THE TIME-DOMAIN RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO AN IMPULSE FORCE
Revision B

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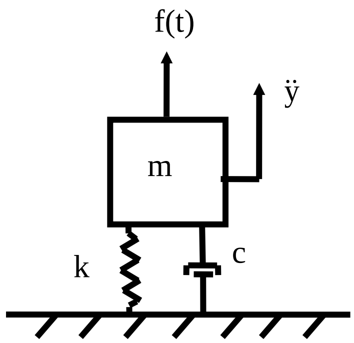
February 8, 2012

Consider a single-degree-of-freedom system.

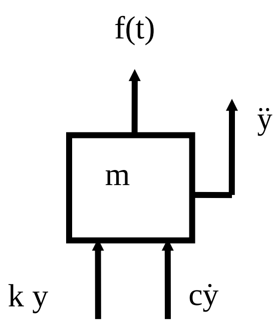


Where

m	=	mass
c	=	viscous damping coefficient
k	=	stiffness
y	=	displacement of the mass
f(t)	=	applied force

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n{}^2 \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)
ξ is the damping ratio

By substitution,

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}f(t) \tag{7}$$

Now apply an impulse force at t=0 via a pair of step functions.

$$f(t) = F[u(t) - u(t-\varepsilon)] \tag{8}$$

where ε is a very small time step.

The governing equation becomes

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{1}{m} F\left[u(t) - u(t-\varepsilon)\right] \tag{9}$$

Now consider that the system undergoes oscillation with $\xi < 1$.

Take the Laplace transform of each side.

$$L\left\{\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y\right\} = L\left\{\frac{1}{m} F\left[u(t) - u(t-\varepsilon)\right]\right\} \tag{10}$$

$$\begin{aligned}
&s^2 Y(s) - s y(0) - y'(0) \\
&+ 2\xi\omega_n s Y(s) - 2\xi\omega_n y(0) \\
&+ \omega_n^2 Y(s) = \qquad \frac{F}{m}\left[\frac{1}{s} - \frac{1}{s}\exp(-\varepsilon s)\right]
\end{aligned} \tag{11}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) - \left\{s + 2\xi\omega_n\right\} y(0) - y'(0) = \frac{F}{m}\left[\frac{1}{s} - \frac{1}{s}\exp(-\varepsilon s)\right] \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) = \frac{F}{m}\left[\frac{1}{s} - \frac{1}{s}\exp(-\varepsilon s)\right] + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \tag{13}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 - \left(\xi\omega_n\right)^2 + \omega_n^2 \tag{14}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2\left(1 - \xi^2\right) \tag{15}$$

Let

$$\omega_d = \omega_n \sqrt{1-\xi^2} \tag{16}$$

Substitute equation (16) into (15).

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s+\xi\omega_n)^2 + \omega_d^2 \tag{17}$$

$$\left\{(s+\xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{F}{m}\left[\frac{1}{s} - \frac{1}{s}\exp(-\varepsilon s)\right] + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \tag{18}$$

$Y(s) =$

$$\frac{F}{ms}\left\{\frac{1}{(s+\xi\omega_n)^2 + \omega_d^2}\right\} - \frac{F}{ms}\left\{\frac{\exp(-\varepsilon s)}{(s+\xi\omega_n)^2 + \omega_d^2}\right\}$$
$$+\left\{\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s+\xi\omega_n)^2 + \omega_d^2}\right\} y'(0)$$

(19)

Divide the right-hand-side of equation (19) into three parts.

$$Y(s) = Y_0(s) + Y_1(s) + Y_2(s) \tag{20}$$

where

$$Y_0(s) = \frac{F}{ms}\left\{\frac{1}{(s+\xi\omega_n)^2 + \omega_d^2}\right\} \tag{21}$$

$$Y_1(s) = -\frac{F}{m\,s}\left\{\frac{\exp(-\varepsilon s)}{(s+\xi\omega_n)^2+\omega_d^2}\right\} \tag{22}$$

$$Y_2(s) = \left\{\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\} y(0) + \left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\} y'(0) \tag{23}$$

Consider $Y_0(s)$ from equation (21).

$$Y_0(s) = \frac{F}{m\,s}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\} \tag{24}$$

Expand into partial fractions using Reference 1.

$$\begin{aligned}\left\{\frac{F}{m\,s}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\} &= \frac{F}{m\,\omega_n^2}\left\{\frac{1}{s}\right\} \\ &\quad -\left(\frac{F}{m\,\omega_n^2}\right)\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\} \\ &\quad -\left(\frac{F}{m\,\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\left\{\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right\}\end{aligned} \tag{25}$$

The inverse Laplace transform per Reference 1 is

$$y_0(t) = \frac{F}{m\omega_n^2}u(t) - \frac{F}{m\omega_n^2}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right],\ t \ge 0 \tag{26}$$

$$y_0(t) = \frac{F}{m\omega_n^2}\left\{u(t) - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\},\ t \ge 0 \tag{27}$$

Now consider the term

$$Y_1(s) = -\frac{F}{ms}\left\{\frac{\exp(-\varepsilon s)}{(s+\xi\omega_n)^2 + \omega_d^2}\right\} \tag{28}$$

Expand into partial fractions using Reference 1.

$$\begin{aligned}\left\{\frac{F}{ms}\right\}\left\{\frac{\exp(-\varepsilon s)}{s^2 + 2\xi\omega_n s + \omega_n^2}\right\} &= -\frac{F}{m\omega_n^2}\left\{\frac{1}{s}\right\}\exp(-\varepsilon s) \\ &+ \left(\frac{F}{m\omega_n^2}\right)\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}\right\}\exp(-\varepsilon s) \\ &+ \left(\frac{F}{m\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\left\{\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}\right\}\exp(-\varepsilon s)\end{aligned} \tag{29}$$

Take the inverse Laplace transform from Appendices A, B & C.

$$y_1(t) = -\frac{F}{m\omega_n^2}\,u(t-\varepsilon) + \left(\frac{F}{m\omega_n^2}\right)\exp[-\xi\omega_n(t-\varepsilon)]\cos\left[\,\omega_d\,(t-\varepsilon)\right]$$

$$+\left(\frac{F}{m\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\exp[-\xi\omega_n\,(t-\varepsilon)]\sin[\omega_d\,(t-\varepsilon)]$$

for $t \geq \varepsilon$

(30)

$$y_1(t) = \frac{F}{m\omega_n^2}\left\{-u(t-\varepsilon) + \exp[-\xi\omega_n(t-\varepsilon)]\left\{\cos\left[\,\omega_d\,(t-\varepsilon)\right] + \left(\frac{\xi\omega_n}{\omega_d}\right)\sin[\omega_d\,(t-\varepsilon)]\right\}\right\}$$

for $t \geq \varepsilon$

(31)

The inverse Laplace transform from Reference 2 for the natural response is

$$y_2(t) = [y(0)]\exp(-\xi\omega_n t)\cos(\omega_d t) + \left[\frac{\xi\omega_n y(0) + y'(0)}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \qquad (32)$$

$$y_2(t) = y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$
$$+ y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{33}$$

The final displacement solution is

$$y(t) = y_0(t) + y_1(t) + y_2(t) \tag{34}$$

$$y(t) = y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$
$$+ y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$
$$+ \frac{F}{m\,\omega_n^2}\left\{u(t) - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\},$$

$0 \le t < \varepsilon$

(35)

$$y(t) = y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+\, y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

$$+\, \frac{F}{m\,\omega_n^2}\left\{u(t) - \exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\}$$

$$+\, \frac{F}{m\,\omega_n^2}\left\{-u(t-\varepsilon) + \exp[-\xi\omega_n(t-\varepsilon)]\left\{\cos\left[\omega_d(t-\varepsilon)\right] + \frac{\xi\omega_n}{\omega_d}\sin[\omega_d(t-\varepsilon)]\right\}\right\},$$

$t \geq \varepsilon$

(36)

The response for a Dirac Delta impulse is derived in Appendix D.

References

1. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Revision G, Vibrationdata, 2011.
2. T. Irvine, Table of Laplace Transforms, Revision I, Vibrationdata, 2011.
3. T. Irvine, Free Vibration of a Single-Degree-of-Freedom System, Revision B, Vibrationdata, 2005.

APPENDIX A

$$L^{-1}\left\{\frac{F}{m\omega_n^2}\frac{1}{s}\exp(-\varepsilon s)\right\} = \frac{F}{m\omega_n^2}\ u(t-\varepsilon) \tag{A-1}$$

APPENDIX B

Consider the generic term

$$F(s) = \left\{\frac{s+\lambda}{(s+\alpha)^2+\beta^2}\right\}\exp(-\rho s) \tag{B-1}$$

Shift in the s-plane.

$$\hat{s} = s+\alpha \tag{B-2}$$

$$F(s) = \left\{\frac{\hat{s}-\alpha+\lambda}{\hat{s}^2+\beta^2}\right\}\exp\left[-\rho\left(\hat{s}-\alpha\right)\right] \tag{B-3}$$

$$F(s) = \left\{\frac{\hat{s}-\alpha+\lambda}{\hat{s}^2+\beta^2}\right\}\exp\left[-\rho\hat{s}\right]\exp\left[\rho\alpha\right] \tag{B-4}$$

$$F(s) = \left\{\frac{\hat{s}}{\hat{s}^2+\beta^2}\right\}\exp\left[-\rho\hat{s}\right]\exp\left[\rho\alpha\right] + \left\{\frac{-\alpha+\lambda}{\hat{s}^2+\beta^2}\right\}\exp\left[-\rho\hat{s}\right]\exp\left[\rho\alpha\right] \tag{B-5}$$

$$f(t) = \exp(\rho\,\alpha)\,\exp(-\alpha\,t)\cos[\beta(t-\rho)] + \left[\frac{-\alpha+\lambda}{\beta}\right]\exp(\rho\,\alpha)\exp(-\alpha\,t)\sin[\beta(t-\rho)],$$

$t > \rho$

(B-6)

$$f(t) = \exp(\rho\,\alpha)\,\exp(-\alpha\,t)\left\{\cos[\beta(t-\rho)] + \left[\frac{-\alpha+\lambda}{\beta}\right]\sin[\beta(t-\rho)]\right\} \tag{B-7}$$

$t > \rho$

$$f(t) = 0 \quad t < \rho \tag{B-8}$$

Note that the $\exp(-\alpha\,t)$ term accounts for the phase shift.

Recall the term

$$Y_{12}(s) = \left(\frac{F}{m\,\omega_n^2}\right)\left\{\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\}\exp(-\varepsilon\,s) \tag{B-9}$$

Compare the term in equation (B-9) to the generic term from (B-1).

$$F(s) = \left\{\frac{s+\lambda}{(s+\alpha)^2+\beta^2}\right\}\exp(-\rho\,s) \tag{B-10}$$

The inverse Laplace transform of equation (B-9) is

$$y_{12}(t) = \left(\frac{F}{m\,\omega_n^2} \right) \exp\left(\varepsilon\,\xi\omega_n\right) \exp\left(-\,\xi\omega_n\, t\right) \cos\left[\,\omega_d \left(t - \varepsilon\right)\right] \qquad \text{(B-11)}$$

$$y_{12}(t) = \left(\frac{F}{m\,\omega_n^2} \right) \exp\left[-\,\xi\omega_n \left(t - \varepsilon\right)\right] \cos\left[\,\omega_d \left(t - \varepsilon\right)\right] \qquad \text{(B-12)}$$

APPENDIX C

Consider the generic term

$$F(s) = \left\{ \frac{\lambda}{(s+\alpha)^2 + \beta^2} \right\} \exp(-\rho s) \qquad \text{(C-1)}$$

Shift in the s-plane.

$$\hat{s} = s + \alpha \qquad \text{(C-2)}$$

$$F(s) = \left\{ \frac{\lambda}{\hat{s}^2 + \beta^2} \right\} \exp\left[-\rho\left(\hat{s} - \alpha\right)\right] \qquad \text{(C-3)}$$

$$F(s) = \left\{ \frac{\lambda}{\hat{s}^2 + \beta^2} \right\} \exp\left[-\rho\hat{s}\right] \exp\left[\rho\alpha\right] \qquad \text{(C-4)}$$

$$f(t) = \left[\frac{\lambda}{\beta}\right] \exp(\rho\alpha) \exp(-\alpha t) \sin\left[\beta(t-\rho)\right]$$

$$t > \rho \qquad \text{(C-5)}$$

$$f(t) = \left[\frac{\lambda}{\beta}\right] \exp\left[-\alpha(t-\rho)\right] \sin\left[\beta(t-\rho)\right]$$

$$t > \rho \qquad \text{(C-6)}$$

$$f(t) = 0 \quad t < \rho \tag{C-7}$$

Note that the $\exp(-\alpha t)$ term accounts for the phase shift.

$$Y_{13}(s) = \left(\frac{F}{m\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\left\{\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right\}\exp(-\varepsilon s) \tag{C-8}$$

Compare with the generic term

$$F(s) = \left\{\frac{\lambda}{(s+\alpha)^2+\beta^2}\right\}\exp(-\rho s) \tag{C-9}$$

$$F(s) = \left\{\frac{\lambda}{\hat{s}^2+\beta^2}\right\}\exp[-\rho\hat{s}]\exp[\rho\alpha] \tag{C-10}$$

$$f(t) = \left[\frac{\lambda}{\beta}\right]\exp(\rho\alpha)\exp(-\alpha t)\sin[\beta(t-\rho)]$$

$$t > \rho \tag{C-11}$$

$$y_{13}(t) = -\left(\frac{F}{m\omega_n^2}\right)\left(\frac{\xi\omega_n}{\omega_d}\right)\exp[-\xi\omega_n(t-\varepsilon)]\sin[\omega_d(t-\varepsilon)]$$

$$t > \varepsilon \tag{C-12}$$

APPENDIX D

<u>Dirac Delta Impulse</u>

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{1}{m} f(t) \qquad \text{(D-1)}$$

Let $\hat{I}$ = total impulse

$$f(t) = \hat{I}\ \delta(t) \qquad \text{(D-2)}$$

.

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{\hat{I}}{m} \delta(t) \qquad \text{(D-3)}$$

$$\begin{aligned} &s^2 Y(s) - sy(0) - y'(0) \\ &+ 2\xi\omega_n sY(s) - 2\xi\omega_n y(0) \\ &+ \omega_n^2 Y(s) = \qquad \frac{\hat{I}}{m} \end{aligned} \qquad \text{(D-4)}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) - \left\{s + 2\xi\omega_n\right\} y(0) - y'(0) = \frac{\hat{I}}{m} \qquad \text{(D-5)}$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{\hat{I}}{m} + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \qquad \text{(D-6)}$$

$$Y(s) =$$

$$\frac{\hat{I}}{m}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

$$+\left\{\frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right\}y(0)+\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\}y'(0)$$

(D-7)

The displacement is

$$y(t) = y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t)+\left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+\,y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

$$+\frac{\hat{I}}{m}\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

(D-8)

The displacement for zero initial conditions is

$$y(t)=\frac{\hat{I}}{m}\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{D-9}$$

The impulse response function is

$$h_d(t) = \frac{1}{m\omega_d}\left[\exp(-\xi\omega_n t)\right]\left[\sin\omega_d t\right] \tag{D-10}$$

The corresponding Laplace transform is

$$H_d(s) = \frac{1}{m}\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = \frac{1}{m}\left[\frac{1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] \tag{D-11}$$

$$\begin{aligned}\dot{y}(t) = &-\xi\omega_n y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &+ y(0)\exp(-\xi\omega_n t)\{-\omega_d \sin(\omega_d t) + \xi\omega_n \cos(\omega_d t)\} \\ &- y'(0)\left[\frac{\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \\ &+ y'(0)\exp(-\xi\omega_n t)\cos(\omega_d t) \\ &- \frac{\hat{I}}{m}\left[\frac{\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \\ &+ \frac{\hat{I}}{m}\exp(-\xi\omega_n t)\cos(\omega_d t)\end{aligned} \tag{D-12}$$

$$\begin{aligned}\dot{y}(t) = &-y(0)\exp(-\xi\omega_n t)\left[\frac{\xi^2\omega_n{}^2}{\omega_d} + \omega_d\right]\sin(\omega_d t) \\ &+ y'(0)\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \\ &+ \frac{\hat{I}}{m}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\end{aligned} \tag{D-13}$$

$$\dot{y}(t) = -y(0)\exp(-\xi\omega_n t)\left[\frac{\xi^2\omega_n{}^2+\omega_d{}^2}{\omega_d{}^2}\right]\sin(\omega_d t)$$

$$+\,y'(0)\exp(-\xi\omega_n t)\left[\cos(\omega_d t)-\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]$$

$$+\,\frac{\hat{I}}{m}\exp(-\xi\omega_n t)\left[\cos(\omega_d t)-\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{D-14}$$

The velocity for zero initial conditions is

$$\dot{y}(t) = \frac{\hat{I}}{m}\exp(-\xi\omega_n t)\left[\cos(\omega_d t)-\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{D-15}$$

The impulse response function is

$$h_v(t) = \frac{1}{m}\exp(-\xi\omega_n t)\left[\cos(\omega_d t)-\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{D-16}$$

The corresponding Laplace transform is

$$H_v(s) = \frac{1}{m}\left[\frac{s}{s^2+2\xi\omega_n s+\omega_n{}^2}\right] = \frac{1}{m}\left[\frac{s}{(s+\xi\omega_n)^2+\omega_d{}^2}\right] \tag{D-17}$$

The acceleration for zero initial conditions is

$$\ddot{y}(t) = \frac{\hat{I}}{m}\delta(t) - 2\xi\omega_n \dot{y}(t) - \omega_n{}^2 y(t) \tag{D-18}$$

$$\ddot{y}(t) =$$

$$\frac{\hat{I}}{m}\left\{ \delta(t) - 2\xi\omega_n \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] - \omega_n{}^2\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \right\}$$

(D-19)

$$\ddot{y}(t) = \frac{\hat{I}}{m}\left\{ \delta(t) + \exp(-\xi\omega_n t)\left[-2\xi\omega_n \cos(\omega_d t) + \frac{(2\xi\omega_n)^2 - \omega_n{}^2}{\omega_d}\sin(\omega_d t)\right]\right\}$$

(D-20)

$$\ddot{y}(t) = \frac{\hat{I}}{m}\left\{ \delta(t) + \exp(-\xi\omega_n t)\left[-2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left[(2\xi)^2 - 1\right]\sin(\omega_d t)\right]\right\}$$

(D-21)

The Laplace transform is

$$Y_a(s) = \frac{\hat{I}}{m}\left\{1 + \frac{-2\xi\omega_n s - 2\xi^2\omega_n{}^2 + \omega_n{}^2\left[(2\xi)^2 - 1\right]}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{D-22}$$

$$Y_a(s) = \frac{\hat{I}}{m}\left\{1 + \frac{-2\xi\omega_n s - \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{D-23}$$

$$Y_a(s) = \frac{\hat{I}}{m}\left\{\frac{s^2 + 2\xi\omega_n s + \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2} + \frac{-2\xi\omega_n s - \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{D-24}$$

$$Y_a(s) = \frac{\hat{I}}{m}\left\{\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{D-25}$$

The corresponding transfer function in terms of the Laplace transform is

$$H_a(s) = \frac{1}{m}\left[\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = \frac{1}{m}\left[\frac{s^2}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right] \tag{D-26}$$