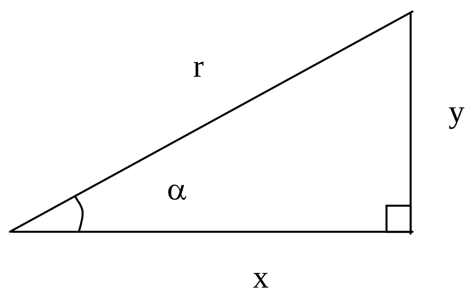
COMPLEX FUNCTIONS AND TRIGONOMETRIC IDENTITIES
Revision E

By Tom Irvine
Email: tomirvine@aol.com

September 14, 2006

Trigonometric Functions of Angle α



$$\sin(\alpha) = \frac{y}{r} \qquad \cos(\alpha) = \frac{x}{r}$$

$$\tan(\alpha) = \frac{y}{x} \qquad \cot(\alpha) = \frac{x}{y}$$

$$\sec(\alpha) = \frac{r}{x} \qquad \csc(\alpha) = \frac{r}{y} \tag{1}$$

Trigonometric Expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{3}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{4}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{5}$$

<u>Exponential Expansion</u>

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6}$$

<u>Trigonometric Identities</u>

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha) \tag{7}$$

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha) \tag{8}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) \tag{9}$$

$$\sin(\alpha)\sin(\beta) = -\frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) \tag{10}$$

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta) \tag{11}$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) \tag{12}$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{13}$$

$$\sec^2(\alpha) - \tan^2(\alpha) = 1 \tag{14}$$

$$\csc^2(\alpha) - \cot^2(\alpha) = 1 \tag{15}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{16}$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \tag{17}$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{18}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{19}$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)} \tag{20}$$

$$A\sin(\alpha t) + B\cos(\alpha t) = \sqrt{A^2 + B^2}\,\left[\sin(\alpha t + \theta)\right]$$

where $\theta = \arctan\left(\frac{B}{A}\right)$

(21)

<u>Euler's Equation</u>

$$\exp(\pm j\alpha) = \cos(\alpha) \pm j\sin(\alpha) \tag{22}$$

$$\sin(\alpha) = \frac{\exp(j\alpha) - \exp(-j\alpha)}{2j} \tag{23}$$

$$\cos(\alpha) = \frac{\exp(j\alpha) + \exp(-j\alpha)}{2} \tag{24}$$

<u>Hyperbolic Functions</u>

$$\sinh(\alpha) = \frac{\exp(\alpha) - \exp(-\alpha)}{2} \tag{25a}$$

$$\cosh(\alpha) = \frac{\exp(\alpha) + \exp(-\alpha)}{2} \tag{25b}$$

$$\cosh^2(\alpha) - \sinh^2(\alpha) = 1 \tag{26}$$

Derivatives

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} \tag{27}$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx} \tag{28}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx} \tag{29}$$

$$\frac{d}{dx}(\cot u) = -\csc u \frac{du}{dx} \tag{30}$$

$$\frac{d}{dx}(\sec u) = \tan u \; \sec u \frac{du}{dx} \tag{31}$$

$$\frac{d}{dx}(\csc u) = -\csc u \; \cot u \frac{du}{dx} \tag{32}$$

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx} \tag{33}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx} \tag{34}$$

$$\frac{d}{dx}(\tanh u) = \frac{1}{\cosh^2 u} \frac{du}{dx} \tag{35}$$

Natural Logarithm of a Complex Number

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2} \; \exp(j\theta)\right], \quad \theta = \arctan\left(\frac{b}{a}\right) \tag{36}$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + \ln[\exp(j\theta)] \quad (37)$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + j\theta \quad (38)$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + j\arctan\left(\frac{b}{a}\right) \quad (39)$$

APPENDIX A

<u>The Square Root of a Complex Number.</u>

Consider

$$x^2 = a + jb \quad \text{(A-1)}$$

Thus

$$x = \pm\sqrt{a + jb} \quad \text{(A-2)}$$

where a and b are real coefficients.

Solve for x.

Let

$$x_1 = (c + jd) \quad \text{(A-3a)}$$

$$x_2 = -(c + jd) \quad \text{(A-3b)}$$

where c and d are real coefficients.

Substitute equation (A-3a) into (A-1).

$$(c + jd)^2 = (a + jb) \quad \text{(A-4)}$$

$$(c + jd)(c + jd) = (a + jb) \tag{A-5}$$

$$c^2 - d^2 + j(2cd) = a + jb \tag{A-6}$$

Equation (A-6) implies two equations. The first is

$$c^2 - d^2 = a \tag{A-7}$$

The second implied equation is

$$2cd = b \tag{A-8}$$

Solve for d using equation (A-8).

$$d = \frac{b}{2c} \tag{A-9}$$

Substitute equation (A-9) into (A-8).

$$c^2 - \left(\frac{b}{2c}\right)^2 = a \tag{A-10}$$

$$c^2 - a - \left(\frac{b}{2c}\right)^2 = 0 \tag{A-11}$$

Multiply through by $4c^2$.

$$4c^4 - 4ac^2 - b^2 = 0 \tag{A-12}$$

Apply the quadratic formula.

$$c^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} \tag{A-13}$$

$$c^2 = \frac{4a \pm 4\sqrt{a^2 + b^2}}{8} \tag{A-14}$$

$$c^2 = \frac{a \pm \sqrt{a^2 + b^2}}{2} \qquad \text{(A-15)}$$

$$c = \sqrt{\frac{a \pm \sqrt{a^2 + b^2}}{2}} \qquad \text{(A-16)}$$

Require c to be real.

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \qquad \text{(A-17)}$$

Substitute equation (A-17) into (A-9).

$$d = \left\{ \frac{b}{2\left[\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}\right]} \right\} \qquad \text{(A-18)}$$

$$d = \left\{ \frac{b}{\sqrt{\frac{4\left(a + \sqrt{a^2 + b^2}\right)}{2}}} \right\} \qquad \text{(A-19)}$$

$$d = \left\{ \frac{b}{\sqrt{2\left(a + \sqrt{a^2 + b^2} \right)}} \right\}$$

(A-20)

Substitute equations (A-20) and (A-17) into (A-3a).

$$x_1 = \left\{ \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \right\} + j \left\{ \frac{b}{\sqrt{2\left(a + \sqrt{a^2 + b^2} \right)}} \right\} \quad \text{(A-21a)}$$

Substitute equations (A-20) and (A-17) into (A-3b).

$$x_2 = -\left\{ \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \right\} - j \left\{ \frac{b}{\sqrt{2\left(a + \sqrt{a^2 + b^2} \right)}} \right\} \quad \text{(A-21b)}$$

Note that equations (A-21a) and (A-21b) cannot be used for the special case:

$a < 0$ and $b = 0$.

For this special case, the roots are

$$x = \pm\, j \sqrt{a} \quad \text{(A-21c)}$$

Example

$$x^2 = 2 + j\,7 \quad \text{(A-22)}$$

$$x = \pm \sqrt{2 + j\,7} \quad \text{(A-23)}$$

Solve for x. Use equation (A-21a).

$$a = 2 \tag{A-24}$$

$$b = 7 \tag{A-25}$$

$$x_1 = 2.154 + j\,1.625 \tag{A-26}$$

$$x_2 = -2.154 - j\,1.625 \tag{A-27}$$

APPENDIX B

<u>Arbitrary Root of a Complex Number</u>

Let

$$x^n = [a + jb] \tag{B-1a}$$

$$x = [a + jb]^{1/n} \tag{B-1b}$$

The coefficients a and b are real numbers. The denominator of the exponent n is also real.

Take the natural logarithm.

$$\ln x = \ln\left\{[a + jb]^{1/n}\right\} \tag{B-2}$$

$$\ln x = \frac{1}{n}\ln[a + jb] \tag{B-3}$$

$$\ln x = \frac{1}{n}\ln\left[\sqrt{a^2 + b^2}\,\exp\left(j\arctan\frac{b}{a}\right)\right] \tag{B-4}$$

$$\ln x = \frac{1}{n}\ln\left[\sqrt{a^2 + b^2}\right] + \frac{1}{n}\ln\left[\exp\left(j\arctan\frac{b}{a}\right)\right] \tag{B-5}$$

$$\ln x = \ln\left[\left(a^2 + b^2\right)^{\frac{1}{2n}}\right] + j\,\frac{1}{n}\arctan\frac{b}{a} \tag{B-6}$$

$$\exp\{\ln x\} = \exp\left\{\ln\left[\left(a^2 + b^2\right)^{\frac{1}{2n}}\right] + j\,\frac{1}{n}\arctan\frac{b}{a}\right\} \tag{B-7}$$

$$x = \exp\left\{\ln\left[\left(a^2 + b^2\right)^{\frac{1}{2n}}\right]\right\}\exp\left\{j\,\frac{1}{n}\arctan\frac{b}{a}\right\} \tag{B-8}$$

$$x = \left(a^2 + b^2\right)^{\frac{1}{2n}}\ \exp\left\{j\,\frac{1}{n}\arctan\frac{b}{a}\right\} \tag{B-9}$$

$$x = \left(a^2 + b^2\right)^{\frac{1}{2n}}\ \left\{\cos\left(\frac{1}{n}\arctan\frac{b}{a}\right) + j\sin\left(\frac{1}{n}\arctan\frac{b}{a}\right)\right\} \tag{B-10}$$

Note that equation (B-10) could be used for the special case of a square root.

APPENDIX C

Cube Root of a Complex Number

Consider

$$x^3 = a + jb \tag{C-1}$$

$$x = [a + jb]^{1/3} \tag{C-2}$$

Equation (C-1) has three roots. The method in Appendix B yields the following formula for one of the cube roots.

$$x_1 = \left(a^2 + b^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3}\arctan\frac{b}{a}\right) + j\sin\left(\frac{1}{3}\arctan\frac{b}{a}\right)\right\} \qquad \text{(C-3)}$$

Rearrange equation (C-1).

$$x^3 - a - jb = 0 \qquad \text{(C-4)}$$

Devise an equation for finding the other two roots.

$$x^3 - a - jb = (x - x_1)(x - x_2)(x - x_3) \qquad \text{(C-5)}$$

Expand the right-hand-side.

$$x^3 - a - jb = \left[x^2 - (x_1 + x_2)x + x_1 x_2\right](x - x_3) \qquad \text{(C-6)}$$

$$x^3 - a - jb = \left[x^2 - (x_1 + x_2)x + x_1 x_2\right](x) + \left[x^2 - (x_1 + x_2)x + x_1 x_2\right](-x_3) \qquad \text{(C-7)}$$

$$x^3 - a - jb = \left[x^3 - (x_1 + x_2)x^2 + x_1 x_2 x\right] - \left[x_3 x^2 - x_3(x_1 + x_2)x + x_1 x_2 x_3\right] \qquad \text{(C-8)}$$

$$x^3 - a - jb = \left[x^3 - (x_1 + x_2)x^2 + x_1 x_2 x\right] + \left[-x_3 x^2 + x_3(x_1 + x_2)x - x_1 x_2 x_3\right] \qquad \text{(C-9)}$$

$$x^3 - a - jb = \left[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1 x_2 + x_1 x_3 + x_2 x_3)x - x_1 x_2 x_3\right] \qquad \text{(C-10)}$$

Equation (C-10) implies three separate equations.

$$(x_1 + x_2 + x_3) = 0 \qquad \text{(C-11)}$$

$$(x_1 x_2 + x_1 x_3 + x_2 x_3) = 0 \qquad \text{(C-12)}$$

$$-x_1 x_2 x_3 = -a - jb \qquad \text{(C-13)}$$

Continue with equation (C-11).

$$x_2 = -x_1 - x_3 \qquad \text{(C-14)}$$

Substitute equation (C-14) into (C-12).

$$\left(x_1\left(-x_1 - x_3\right) + x_1 x_3 + \left(-x_1 - x_3\right)x_3\right) = 0 \tag{C-15}$$

$$-x_1^2 - x_1 x_3 + x_1 x_3 - x_1 x_3 - x_3^2 = 0 \tag{C-16}$$

$$-x_1^2 - x_1 x_3 - x_3^2 = 0 \tag{C-17}$$

$$x_3^2 + x_1 x_3 + x_1^2 = 0 \tag{C-18}$$

Use the quadratic formula.

$$x_3 = \frac{-x_1 \pm \sqrt{x_1^2 - 4x_1^2}}{2} \tag{C-19}$$

$$x_3 = \frac{-x_1 \pm \sqrt{-3x_1^2}}{2} \tag{C-20}$$

$$x_3 = \frac{-x_1 \pm x_1\sqrt{-3}}{2} \tag{C-21}$$

$$x_3 = x_1\left[\frac{-1 \pm j\sqrt{3}}{2}\right] \tag{C-22}$$

Choose

$$x_3 = x_1\left[\frac{-1 - j\sqrt{3}}{2}\right] \tag{C-23}$$

Recall equation (C-14).

$$x_2 = -x_1 - x_3 \tag{C-24}$$

$$x_2 = -x_1 - x_1\left[\frac{-1 - j\sqrt{3}}{2}\right] \tag{C-25}$$

$$x_2 = x_1\left\{-1-\left[\frac{-1-j\sqrt{3}}{2}\right]\right\} \quad \text{(C-26)}$$

$$x_2 = x_1\left\{-1+\left[\frac{1+j\sqrt{3}}{2}\right]\right\} \quad \text{(C-27)}$$

$$x_2 = x_1\left\{-\frac{2}{2}+\frac{1+j\sqrt{3}}{2}\right\} \quad \text{(C-28)}$$

$$x_2 = x_1\left[\frac{-1+j\sqrt{3}}{2}\right] \quad \text{(C-29)}$$

The roots x_2 and x_3 thus form a complex conjugate pair.

Summarize the roots.

$$x_1 = \left(a^2+b^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3}\arctan\frac{b}{a}\right) + j\sin\left(\frac{1}{3}\arctan\frac{b}{a}\right)\right\} \quad \text{(C-30)}$$

$$x_2 = x_1\left[\frac{-1+j\sqrt{3}}{2}\right] \quad \text{(C-31)}$$

$$x_3 = x_1\left[\frac{-1-j\sqrt{3}}{2}\right] \quad \text{(C-32)}$$

<u>Example</u>

Solve for x.

$$x^3 = 2 + j7 \quad \text{(C-33)}$$

$$x = [2 + j7]^{1/3} \quad \text{(C-34)}$$

$$a = 2 \quad \text{(C-35)}$$

$$b = 7 \quad \text{(C-36)}$$

$$n = 3 \quad \text{(C-37)}$$

There are three roots. The first root is

$$x_1 = \left(a^2 + b^2\right)^{\frac{1}{6}} \left\{\cos\left(\frac{1}{3}\arctan\frac{b}{a}\right) + j\sin\left(\frac{1}{3}\arctan\frac{b}{a}\right)\right\} \quad \text{(C-38)}$$

$$x_1 = 1.938\left[0.909 + j418\right] \quad \text{(C-39)}$$

$$x_1 = 1.761 + j\,0.809 \quad \text{(C-40)}$$

The second root is a coordinate transformation of the first root.

$$x_2 = x_1\left[\frac{-1 + j\sqrt{3}}{2}\right] \quad \text{(C-41)}$$

$$x_2 = \left[1.762 + j\,0.809\right]\left[\frac{-1 + j\sqrt{3}}{2}\right] \quad \text{(C-42)}$$

$$x_2 = -1.581 + j1.120 \quad \text{(C-43)}$$

$$x_3 = x_1\left[\frac{-1 - j\sqrt{3}}{2}\right] \quad \text{(C-44)}$$

$$x_3 = \left[1.761 + j\,0.809\right]\left[\frac{-1 - j\sqrt{3}}{2}\right] \quad \text{(C-45)}$$

$$x_3 = -0.180 - j1.930 \quad \text{(C-46)}$$

In summary, the cube roots of (2 + j 7) are

$$x_1 = 1.761 + j\,0.809 \tag{C-47}$$

$$x_2 = -1.581 + j1.120 \tag{C-48}$$

$$x_3 = -0.180 - j1.930 \tag{C-49}$$

APPENDIX D

<u>Derivation of the Quadratic Formula</u>

$$ax^2 + bx + c = 0 \tag{D-1}$$

$$x^2 + (b/a)x + (c/a) = 0 \tag{D-2}$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \tag{D-3}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \tag{D-4}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{D-5}$$

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{D-6}$$

$$\left(x+\frac{b}{2a}\right)=\frac{\pm\sqrt{b^2-4ac}}{2a} \qquad \text{(D-7)}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \qquad \text{(D-8)}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \text{(D-9)}$$