This paper applies Smallwood’s methodology for base excitation in Reference 1 to the case of an applied force. It also extends the method to multi-degree-of-freedom systems.

Ahlin & Brandt purported in Reference 2 to use the method given in the present paper, but they omitted a derivation and the resulting filtering coefficients.

This paper fills the gap and gives accompanying Matlab scripts. The goals are to publicize the method and to place the filter coefficients equations in the public domain.

Introduction

The purpose of this paper is to derive an efficient and accurate method for the modal transient analysis of a dynamic system subjected to an external force or forces. The method will be a digital recursive filtering relationship using the ramp invariant technique which models the slope between adjacent points of the input force.

Assumptions

A modal analysis of the system has been previously performed. The system has been reduced to uncoupled mass, damping and stiffness matrices per the method given in Reference 3, as well as in common structural dynamics textbooks. The physical responses can then be recovered from the modal responses after the transient analysis.

The initial conditions are zero. Note that the response to initial conditions can be solved exactly using Laplace transforms, as needed.

Forcing Function

The forcing function may vary arbitrarily with time.

The forcing function may be either from measured data\(^1\) or from a synthesis.

\(^1\) Measured data should be collected using the proper sample rate and analog anti-aliasing filter.
The time step should be chosen so that there are at least ten points per cycle for the highest natural frequency of interest. In other words, the sample rate should be at least ten times the highest natural frequency of interest. This is an industry standard rule-of-thumb for time domain calculations per Reference 4.

Smallwood wrote in his seminal paper that the ramp invariant method allows for analysis frequencies approaching the Nyquist frequency, which in one-half the sample rate. But the x10 rule is still recommend in the present paper.

Note that the time step and corresponding sample rate are fixed for measured data as the data is being collected. The time step can be shortened via interpolation or a cubic spline fit as a post-processing step, but caution must be exercise in either case.

The time step can be set as small as needed for the case where a function is to be synthesized. Alternatively, the number of modes included in analysis can be truncated if needed to meet the rule-of-thumb. The highest frequency of interest is thus lowered as a trade-off.

Candidate Methods

The methods for the modal transient analysis are:

1. Convolution integral converted to a nested series for digital data
2. Convolution integral represented as digital cursive filtering relationship
3. Runge-Kutta fourth order method
4. Newmark-beta implicit numerical integration method

Convolution Integral

The Convolution method would yield an exact answer for the response if the system and the forcing function could be analyzed in analog rather than digital form. This is impractical, however.

The nested series representation of the Convolution integral is not commonly used because it is numerically inefficient.\(^2\)

The Convolution integral represented as digital cursive filtering relationship. The derivation is performed using a Z-transform.

\(^2\) Convolution can also be performed in the frequency domain my by multiplying the Fourier transform of the applied force by the frequency response function of the system. The inverse Fourier transform of the product gives the response time history. A Fast Fourier transform (FFT) and its corresponding inverse can be used to expedite the calculation. Note that there are some potential error sources with this method such as leakage.
There are two Z-transform approaches: the impulse invariant and ramp invariant simulations. The ramp invariant simulation is preferred because it adds a filtering term and has better accuracy at frequencies approach the Nyquist frequency per Reference 1.

Note that either digital recursive filtering relationship requires a constant time step.

**Newmark-beta Method**

The Newmark-beta method is favored in structural dynamics books for multi-degree-of-freedom systems, as given in References 5 and 6. It can accept a variable time step for the force input. Its true strength is that it can be used for the direct integration of a system of uncoupled mass, damping, and stiffness matrices. Obviously, it can be applied to an uncoupled system as well.

The Newmark-beta method is derived from the continuous time equation familiar to high school physics students.

\[
u = \dot{u} + \frac{1}{2} \ddot{u} t^2
\]

It is typically implemented with \( \beta = 1/4 \), thus assuming constant average acceleration over the time step.

**Runge-Kutta Method**

The Runge-Kutta method extends the Taylor series method by estimating higher order derivatives at points within the time step.

There are many types of Runge-Kutta algorithms. A common type which may be used for an arbitrary forcing function is the Runge-Kutta fourth order method. This method is given in Reference 7.

But the Runge-Kutta fourth order method may give unstable results\(^3\) for stiff systems with higher natural frequencies.

The probability that instability will occur is difficult to predict. It depends on both the natural frequency and the time step. The problem can be mitigated by using a smaller time step, but this requires a longer analysis time.

If an instability occurs, then the analysis be repeated using a fewer number of modes for the case of a multi-degree-of-freedom system.

\(^3\) “To infinity and beyond,” recalling Buzz Lightyear.
**SDOF Equation of Motion**

Consider a single-degree-of-freedom system.

![Free-body diagram of a single-degree-of-freedom system](image)

**Figure 1.**

The variables are:

- $M$ is the mass
- $C$ is the viscous damping coefficient
- $K$ is the stiffness
- $X$ is the absolute displacement of the mass
- $f(t)$ is the applied force

Note that the double-dot denotes acceleration.

The free-body diagram is

![Free-body diagram with forces](image)

**Figure 2.**
Summation of forces in the vertical direction

\[
\sum F = m\ddot{x} \tag{1}
\]

\[
m\ddot{x} = -c\dot{x} - kx + f(t) \tag{2}
\]

\[
m\ddot{x} + cx + kx = f(t) \tag{3}
\]

Divide through by \(m\),

\[
\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)f(t) \tag{4}
\]

By convention,

\[
(c/m) = 2\xi\omega_n \tag{5}
\]

\[
(k/m) = \omega_n^2 \tag{6}
\]

where

\(\omega_n\) is the natural frequency in (radians/sec)

\(\xi\) is the damping ratio.

By substitution,

\[
\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}f(t) \tag{7}
\]

Equation (7) does not have a closed-form solution for the general case in which \(f(t)\) is an arbitrary function. A convolution integral approach must be used to solve the equation.

The solution method proceeds by finding a solution to the *homogeneous* form of equation (7). In other words, the solution is found for \(f(t)=0\).

\[
\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{8}
\]

Note that equation (8) is essentially the same as equation (A-1) in Reference 8, except that equation (8) is expressed in terms of absolute displacement.
Displacement

The damped natural frequency $\omega_d$ is

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$ \hspace{1cm} (9)

The displacement equation via a convolution integral is

$$x(t) = \frac{1}{m\omega_d} \int_0^1 f(\tau) \left[ \exp \left\{ -\xi\omega_n (t - \tau) \right\} \right] \left[ \sin \omega_d (t - \tau) \right] d\tau$$ \hspace{1cm} (10)

The corresponding impulse response function for the displacement is

$$\hat{h}_d(t) = \frac{1}{m\omega_d} \left[ \exp \left\{ -\xi\omega_n t \right\} \right] \left[ \sin \omega_d t \right]$$ \hspace{1cm} (11)

Further details regarding this derivation are given in Reference 9.

The corresponding Laplace transform from Reference 10 is

$$H_d(s) = \frac{1}{m} \left[ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right]$$ \hspace{1cm} (12)

The Z-transform for the ramp invariant simulation is

$$\hat{H}_d(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] Z \left\{ \text{L}^{-1} \left[ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right] \right\}$$

where $T$ is the time step \hspace{1cm} (13)
Evaluate the inverse Laplace transform per References 10 and 11.

\[
L^{-1}\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] = \frac{1}{\omega_n} \left\{ -2\xi\omega_n + \omega_n^2 t + \exp\left(-\xi\omega_n t\right) \left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right] \sin(\omega_d t)\right]\right\}
\]

(14)

The Z-transform becomes

\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4} \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ -2\xi\omega_n + \omega_n^2 T + \exp\left(-\xi\omega_n T\right) \left[2\xi\omega_n \cos(\omega_d T) + \frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right] \sin(\omega_d T)\right]\right\}
\]

(15)

Let

\[
\alpha = \frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right]
\]

(16)

\[
\beta = 2\xi\omega_n
\]

(17)
The Z-transform is evaluated using the method in Reference 12.

\[
\hat{H}_d(z) = -\frac{\beta}{4\omega_n} \left[ \frac{(z-1)^2}{Tz} \right] \left[ \frac{z}{z-1} \right] + \frac{\omega_n^2}{\omega_n} \left[ \frac{(z-1)^2}{Tz} \right] \left[ \frac{Tz}{(z-1)^2} \right]
\]

\[
\hat{H}_d(z) =\frac{1}{m_\omega_n} \left[ (-\beta)\left(\frac{z-1}{T}\right) + \omega_n^2 \right] + \frac{1}{m_\omega_n} \left[ \frac{(z-1)^2}{T} \right] \left[ \frac{\beta\{z - \exp[-\xi \omega_n T]\}[\cos[\omega_d T]]}{z^2 - 2z\{\exp[-\xi \omega_n T]\}[\cos[\omega_d T]] + \{\exp[-2\xi \omega_n T]\}} \right]
\]

\[
\hat{H}_d(z) = \frac{1}{m_\omega_n} \left[ \frac{(z-1)^2}{T} \right] \left[ \frac{\alpha\{z - \exp[-\xi \omega_n T]\}[\sin[\omega_d T]]}{z^2 - 2z\{\exp[-\xi \omega_n T]\}[\cos[\omega_d T]] + \{\exp[-2\xi \omega_n T]\}} \right]
\]

\[
(18)
\]

\[
(19)
\]
\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4} \left[ -\beta \left( \frac{z-1}{T} \right) + \omega_n^2 \right] \\
+ \frac{1}{m\omega_n^4} \left[ \frac{(z-1)^2}{T} \left[ \beta \{ z - \exp[-\xi \omega_n T] \} \{ \cos[\omega_d T] \} + \alpha \{ \exp[-\xi \omega_n T] \} \{ \sin[\omega_d T] \} \right] \right] \\
+ \frac{1}{m\omega_n^4} \left[ \frac{(z-1)^2}{T} \left[ \beta \{ z + \exp(-\xi \omega_n T) \} \{ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \} \right] \right] \\
+ \frac{1}{m\omega_n^4} \left[ \frac{(z-1)^2}{T} \left[ \frac{\beta z + \exp(-\xi \omega_n T) \{ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \}}{z^2 - 2z \{ \exp[-\xi \omega_n T] \} \{ \cos(\omega_d T) \} + \{ \exp[-2\xi \omega_n T] \}} \right] \right] \\
\]

(20)

\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4 T} \left[ -\beta z + \beta + \omega_n^2 T \right] \\
+ \frac{1}{m\omega_n^4 T} \left[ \frac{(z-1)^2}{T} \left[ \beta \{ z + \exp(-\xi \omega_n T) \} \{ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \} \right] \right] \\
+ \frac{1}{m\omega_n^4 T} \left[ \frac{(z-1)^2}{T} \left[ \frac{\beta z + \exp(-\xi \omega_n T) \{ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \}}{z^2 - 2z \{ \exp[-\xi \omega_n T] \} \{ \cos(\omega_d T) \} + \{ \exp[-2\xi \omega_n T] \}} \right] \right] \\
\]

(21)

Let

\[
\psi = \exp(-\xi \omega_n T) \{ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \} \\
\rho = -2 \{ \exp(-\xi \omega_n T) \} \{ \cos(\omega_d T) \} \\
\lambda = \exp(-2\xi \omega_n T) \\
\eta = \beta + \omega_n^2 T
\]

(22) (23) (24) (25)

By substitution,

\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4 T} \left[ -\beta z + \eta \right] \\
+ \frac{1}{m\omega_n^4 T} \left[ \frac{(z-1)^2}{T} \left[ \frac{\beta z + \psi}{z^2 + \rho + \lambda} \right] \right] \\
\]

(26)

\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4 T} \left[ -\beta z + \eta \right] \left[ \frac{z^2 + \rho + \lambda}{z^2 + \rho + \lambda} \right] \\
+ \frac{1}{m\omega_n^4 T} \left[ \frac{(z-1)^2}{T} \left[ \frac{\beta z + \psi}{z^2 - 2z + 1} \right] \right] \\
\]

(27)
\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4 T} \left[ -\beta z \left( z^2 + z\rho + \lambda \right) \right] + \frac{1}{m\omega_n^4 T} \left[ \eta \left( z^2 + z\rho + \lambda \right) \right] + \frac{1}{m\omega_n^4 T} \left[ \frac{\beta z + \psi}{z^2 + z\rho + \lambda} \right] + \frac{1}{m\omega_n^4 T} \left[ -2z \left( z^2 + z\rho + \lambda \right) \right] + \frac{1}{m\omega_n^4 T} \left[ \frac{\beta z + \psi}{z^2 + z\rho + \lambda} \right] \]
\]
\[
(28)
\]
\[
\hat{H}_d(z) = \frac{1}{m\omega_n^4 T} \left[ -\beta z^3 - \beta \rho z^2 - \beta \lambda z \right] + \frac{1}{m\omega_n^4 T} \left[ \eta z^2 + \eta \rho z + \eta \lambda \right] + \frac{1}{m\omega_n^4 T} \left[ -2z^2 - 2\psi z \right] + \frac{1}{m\omega_n^4 T} \left[ \frac{\beta z + \psi}{z^2 + z\rho + \lambda} \right] + \frac{1}{m\omega_n^4 T} \left[ \frac{\beta z + \psi}{z^2 + z\rho + \lambda} \right] \]
\[
(29)
\]
\[
\hat{H}_d(z) = \frac{-\beta z^3 - \beta \rho z^2 - \beta \lambda z + \eta z^2 + \eta \rho z + \eta \lambda + \beta z^3 + \psi z^2 - 2\beta z^2 - 2\psi z + \beta z + \psi}{m\omega_n^4 T \left( z^2 + z\rho + \lambda \right)} \]
\[
(30)
\]
\[
\hat{H}_d(z) = \frac{(-\beta \rho + \psi + \eta - 2\beta) z^2 + (-\beta \lambda + \eta \rho - 2\psi + \beta) z + (\eta \lambda + \psi)}{m\omega_n^4 T \left( z^2 + z\rho + \lambda \right)} \]
\[
(31)
\]
Solve for the filter coefficients using the method in Reference 1.

\[
\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{[-\beta \rho + \psi + \eta - 2\beta]z^2 + (-\beta \lambda + \eta \rho - 2\psi + \beta)z + (\eta \lambda + \psi)}{z^2 + z \rho + \lambda} \left[ \frac{m \omega_n^4 T}{m \omega_n^4 T} \right]
\]

(32)

Solve for \(a_1\).

\[
a_1 = \rho = -2 \exp(-\xi \omega_n T) \cos(\omega_d T)
\]

(33)

Solve for \(a_2\).

\[
a_2 = \lambda = \exp(-2 \xi \omega_n T)
\]

(34)

Note that the \(a_1\) and \(a_2\) coefficients are common for displacement, velocity and acceleration.

Solve for \(b_0\).

\[
b_0 = [-\beta \rho + \psi + \eta - 2\beta] \left[ \frac{m \omega_n^4 T}{m \omega_n^4 T} \right]
\]

(35)

\[
b_0 = [\psi + \eta - \beta (\rho + 2)] \left[ \frac{m \omega_n^4 T}{m \omega_n^4 T} \right]
\]

(36)

\[
b_0 = \frac{\exp(-\xi \omega_n T)[\alpha \sin(\omega_d T) - \beta \cos(\omega_d T)] + \beta + \omega_n^2 T - \beta [-2 \exp(-\xi \omega_n T) \cos(\omega_d T) + 2]}{m \omega_n^4 T}
\]

(37)

\[
b_0 = \frac{\exp(-\xi \omega_n T)[\alpha \sin(\omega_d T) - \beta \cos(\omega_d T)] + \beta + \omega_n^2 T + 2\beta [\exp(-\xi \omega_n T) \cos(\omega_d T) - 1]}{m \omega_n^4 T}
\]

(38)
\[ b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right]\sin(\omega_d T) - 2\xi\omega_n \cos(\omega_d T)\right]}{m\omega_n^4 T} \]

\[ + \frac{2\xi\omega_n + \omega_n^2 T + 4\xi\omega_n \left[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1\right]}{m\omega_n^4 T} \]

(39)

\[ b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right]\sin(\omega_d T)\right] + \omega_n^2 T + 2\xi\omega_n \left[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1\right]}{m\omega_n^3 T} \]

(40)

\[ b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right]\sin(\omega_d T)\right] + \omega_n T + 2\xi\left[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1\right]}{m\omega_n^3 T} \]

(41)

\[ b_0 = \frac{2\xi\left[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1\right] + \exp(-\xi\omega_n T)\left[\frac{\omega_n}{\omega_d} \left[2\xi^2 - 1\right]\sin(\omega_d T)\right] + \omega_n T}{m\omega_n^3 T} \]

(42)
Solve for $b_1$.

$$b_1 = -\beta \lambda + \eta \rho - 2\psi + \beta$$

$$\text{m} \omega_n^4 T$$

(43)

$$b_1 = \eta \rho - 2\psi + \beta (1 - \lambda)$$

$$\text{m} \omega_n^4 T$$

(44)

$$b_1 =$$

$$\frac{-2 \left( \beta + \omega_n^2 T \exp(-\xi \omega_n T) \cos(\omega_d T) - 2 \exp(-\xi \omega_n T) \left[ \alpha \sin(\omega_d T) - \beta \cos(\omega_d T) \right] \right)}{\text{m} \omega_n^4 T}$$

$$+ \frac{\beta \left[ 1 - \exp(-2\xi \omega_n T) \right]}{\text{m} \omega_n^4 T}$$

(45)

$$b_1 = \frac{-2 \omega_n^2 T \exp(-\xi \omega_n T) \cos(\omega_d T) - 2 \exp(-\xi \omega_n T) \alpha \sin(\omega_d T) + \beta \left[ 1 - \exp(-2\xi \omega_n T) \right]}{\text{m} \omega_n^4 T}$$

(46)

$$b_1 = \frac{-2 \omega_n^2 T \exp(-\xi \omega_n T) \cos(\omega_d T) - 2 \exp(-\xi \omega_n T) \alpha \sin(\omega_d T) + \beta \left[ 1 - \exp(-2\xi \omega_n T) \right]}{\text{m} \omega_n^4 T}$$

(47)
\[ b_1 = \frac{-2\omega_n^2 T \exp(-\xi \omega_n T) \cos(\omega_d T) - 2 \frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right] \exp(-\xi \omega_n T) \sin(\omega_d T)}{m\omega_n^4 T} + \frac{2\xi \omega_n \left[1 - \exp(-2\xi \omega_n T)\right]}{m\omega_n^4 T} \]  
\[ (48) \]

\[ b_1 = \frac{-2\omega_n T \exp(-\xi \omega_n T) \cos(\omega_d T) + 2\xi \left[1 - \exp(-2\xi \omega_n T)\right] - 2 \frac{\omega_n}{\omega_d} \left[2\xi^2 - 1\right] \exp(-\xi \omega_n T) \sin(\omega_d T)}{m\omega_n^3 T} \]

Solve for \( b_2 \).

\[ b_2 = \frac{\eta \lambda + \psi}{m\omega_n^4 T} \]
\[ (50) \]

\[ b_2 = \frac{\left(\beta + \omega_n^2 T\right) \exp(-2\xi \omega_n T) + \exp(-\xi \omega_n T) \alpha \sin(\omega_d T) - \beta \cos(\omega_d T)}{m\omega_n^4 T} \]
\[ (51) \]

\[ b_2 = \frac{\left(2\xi \omega_n + \omega_n^2 T\right) \exp(-2\xi \omega_n T) + \exp(-\xi \omega_n T) \left\{\frac{\omega_n^2}{\omega_d} \left[-1 + 2\xi^2\right] \sin(\omega_d T) - 2\xi \omega_n \cos(\omega_d T)\right\}}{m\omega_n^4 T} \]
\[ (52) \]
\[
\begin{align*}
\frac{(2\xi + \omega_n T) \exp(-2\xi \omega_n T) + \exp(-\xi \omega_n T) \left\{ \frac{\omega_n}{\omega_d} \left[ 2\xi^2 - 1 \right] \sin(\omega_d T) - 2\xi \cos(\omega_d T) \right\}}{m \omega_n^3 T} & = b_2 \\
\text{(53)}
\end{align*}
\]

The digital recursive filtering relationship for the displacement is

\[
\begin{align*}
\dot{x}_i & = -a_1 x_{i-1} - a_2 x_{i-2} \\
& \quad + b_0 f_i + b_1 f_{i-1} + b_2 f_{i-2} \\
\text{(54)}
\end{align*}
\]
The digital recursive relationship for the displacement is thus

\[ x_i = \]
\[ + 2 \exp[-\xi \omega_n T] \cos(\omega_d T) x_{i-1} \]
\[ - \exp[-2\xi \omega_n T] x_{i-2} \]
\[ + \frac{1}{m \omega_n^3 T} \left\{ 2\xi \left[ \exp(-\xi \omega_n T) \cos(\omega_d T) - 1 \right] + \exp(-\xi \omega_n T) \left[ \frac{\omega_n}{\omega_d} \left[ 2\xi^2 - 1 \right] \sin(\omega_d T) \right] + \omega_n T \right\} f_i \]
\[ + \frac{1}{m \omega_n^3 T} \left\{ -2\omega_n T \exp(-\xi \omega_n T) \cos(\omega_d T) + 2\xi \left[ 1 - \exp(-2\xi \omega_n T) \right] - 2 \frac{\omega_n}{\omega_d} \left[ 2\xi^2 - 1 \right] \exp(-\xi \omega_n T) \sin(\omega_d T) \right\} f_{i-1} \]
\[ + \frac{1}{m \omega_n^3 T} \left\{ \left( 2\xi + \omega_n \right) \exp(-2\xi \omega_n T) + \exp(-\xi \omega_n T) \left[ \frac{\omega_n}{\omega_d} \left[ 2\xi^2 - 1 \right] \sin(\omega_d T) - 2\xi \cos(\omega_d T) \right] \right\} f_{i-2} \]

(55)
Velocity

The impulse response function for the velocity response is

\[
\hat{h}_v(t) = \frac{1}{m} \left[ \exp(-\xi \omega_n t) \left[ \left( -\frac{\xi \omega_n}{\omega_d} \right) \sin \omega_d t + \cos \omega_d t \right] \right]
\]  

(56)

The corresponding Laplace transform is

\[
H_v(s) = \frac{1}{m} \left[ \frac{s}{s^2 + 2\xi \omega_n s + \omega_n^2} \right]
\]

(57)

The Z-transform for the ramp invariant simulation is

\[
\hat{H}_v(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] \left\{ \mathcal{Z}^{-1} \left[ \frac{s}{s^2 + 2\xi \omega_n s + \omega_n^2} \right] \right\}
\]

(58)

\[
\hat{H}_v(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] \left\{ \mathcal{Z}^{-1} \left[ \frac{1}{s^2 + 2\xi \omega_n s + \omega_n^2} \right] \right\}
\]

(59)

Evaluate the inverse Laplace transform per References 10 and 11.

\[
\mathcal{L}^{-1} \left[ \frac{1}{s \left( s^2 + 2\xi \omega_n s + \omega_n^2 \right)} \right] = \frac{1}{2 \omega_n} - \frac{1}{2 \omega_n} \exp(-\xi \omega_n t) \left[ \cos(\omega_d t) + \frac{\xi \omega_n}{\omega_d} \sin(\omega_d t) \right]
\]

(60)

The Z-transform for the ramp invariant simulation is

\[
\hat{H}_v(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] \left\{ \frac{1}{\omega_n^2} - \frac{1}{\omega_n^2} \exp(-\xi \omega_n T) \left[ \cos(\omega_d T) + \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] \right\}
\]

(61)
\[ \hat{H}_v(z) = \frac{1}{m\omega_n^2 z} \left[ \chi_0^2 \left\{ 1 - \exp(-\xi\omega_n T) \left[ \cos(\omega_d T) + \frac{\xi\omega_n}{\omega_d} \sin(\omega_d T) \right] \right\} \right] \] 

(62)

The Z-transform is evaluated using the method in Reference 12.

\[ \hat{H}_v(z) = \frac{1}{m\omega_n^2 T z} \left[ \frac{z}{z-1} - \frac{z \left[ z - \exp(-\xi\omega_n T) \cos(\omega_d T) \right] + z \frac{\xi\omega_n}{\omega_d} \left[ \exp(-\xi\omega_n T) \sin(\omega_d T) \right]} {z^2 - 2 \exp(-\xi\omega_n T) \cos(\omega_d T) \sin(\omega_d T) + \exp(-2\xi\omega_n T)} \right] \] 

(63)

\[ \hat{H}_v(z) = \frac{1}{m\omega_n^2 T} \left[ \frac{z}{z-1} - \frac{z \left[ z - \exp(-\xi\omega_n T) \cos(\omega_d T) \right] + z \frac{\xi\omega_n}{\omega_d} \left[ \exp(-\xi\omega_n T) \sin(\omega_d T) \right]} {z^2 - 2 \exp(-\xi\omega_n T) \cos(\omega_d T) \sin(\omega_d T) + \exp(-2\xi\omega_n T)} \right] \] 

(64)

\[ \hat{H}_v(z) = \frac{1}{m\omega_n^2 T} \left[ \frac{z - \exp(-\xi\omega_n T) \cos(\omega_d T) + \frac{\xi\omega_n}{\omega_d} \left[ \exp(-\xi\omega_n T) \sin(\omega_d T) \right]} {z^2 - 2 \exp(-\xi\omega_n T) \cos(\omega_d T) \sin(\omega_d T) + \exp(-2\xi\omega_n T)} \right] \] 

(65)
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z - \psi}{z^2 + \rho z + \lambda} \right] \]

Let
\[
\psi = \exp(-\xi \omega_n T) \left[ \cos(\omega_d T) - \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] \quad \text{(67)}
\]
\[
\rho = -2 \{ \exp(-\xi \omega_n T) \} \{ \cos(\omega_d T) \} \quad \text{(68)}
\]
\[
\lambda = \exp(-2 \xi \omega_n T) \quad \text{(69)}
\]

\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z - \psi}{z^2 + \rho z + \lambda} \right] \quad \text{(70)}
\]
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z^2 - 2z + 1 - \psi(z^2 - 2z + 1)}{z^2 + \rho z + \lambda} \right] \quad \text{(71)}
\]
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z^2 - 2z + 1 - \psi(z^2 - 2z + 1)}{z^2 + \rho z + \lambda} \right] \quad \text{(72)}
\]
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z^3 - 2z^2 + z - \psi z^2 - 2\psi z + \psi}{z^2 + \rho z + \lambda} \right] \quad \text{(73)}
\]
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)^2 - \frac{z^3 - 2z^2 + z - \psi z^2 + 2\psi z - \psi}{z^2 + \rho z + \lambda} \right] \quad \text{(74)}
\]
\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1) - \frac{z^3 + (-2 - \psi)z^2 + (1 + 2\psi)z - \psi}{z^2 + \rho z + \lambda} \right] \] (75)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1) + \frac{-z^3 + (2 + \psi)z^2 - (1 + 2\psi)z + \psi}{z^2 + \rho z + \lambda} \right] \] (76)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ (z-1)z^2 + \rho z + \lambda + \frac{-z^3 + (2 + \psi)z^2 - (1 + 2\psi)z + \psi}{z^2 + \rho z + \lambda} \right] \] (77)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ z^3 + \rho z^2 + \lambda z - z^2 - \rho z - \lambda + \frac{-z^3 + (2 + \psi)z^2 - (1 + 2\psi)z + \psi}{z^2 + \rho z + \lambda} \right] \] (78)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ z^3 + (\rho - 1)z^2 + (\lambda - \rho)z - \lambda + \frac{-z^3 + (2 + \psi)z^2 - (1 + 2\psi)z + \psi}{z^2 + \rho z + \lambda} \right] \] (79)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ \frac{(2 + \psi + 1)z^2 + (\lambda - \rho - 1 - 2\psi)z + \psi - \lambda}{z^2 + \rho z + \lambda} \right] \] (80)

\[ \hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ \frac{(\psi + 1)z^2 + (\lambda - \rho - 2\psi + 1)z + \psi - \lambda}{z^2 + \rho z + \lambda} \right] \] (81)
\[
\hat{H}_v(z) = \frac{1}{m \omega_n^2 T} \left[ \frac{(\psi + \rho + 1)z^2 + (\lambda - \rho - 2\psi - 1)z + \psi - \lambda}{z^2 + \rho z + \lambda} \right]
\] (82)

Solve for the filter coefficients using the method in Reference 1.

\[
\frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = \frac{1}{m \omega_n^2 T} \left[ \frac{(\psi + \rho + 1)z^2 + (\lambda - \rho - 2\psi - 1)z + \psi - \lambda}{z^2 + \rho z + \lambda} \right]
\] (83)

Solve for \(a_1\).

\[
a_1 = \rho = -2\exp(-\xi \omega_n T) \cos(\omega_d T)
\] (84)

Solve for \(a_2\).

\[
a_2 = \lambda = \exp(-2\xi \omega_n T)
\] (85)

Solve for \(c_0\).

\[
c_0 = \frac{\psi + \rho + 1}{m \omega_n^2 T}
\] (86)

\[
c_0 = \frac{\exp(-\xi \omega_n T) \left[ \cos(\omega_d T) - \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] - 2 \exp(-\xi \omega_n T) \cos(\omega_d T) + 1}{m \omega_n^2 T}
\] (87)

\[
c_0 = \frac{\exp(-\xi \omega_n T) \left[ -\cos(\omega_d T) - \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] + 1}{m \omega_n^2 T}
\] (88)
Solve for $c_1$.

$$c_1 = \frac{\lambda - \rho - 2\psi - 1}{m\omega_n^2 T}$$

(89)

$$c_1 = \frac{\exp(-2\xi\omega_n T) + 2\exp(-\xi\omega_n T)\cos(\omega_d T) - 2\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d} \sin(\omega_d T)\right]}{m\omega_n^2 T} - 1$$

(90)

Solve for $c_2$.

$$c_2 = \frac{\psi - \lambda}{m\omega_n^2 T}$$

(92)

$$c_2 = \frac{\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d} \sin(\omega_d T)\right] - \exp(-2\xi\omega_n T)}{m\omega_n^2 T}$$

(93)

The digital recursive filtering relationship for the velocity is

$$\dot{x}_i = -a_1 \dot{x}_{i-1} - a_2 \dot{x}_{i-2} + c_0 f_i + c_1 f_{i-1} + c_2 f_{i-2}$$

(94)
The digital recursive relationship for the velocity is thus

\[ \dot{x}_i = \]

\[ + 2 \exp[-\xi \omega_n T] \cos(\omega_d T) \dot{x}_{i-1} \]

\[ - \exp[-2\xi \omega_n T] \dot{x}_{i-2} \]

\[ + \frac{1}{m \omega_n^2 T} \left\{ \exp(-\xi \omega_n T) \left[ - \cos(\omega_d T) - \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] + 1 \right\} f_i \]

\[ + \frac{1}{m \omega_n^2 T} \left\{ \exp(-2\xi \omega_n T) + 2 \exp(-\xi \omega_n T) \left[ \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] - 1 \right\} f_{i-1} \]

\[ + \frac{1}{m \omega_n^2 T} \left\{ \exp(-\xi \omega_n T) \left[ \cos(\omega_d T) - \frac{\xi \omega_n}{\omega_d} \sin(\omega_d T) \right] - \exp(-2\xi \omega_n T) \right\} f_{i-2} \]

\[ (95) \]

**Acceleration**

The acceleration response for a unit impulse is

\[ \ddot{y}(t) = \frac{1}{m} \left\{ \delta(t) + \exp(-\xi \omega_n T) \left[ - 2\xi \omega_n \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left( 2\xi \right)^2 - 1 \right] \sin(\omega_d t) \right\} \]

\[ (96) \]

The corresponding Laplace transform is

\[ H_a(s) = \frac{1}{m} \left[ \frac{s^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \right] \]

\[ (97) \]

The Z-transform is

\[ \hat{H}_a(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] Z \left\{ L^{-1} \left[ \frac{s^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \right] \right\} \]

\[ (98) \]
\[
\hat{H}_a(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] Z \left\{ \frac{1}{L^{-1} \left[ \frac{s^2}{s^2 + 2 \xi \omega_n s + \omega_n^2} \right]} \right\} 
\]

Evaluate the inverse Laplace transform per Reference 10 and 11.

\[
L^{-1} \left[ \frac{1}{s^2 + 2 \xi \omega_n s + \omega_n^2} \right] = L^{-1} \left[ \frac{1}{(s + \xi \omega_n)^2 + \omega_d^2} \right] 
\]

The Z-transform for the ramp invariant simulation is

\[
\hat{H}_a(z) = \frac{1}{m} \left[ \frac{(z-1)^2}{Tz} \right] Z \left\{ \frac{1}{\omega_d} \exp(-\xi \omega_n T) \sin(\omega_d T) \right\} 
\]

The Z-transform is evaluated using the method in Reference 12.

\[
\hat{H}_a(z) = \frac{1}{m \omega_d} \left[ \frac{(z-1)^2}{Tz} \right] \left[ \frac{z \left[ \exp(-\xi \omega_n T) \sin(\omega_d T) \right]}{z^2 - 2z \exp(-\xi \omega_n T) \cos(\omega_d T) + \exp(-2 \xi \omega_n T)} \right] 
\]
\[ \hat{H}_a(z) = \frac{1}{m_\omega T} \left[ \frac{(z-1)^2 \left[ \exp(-\xi \omega T) \sin(\omega_d T) \right]}{z^2 - 2z \exp(-\xi \omega T) \cos(\omega_d T) + \exp(-2\xi \omega T)} \right] \] (106)

\[ \hat{H}_a(z) = \frac{\exp(-\xi \omega T) \sin(\omega_d T)}{m_\omega T} \left[ \frac{(z-1)^2}{z^2 - 2z \exp(-\xi \omega T) \cos(\omega_d T) + \exp(-2\xi \omega T)} \right] \] (107)

\[ \hat{H}_a(z) = \frac{\exp(-\xi \omega T) \sin(\omega_d T)}{m_\omega T} \left[ \frac{z^2 - 2z + 1}{z^2 - 2z \exp(-\xi \omega T) \cos(\omega_d T) + \exp(-2\xi \omega T)} \right] \] (108)

Solve for the filter coefficients using the method in Reference 1.

\[ \frac{d_0 z^2 + d_1 z + d_2}{z^2 + a_1 z + a_2} = \frac{\exp(-\xi \omega T) \sin(\omega_d T)}{m_\omega T} \left[ \frac{z^2 - 2z + 1}{z^2 - 2z \exp(-\xi \omega T) \cos(\omega_d T) + \exp(-2\xi \omega T)} \right] \] (109)

Solve for \( a_1 \).

\[ a_1 = \rho = -2\exp(-\xi \omega T) \cos(\omega_d T) \] (110)

Solve for \( a_2 \).

\[ a_2 = \lambda = \exp(-2\xi \omega T) \] (111)
Solve for \( d_0 \).
\[
d_0 = \frac{\exp(-\xi \omega_n T) \sin(\omega_d T)}{m \omega_d T}
\]  
(112)

Solve for \( d_1 \).
\[
d_1 = -2d_0
\]  
(113)

Solve for \( d_2 \).
\[
d_2 = d_0
\]  
(114)

The digital recursive filtering relationship for the acceleration is
\[
\ddot{x}_i = -a_1 \ddot{x}_{i-1} - a_2 \ddot{x}_{i-2} + d_0 f_i + d_1 f_{i-1} + d_2 f_{i-2}
\]  
(115)

\[
\ddot{x}_i = +2 \exp[-\xi \omega_n T] \cos(\omega_d T) \ddot{x}_{i-1} - \exp[-2\xi \omega_n T] \ddot{x}_{i-2}
\]
\[
+ \frac{\exp(-\xi \omega_n T) \sin(\omega_d T)}{m \omega_d T} \left\{ f_i - 2f_{i-1} + f_{i-2} \right\}
\]  
(116)

There are three response parameters: displacement, velocity and acceleration. If two are known, the third can be calculated via equation (7).

Transmitted Force
The impulse response function for transmitted force from Reference 14 is
\[
\hat{h}_{ff}(t) = \exp\left(-\xi \omega_n t\right) \left[ 2 \xi \omega_n \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} (1 - 2 \xi^2) \sin(\omega_d t) \right]
\]  
(117)
The corresponding Laplace transform is

\[ H_{ft}(s) = \left[ \frac{2\xi_2\omega_n s + \omega_n^2}{s^2 + 2\xi_2\omega_n s + \omega_n^2} \right] \]  

The Z-transform is

\[ \hat{H}_{ft}(z) = \left[ \frac{(z-1)^2}{Tz} \right] Z \left\{ L^{-1} \left[ \frac{1}{s^2} \left( \frac{2\xi_2\omega_n s + \omega_n^2}{s^2 + 2\xi_2\omega_n s + \omega_n^2} \right) \right] \right\} \]

Note that the transmitted force impulse response function is essentially the same as the absolute acceleration impulse response function for the case of base excitation. The digital recursive filtering relationship can thus be taken from Reference 15.

\[ f_{t,i} = +2\exp\left[-\xi_2\omega_n \Delta t\right]\cos(\omega_d \Delta t) f_{t,i-1} \]

\[ -\exp\left[-2\xi_2\omega_n \Delta t\right] f_{t,i-2} \]

\[ + \left\{ 1 - \left( \frac{1}{\omega_d T} \right) \exp\left(-\xi_2\omega_n T\right) \sin(\omega_d T) \right\} f_i \]

\[ + \left\{ 2\exp\left(-\xi_2\omega_n T\right) \left( -\cos(\omega_d T) + \left( \frac{1}{\omega_d T} \right) \sin(\omega_d T) \right) \right\} f_{i-1} \]

\[ + \left\{ \exp\left(-2\xi_2\omega_n T\right) - \left( \frac{1}{\omega_d T} \right) \exp\left(-\xi_2\omega_n T\right) \sin(\omega_d T) \right\} f_{i-2} \]  

(120)
SDOF Example

An SDOF system has a natural frequency of 10 Hz with an amplification of $Q=10$. Its mass is 1 lbm. It is subjected to 1 lbf sinusoidal excitation at its natural frequency as shown in Figure 3.

The analysis is performed using the ramp invariant digital recursive filters derived in this paper as implemented in Matlab script: arbit_force.m.

This is a simple problem which does not demonstrate the full power of the ramp invariant filtering method for calculating the response to a force which varies arbitrarily with time.

But the simple example is useful for checking the accuracy of the method. The peak results agree with the expected values from the corresponding frequency response functions.

The descriptive statistics for the response from the Matlab script are:

Displacement Response

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum</td>
<td>0.97 inch</td>
</tr>
<tr>
<td>minimum</td>
<td>-0.971 inch</td>
</tr>
<tr>
<td>overall</td>
<td>0.601 inch RMS</td>
</tr>
</tbody>
</table>

Velocity Response

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum</td>
<td>61.6 in/sec</td>
</tr>
<tr>
<td>minimum</td>
<td>-61.6 in/sec</td>
</tr>
<tr>
<td>overall</td>
<td>38.1 in/sec RMS</td>
</tr>
</tbody>
</table>

Acceleration Response

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum</td>
<td>9.98 G</td>
</tr>
<tr>
<td>minimum</td>
<td>-9.97 G</td>
</tr>
<tr>
<td>overall</td>
<td>6.17 G RMS</td>
</tr>
</tbody>
</table>
Figure 3.
The resulting displacement, velocity and acceleration responses for the ramp invariant method are shown in Figures 4, 5 and 6, respectively. The exact results from the Laplace transform calculation are superimposed for comparison, as calculated from the formulas Reference 13 as implemented via Matlab script: sdof_sine_force.m.

There is excellent agreement between the two displacement curves as shown in Figure 4.
Figure 5.

There is excellent agreement between the two velocity curves as shown in Figure 5.
There is excellent agreement between the two acceleration curves as shown in Figure 6.
MDOF Application

The ramp invariant digital recursive filtering relationship can be readily used as the numerical engine in an MDOF modal transient analysis for each of the respective response parameters.

An example is shown in Appendix A.

References


APPENDIX A

MDOF Example

The example from Reference 2 is repeated here.

Assume 5% damping for each mode. Assume zero initial conditions.
The parameters are

<table>
<thead>
<tr>
<th>Table A-1. Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>m₁</td>
</tr>
<tr>
<td>m₂</td>
</tr>
<tr>
<td>k₁</td>
</tr>
<tr>
<td>k₂</td>
</tr>
<tr>
<td>k₃</td>
</tr>
<tr>
<td>B₁</td>
</tr>
<tr>
<td>B₂</td>
</tr>
<tr>
<td>α</td>
</tr>
<tr>
<td>β</td>
</tr>
</tbody>
</table>

The mass matrix is

\[
M = \begin{bmatrix}
m₁ & 0 \\
0 & m₂
\end{bmatrix} = \begin{bmatrix}
3 & 0 \\
0 & 2
\end{bmatrix} \tag{A-1}
\]

The stiffness matrix is

\[
K = \begin{bmatrix}
k₁ + k₂ & -k₂ \\
-k₂ & k₂ + k₃
\end{bmatrix} = \begin{bmatrix}
700,000 & -300,000 \\
-300,000 & 400,000
\end{bmatrix} \tag{A-2}
\]

The applied forces are

\[
\begin{bmatrix}
f₁(t) \\
f₂(t)
\end{bmatrix} = \begin{bmatrix}
B₁ \sin(2\pi α t) \\
B₂ \sin(2\pi β t)
\end{bmatrix} = \begin{bmatrix}
100\sin(2\pi 55 t) \\
200\sin(2\pi 100 t)
\end{bmatrix} \tag{A-3}
\]

Each of the two forcing functions is synthesized using Matlab script: generate.m, as a pre-processing step.

The modal transient analysis is performed using Matlab script: mdof_modal_arbit_force_ri.m
This program calculates the response of an MDOF system to arbitrary force excitation via the ramp invariant digital recursive filtering relationship.

The system is decoupled using normal modes as an intermediate step.

Enter the units system
1=English  2=metric
1
Assume symmetric mass and stiffness matrices.
Select input mass unit
1=lbm  2=lbf sec^2/in
2
stiffness unit = lbf/in

Select file input method
1=file preloaded into Matlab  2=Excel file
1

Mass Matrix
Enter the matrix name:  mass_case5

Select damping input method
1=uniform damping ratio  2=damping ratio vector
1
Enter damping ratio
0.05

Stiffness Matrix
Enter the matrix name:  stiff_case5

Natural Frequencies
No.  f(Hz)
1.   48.552
2.   92.839

Modes Shapes (column format)

ModeShapes =

   0.3797   -0.4349
   0.5326    0.4651
Enter duration(sec)
0.3

Enter sample rate (samples/sec)
(recommend 1857)
2000

Each force file must have two columns: time(sec) & force(lbf)

Enter the number of force files
2

Note: the first dof is 1

Enter force file 1
Enter the matrix name: sine1

Enter the number of dofs at which this force is applied
1

Enter the dof number for this force
1

Enter force file 2
Enter the matrix name: sine2

Enter the number of dofs at which this force is applied
1

Enter the dof number for this force
2

begin interpolation
end interpolation

Calculating response...
The displacement, velocity and acceleration responses are shown in Figures A-2, A-3 and A-4, respectively.

The results appear to be the same as the Laplace transform results in Reference 3. A formal comparison will be given in the next revision of this paper.
Figure A-3.

Figure A-4.