THE GENERALIZED COORDINATE METHOD FOR DISCRETE SYSTEMS
Revision F

By Tom Irvine
Email: tomirvine@aol.com

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Two-degree-of-freedom System

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

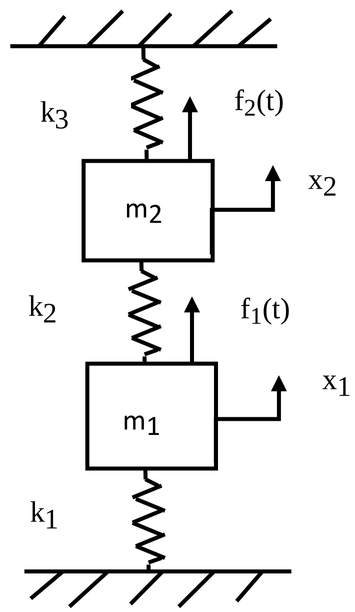


Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

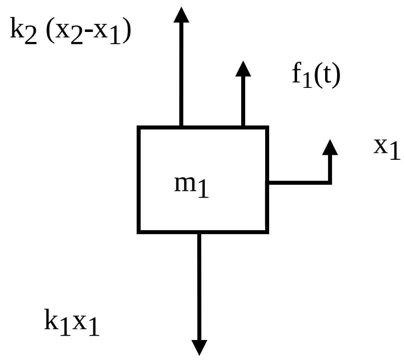


Figure 2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \quad (1)$$

$$m_1\ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1 x_1 \quad (2)$$

$$m_1\ddot{x}_1 - k_2(x_2 - x_1) + k_1 x_1 = f_1(t) \quad (3)$$

$$m_1\ddot{x}_1 + k_2(-x_2 + x_1) + k_1 x_1 = f_1(t) \quad (4)$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t) \quad (5)$$

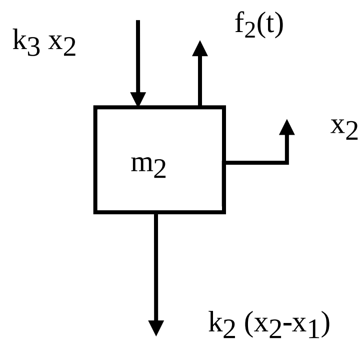


Figure 3.

Derive the equation of motion for mass 2.

$$\sum F = m_2\,\ddot{x}_2 \tag{6}$$

$$m_2\ddot{x}_2 = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 \tag{7}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 = f_2(t) \tag{8}$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = f_2(t) \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{10}$$

Decoupling

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\ddot{\bar{x}} + K\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\bar{q}\exp(j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2\bar{q}\exp(j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M \bar{q} \exp(j\omega t) + K \bar{q} \exp(j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M \bar{q} + K \bar{q}\right\} \exp(j\omega t) = \bar{0} \tag{21}$$

$$-\omega^2 M \bar{q} + K \bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M + K\right\} \bar{q} = \bar{0} \tag{23}$$

$$\left\{K - \omega^2 M\right\} \bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = \bar{0} \tag{25}$$

$$\det\left\{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{26}$$

$$\det\left\{\begin{bmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix}\right\} = 0 \tag{27}$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_2^2 = 0 \tag{28}$$

$$(k_1+k_2)(k_2+k_3)-\omega^2 m_1(k_2+k_3)-\omega^2 m_2(k_1+k_2)+\omega^4 m_1 m_2-k_2{}^2=0 \tag{29}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+(k_1+k_3)k_2+k_2{}^2-k_2{}^2=0 \tag{30}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+k_1 k_2+k_2 k_3=0 \tag{31}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a} \tag{32}$$

$$\omega_2^2=\frac{-b+\sqrt{b^2-4ac}}{2a} \tag{33}$$

where

$$a=m_1 m_2 \tag{34}$$

$$b=[-m_1(k_2+k_3)-m_2(k_1+k_2)] \tag{35}$$

$$c=k_1 k_2+k_1 k_3+k_2 k_3 \tag{36}$$

The eigenvectors are found via the following equations.

$$\left\{K-\omega_1{}^2\,M\right\}\bar{q}_1=\bar{0} \tag{37}$$

$$\left\{K-\omega_2{}^2\,M\right\}\bar{q}_2=\bar{0} \tag{38}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{39}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{40}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{41}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{42}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{43}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{44}$$

where

I	is the identity matrix
Ω	is a diagonal matrix of eigenvalues

The superscript T represents transpose.

Note the mass-normalized forms

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{45}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{46}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial.

Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{47}$$

Substitute equation (47) into the equation of motion, equation (11).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{48}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T\,\bar{F} \tag{49}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T\,\bar{F} \tag{50}$$

The equations of motion along with an added damping matrix become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix}+\begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix}+\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=\begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{51}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Equation (51) yields two equations

$$\ddot{\eta}_1+2\xi_1\omega_1\dot{\eta}_1+\omega_1^2\eta_1=\hat{v}_1 f_1(t)+\hat{v}_2 f_2(t) \tag{52}$$

$$\ddot{\eta}_2+2\xi_2\omega_2\dot{\eta}_2+\omega_2^2\eta_2=\hat{w}_1 f_1(t)+\hat{w}_2 f_2(t) \tag{53}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x}=\hat{Q}\,\bar{\eta} \tag{54}$$

Thus

$$\bar{x}(0)=\hat{Q}\,\bar{\eta}(0) \tag{55}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{x}(0)=\hat{Q}^T M\hat{Q}\,\bar{\eta}(0) \tag{56}$$

Recall

$$\hat{Q}^T M\,\hat{Q}=I \tag{57}$$

$$\hat{Q}^T M \bar{x}(0) = I \bar{\eta}(0) \tag{58}$$

$$\hat{Q}^T M \bar{x}(0) = \bar{\eta}(0) \tag{59}$$

Finally, the transformed initial displacement is

$$\bar{\eta}(0) = \hat{Q}^T M \bar{x}(0) \tag{60}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^T M \bar{\dot{x}}(0) \tag{61}$$

A basis for a solution is thus derived.

Harmonic Force

Now consider the special case of harmonic forcing functions

$$f_1(t) = B_1 \sin(\alpha t) \tag{62}$$

$$f_2(t) = B_2 \sin(\beta t) \tag{63}$$

Thus,

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = \hat{v}_1 B_1 \sin(\alpha t) + \hat{v}_2 B_2 \sin(\beta t) \tag{64}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \dot{\eta}_2 + \omega_2^2 \eta_2 = \hat{w}_1 B_1 \sin(\alpha t) + \hat{w}_2 B_2 \sin(\beta t) \tag{65}$$

Take the Laplace transform of equation (64).

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = \hat{v}_1 B_1 \sin(\alpha t) + \hat{v}_2 B_2 \sin(\beta t) \tag{66}$$

$$L\left\{\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1\right\} = L\{\hat{v}_1 B_1 \sin(\alpha t)\} + L\{\hat{v}_2 B_2 \sin(\beta t)\} \tag{67}$$

$$
\begin{aligned}
& s^2\hat{\eta}_1(s) - s\eta_1(0) - \dot{\eta}_1(0) \\
& + 2\xi_1\omega_1 s\hat{\eta}_1(s) - 2\xi_1\omega_1\eta_1(0) \\
& + \omega_1^2\hat{\eta}_1(s) = \hat{v}_1 B_1\left\{\frac{\alpha}{s^2+\alpha^2}\right\} + \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\}
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
& \left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}\hat{\eta}_1(s) - \left\{s + 2\xi_1\omega_1\right\}\eta_1(0) - \dot{\eta}_1(0) \\
& \qquad = \hat{v}_1 B_1\left\{\frac{\alpha}{s^2+\alpha^2}\right\} + \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\}
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
& \left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}\hat{\eta}_1(s) \\
& \qquad = \hat{v}_1 B_1\left\{\frac{\alpha}{s^2+\alpha^2}\right\} + \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\} + \left\{s + 2\xi_1\omega_1\right\}\eta_1(0) + \dot{\eta}_1(0)
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
\hat{\eta}_1(s) = & \frac{\hat{v}_1 B_1}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}}\left\{\frac{\alpha}{s^2+\alpha^2}\right\} + \frac{\hat{v}_2 B_2}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}}\left\{\frac{\beta}{s^2+\beta^2}\right\} \\
& + \frac{\left\{s + 2\xi_1\omega_1\right\}\eta_1(0) + \dot{\eta}_1(0)}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}}
\end{aligned}
\tag{71}
$$

The solution is found via References 3 and 4. The inverse Laplace transform for the first modal coordinate is

$$
\begin{aligned}
\eta_1(t) = {} & + \frac{\hat{v}_1 B_1}{\left[\left(\alpha^2-\omega_1^2\right)^2+\left(2\xi_1\alpha\omega_1\right)^2\right]}\left\{-\left[2\xi_1\omega_1\alpha\right]\cos(\alpha t)-\left[\alpha^2-\omega_1^2\right]\sin(\alpha t)\right\} \\
& + \frac{\hat{v}_1 B_1\left[\dfrac{\alpha}{\omega_{d,1}}\right]\exp\left(-\xi_1\omega_1 t\right)}{\left[\left(\alpha^2-\omega_1^2\right)^2+\left(2\xi_1\alpha\omega_1\right)^2\right]}\left\{\left[2\xi_1\omega_1\omega_{d,1}\right]\cos\left(\omega_{d,1}t\right)+\left[\alpha^2-\omega_1^2\left(1-2\xi_1^2\right)\right]\sin\left(\omega_{d,1}t\right)\right\} \\
& + \frac{\hat{v}_2 B_2}{\left[\left(\beta^2-\omega_1^2\right)^2+\left(2\xi_1\beta\omega_1\right)^2\right]}\left\{-\left[2\xi_1\omega_1\beta\right]\cos(\beta t)-\left[\beta^2-\omega_1^2\right]\sin(\beta t)\right\} \\
& + \frac{\hat{v}_2 B_2\left[\dfrac{\beta}{\omega_{d,1}}\right]\exp\left(-\xi_1\omega_1 t\right)}{\left[\left(\beta^2-\omega_1^2\right)^2+\left(2\xi_1\beta\omega_1\right)^2\right]}\left\{\left[2\xi_1\omega_1\omega_{d,1}\right]\cos\left(\omega_{d,1}t\right)+\left[\beta^2-\omega_1^2\left(1-2\xi_1^2\right)\right]\sin\left(\omega_{d,1}t\right)\right\} \\
& + \exp\left(-\xi_1\omega_1 t\right)\left\{\eta_1(0)\cos\left(\omega_{d,1}t\right)+\left\{\frac{\dot{\eta}_1(0)+\left(\xi_1\omega_1\right)\eta_1(0)}{\omega_{d,1}}\right\}\sin\left(\omega_{d,1}t\right)\right\}
\end{aligned}
\tag{72}
$$

Similarly,

$$\eta_2(t) =$$

$$+\frac{\hat{w}_1 B_1}{\left[\left(\alpha^2-\omega_2^2\right)^2+\left(2\xi_2\,\alpha\,\omega_2\right)^2\right]}\left\{-\left[2\xi_2\omega_2\,\alpha\right]\cos(\alpha t)-\frac{1}{\alpha}\left[\alpha^2-\omega_2^2\right]\sin(\alpha t)\right\}$$

$$+\frac{\hat{w}_1 B_1\left[\dfrac{\alpha}{\omega_{d,2}}\right]\exp\left(-\xi_2\omega_2 t\right)}{\left[\left(\alpha^2-\omega_2^2\right)^2+\left(2\xi_2\,\alpha\,\omega_2\right)^2\right]}\left\{\left[2\xi_2\omega_2\,\omega_{d,2}\right]\cos\left(\omega_{d,2}t\right)+\left[\alpha^2-\omega_2{}^2\left(1-2\xi_2{}^2\right)\right]\sin\left(\omega_{d,2}t\right)\right\}$$

$$+\frac{\hat{w}_2 B_2}{\left[\left(\beta^2-\omega_2^2\right)^2+\left(2\xi_2\,\beta\,\omega_2\right)^2\right]}\left\{-\left[2\xi_2\omega_2\,\beta\right]\cos(\beta t)-\frac{1}{\beta}\left[\beta^2-\omega_2^2\right]\sin(\beta t)\right\}$$

$$+\frac{\hat{w}_2 B_2\left[\dfrac{\beta}{\omega_{d,2}}\right]\exp\left(-\xi_2\omega_2 t\right)}{\left[\left(\beta^2-\omega_2^2\right)^2+\left(2\xi_2\,\beta\,\omega_2\right)^2\right]}\left\{\left[2\xi_2\omega_2\,\omega_{d,2}\right]\cos\left(\omega_{d,2}t\right)+\left[\beta^2-\omega_2{}^2\left(1-2\xi_2{}^2\right)\right]\sin\left(\omega_{d,2}t\right)\right\}$$

$$+\exp\left(-\xi_2\omega_2 t\right)\left\{\eta_2(0)\cos\left(\omega_{d,2}t\right)+\left\{\frac{\dot{\eta}_2(0)+\left(\xi_2\,\omega_2\right)\eta_2(0)}{\omega_{d,2}}\right\}\sin\left(\omega_{d,2}t\right)\right\}$$

(73)

The physical displacements are found via

$$\bar{x} = \hat{Q}\,\bar{\eta} \qquad (74)$$

References

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.

2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.

3. T. Irvine, Table of Laplace Transforms, Vibrationdata, 2000.

4. T. Irvine, Partial Fraction Expansion, Rev F, Vibrationdata, 2010.

APPENDIX A

Example

Consider the system in Figure 1 with the values in Table A-1.

Assume 5% damping for each mode. Assume zero initial conditions.

Table A-1. Parameters		
Variable	Value	Unit
m_1	3.0	lbf sec^2/in
m_2	2.0	lbf sec^2/in
k_1	400,000	lbf/in
k_2	300,000	lbf/in
k_3	100,000	lbf/in
B_1	100	lbf
B_2	200	lbf
α	55	Hz
β	100	Hz

The mass matrix is

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(A-1)}$$

The stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 700{,}000 & -300{,}000 \\ -300{,}000 & 400{,}000 \end{bmatrix} \tag{A-2}$$

The analysis is performed using a Matlab script.

```
>> twodof_sine_force
 twodof_sine_force.m   ver 1.4  January 17, 2012
 by Tom Irvine  Email: tomirvine@aol.com

 This script calculates the response of a two-degree-of-freedom
 system to sinusoidal force excitation.


 Enter the units system
 1=English  2=metric
 1
 Assume symmetric mass and stiffness matrices.
 Select input mass unit
  1=lbm  2=lbf sec^2/in
 1
 stiffness unit = lbf/in

 Select file input method
   1=file preloaded into Matlab
   2=Excel file
1

 Mass Matrix
 Enter the matrix name:  m_two

 Stiffness Matrix
 Enter the matrix name:  k_two
 Input Matrices

mass =

     3     0
     0     2


stiff =

      700000     -300000
     -300000      400000

 Natural Frequencies
 No.      f(Hz)
```

```
1.        48.552
2.        92.839

  Modes Shapes (column format)

ModeShapes =

    0.3797   -0.4349
    0.5326    0.4651


 Enter the damping ratio for mode 1 0.05
 Enter the damping ratio for mode 2 0.05

 Particpation Factors =

     2.204
   -0.3746

 Enter the first force amplitude (lbf)  100
 Enter the first force frequency (Hz)   55

 Enter the second force amplitude (lbf)  200
 Enter the second force frequency (Hz)   100

 Enter the  first initial displacement (in)  0
 Enter the second initial displacement (in)  0

 Enter the  first initial velocity (in/sec)  0
 Enter the second initial velocity (in/sec)  0

 Enter the sample rate (samples/sec) 10000

 Enter the duration (sec) 0.3

 dof 1 displacement (in)
 max= 0.001015
 min= -0.001373

 dof 2 displacement (in)
 max= 0.002139
 min= -0.00166
```

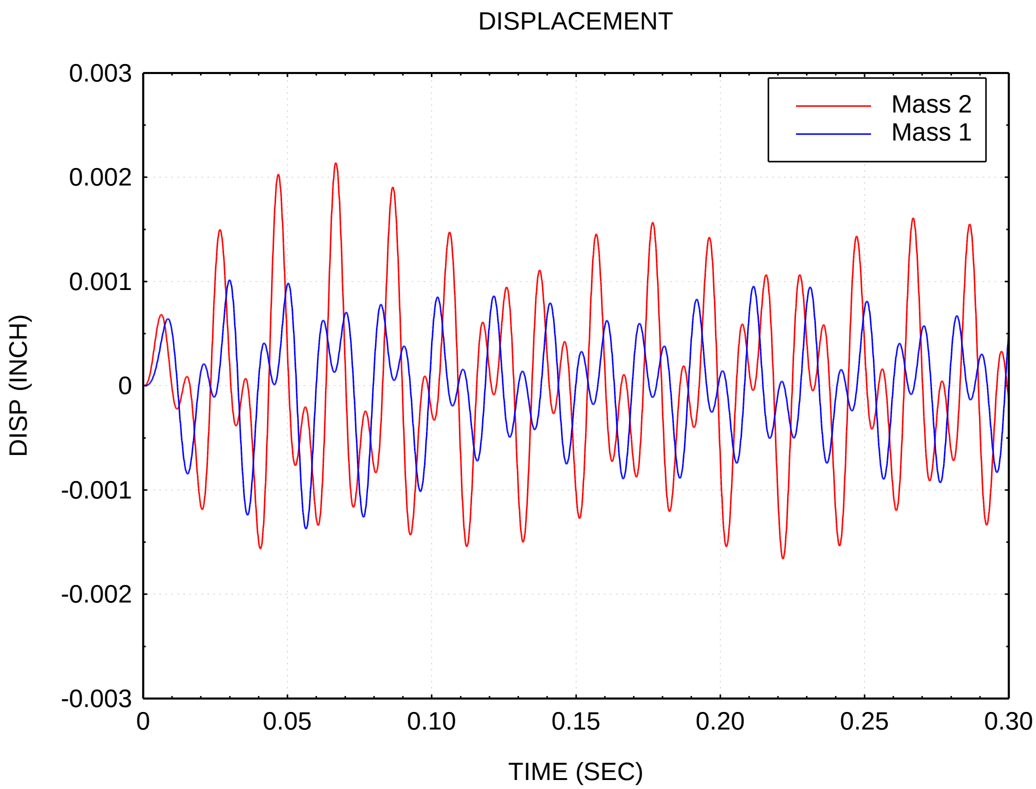
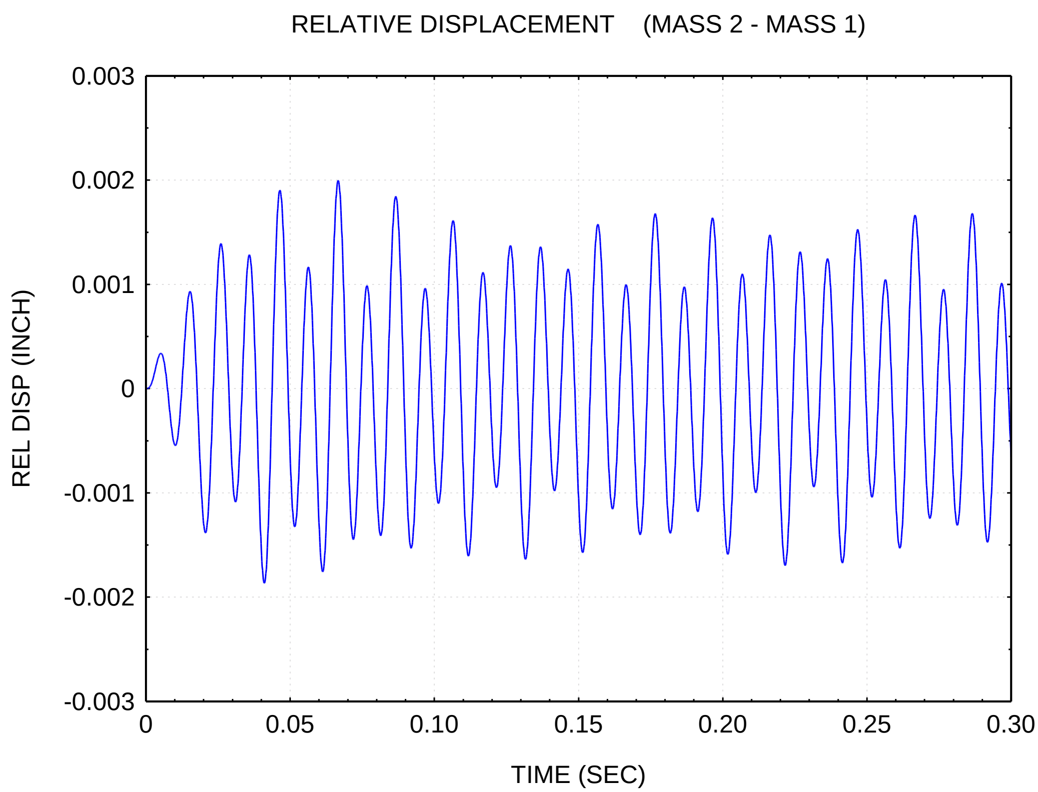

DISPLACEMENT
Mass 2
Mass 1
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
DISP (INCH)
0
0.05
0.10
0.15
0.20
0.25
0.30
TIME (SEC)

Figure A-1.



RELATIVE DISPLACEMENT (MASS 2 - MASS 1)
REL DISP (INCH)
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
0
0.05
0.10
0.15
0.20
0.25
0.30
TIME (SEC)

Figure A-2.

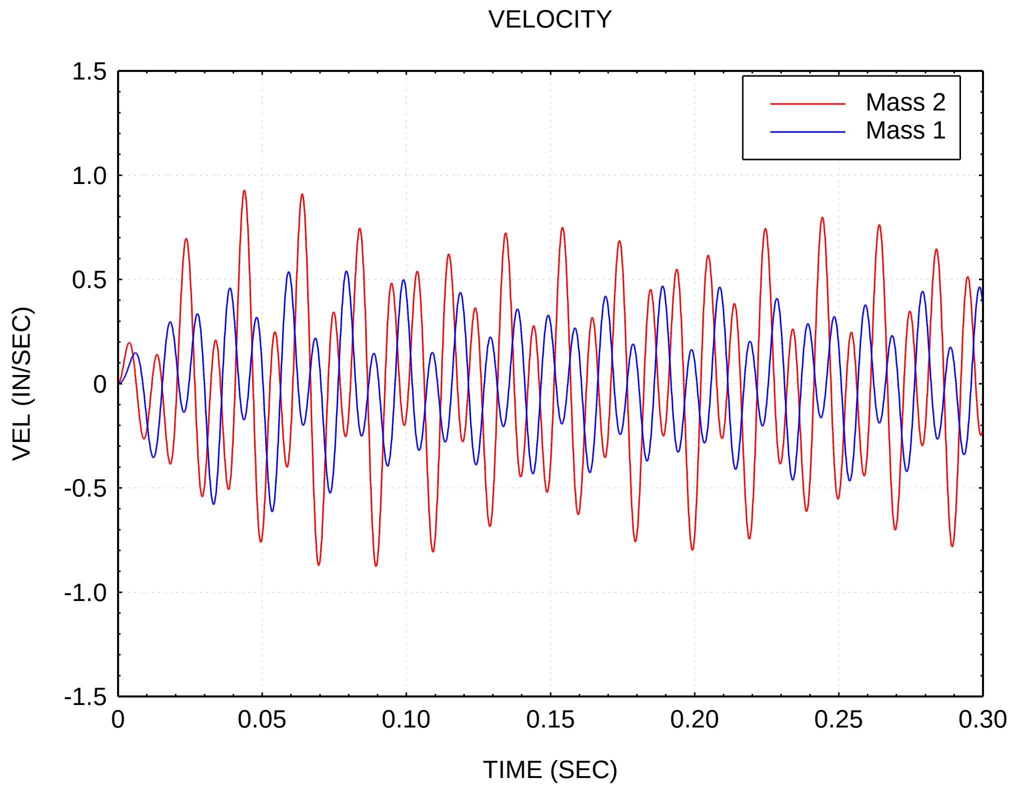

VELOCITY
Mass 2
Mass 1
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
VEL (IN/SEC)
0
0.05
0.10
0.15
0.20
0.25
0.30
TIME (SEC)

Figure A-3.

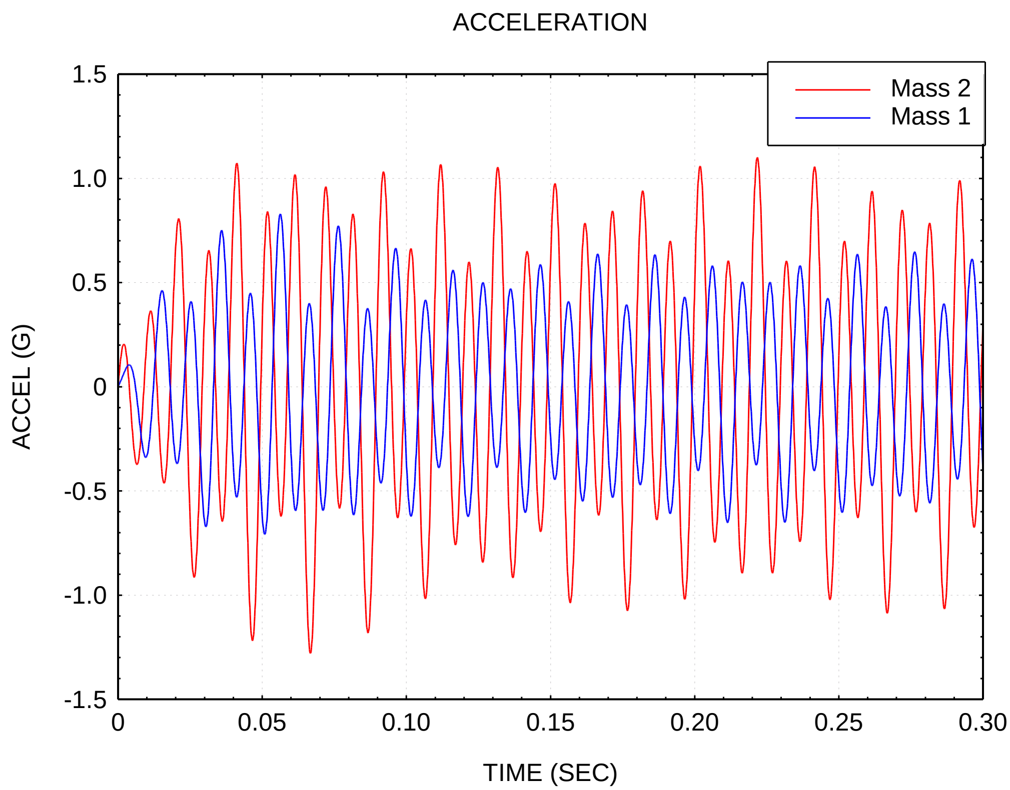

ACCELERATION
Mass 2
Mass 1
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
ACCEL (G)
0
0.05
0.10
0.15
0.20
0.25
0.30
TIME (SEC)

Figure A-4.