right] \right\}
\end{aligned} \tag{G-10}$$

$$\begin{aligned}
& \frac{s^4}{(s+a)(s+b)(s+c)(s+d)} \\
&= \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\} \\
&+ A \left\{ \left[\frac{1}{s+a} \right] + C \left[\frac{1}{a-c} \right] \left[\frac{-1}{s+a} + \frac{1}{s+c} \right] + D \left[\frac{1}{a-d} \right] \left[\frac{-1}{s+a} + \frac{1}{s+d} \right] \right\} \\
&+ B \left\{ \left[\frac{1}{s+b} \right] + C \left[\frac{1}{b-c} \right] \left[\frac{-1}{s+b} + \frac{1}{s+c} \right] + D \left[\frac{1}{b-d} \right] \left[\frac{-1}{s+b} + \frac{1}{s+d} \right] \right\}
\end{aligned} \tag{G-11}$$

$$\begin{aligned}
& \frac{s^4}{(s+a)(s+b)(s+c)(s+d)} \\
&= \left\{ 1 + C \left[\frac{1}{s+c} \right] + D \left[\frac{1}{s+d} \right] \right\} \\
&+ \left\{ \left[\frac{A}{s+a} \right] + \left[\frac{AC}{a-c} \right] \left[\frac{-1}{s+a} \right] + \left[\frac{AC}{a-c} \right] \left[\frac{1}{s+c} \right] + \left[\frac{AD}{a-d} \right] \left[\frac{-1}{s+a} \right] + \left[\frac{AD}{a-d} \right] \left[\frac{1}{s+d} \right] \right\} \\
&+ \left\{ \left[\frac{B}{s+b} \right] + \left[\frac{BC}{b-c} \right] \left[\frac{-1}{s+b} \right] + \left[\frac{BC}{b-c} \right] \left[\frac{1}{s+c} \right] + \left[\frac{BD}{b-d} \right] \left[\frac{-1}{s+b} \right] + \left[\frac{BD}{b-d} \right] \left[\frac{1}{s+d} \right] \right\}
\end{aligned} \tag{G-12}$$

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$1 + A \left[1 - \left[\frac{C}{a-c} \right] - \left[\frac{D}{a-d} \right] \right] \left[\frac{1}{s+a} \right]$$

$$+ B \left[1 - \left[\frac{C}{b-c} \right] - \left[\frac{D}{b-d} \right] \right] \left[\frac{1}{s+b} \right]$$

$$+ C \left[1 + \left[\frac{A}{a-c} \right] + \left[\frac{B}{b-c} \right] \right] \left[\frac{1}{s+c} \right]$$

$$+ D \left[1 + \left[\frac{A}{a-d} \right] + \left[\frac{B}{b-d} \right] \right] \left[\frac{1}{s+d} \right]$$

(G-13)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$1 + \left[\frac{-a^2}{a-b} \right] \left[1 - \left[\frac{-c^2}{(a-c)(c-d)} \right] - \left[\frac{d^2}{(a-d)(c-d)} \right] \right] \left[\frac{1}{s+a} \right]$$

$$+ \left[\frac{b^2}{a-b} \right] \left[1 - \left[\frac{-c^2}{(b-c)(c-d)} \right] - \left[\frac{d^2}{(c-d)(b-d)} \right] \right] \left[\frac{1}{s+b} \right]$$

$$+ \left[\frac{-c^2}{c-d} \right] \left[1 + \left[\frac{-a^2}{(a-b)(a-c)} \right] + \left[\frac{b^2}{(a-b)(b-c)} \right] \right] \left[\frac{1}{s+c} \right]$$

$$+ \left[\frac{d^2}{c-d} \right] \left[1 + \left[\frac{-a^2}{(a-b)(a-d)} \right] + \left[\frac{b^2}{(a-b)(b-d)} \right] \right] \left[\frac{1}{s+d} \right]$$

(G-14)

$$\frac{s^4}{(s+a)(s+b)(s+c)(s+d)} =$$

$$1 + \left[\frac{-a^2}{a-b} \right] \left[1 + \left[\frac{c^2}{(a-c)(c-d)} \right] - \left[\frac{d^2}{(a-d)(c-d)} \right] \right] \left[\frac{1}{s+a} \right]$$

$$+ \left[\frac{b^2}{a-b} \right] \left[1 + \left[\frac{c^2}{(b-c)(c-d)} \right] - \left[\frac{d^2}{(c-d)(b-d)} \right] \right] \left[\frac{1}{s+b} \right]$$

$$+ \left[\frac{-c^2}{c-d} \right] \left[1 - \left[\frac{a^2}{(a-b)(a-c)} \right] + \left[\frac{b^2}{(a-b)(b-c)} \right] \right] \left[\frac{1}{s+c} \right]$$

$$+ \left[\frac{d^2}{c-d} \right] \left[1 - \left[\frac{a^2}{(a-b)(a-d)} \right] + \left[\frac{b^2}{(a-b)(b-d)} \right] \right] \left[\frac{1}{s+d} \right]$$

(G-15)