Derivation of the Filter Coefficients for the Ramp Invariant Method as Applied to Base Excitation of a Single-degree-of-Freedom System
Revision B

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

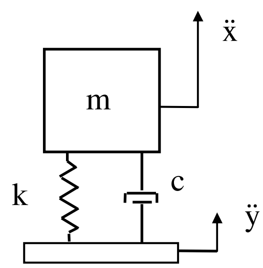

Figure 1.

where

m	=	Mass
c	=	viscous damping coefficient
k	=	Stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

A free-body diagram is shown in Figure 2.

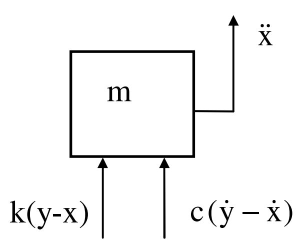

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

Let

$$u = x - y$$
$$\dot{u} = \dot{x} - \dot{y}$$
$$\ddot{u} = \ddot{x} - \ddot{y}$$
$$\ddot{x} = \ddot{u} + \ddot{y}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{u} + \ddot{y}) = -c\dot{u} - ku \tag{3}$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{u} + (c/m)\dot{u} + (k/m)u = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{6}$$

$$(k/m) = \omega_n{}^2 \tag{7}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n{}^2 u = -\ddot{y} \tag{8}$$

Equation (8) does not have a closed-form solution for the general case in which $\ddot{y}$ is an arbitrary function. A convolution integral approach must be used to solve the equation. Note that the impulse response function is embedded in the convolution integral.

Displacement

The damped natural frequency ω_d is

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{9}$$

The displacement equation via a convolution integral is

$$u(t) = -\frac{1}{\omega_d}\int_0^t y(\tau)\left[\exp\{-\xi\omega_n (t - \tau)\}\right]\left[\sin \omega_d (t - \tau)\right]d\tau \tag{10}$$

The corresponding impulse response function for the displacement is

$$\hat{h}_d(t) = \frac{1}{\omega_d}\left[\exp(-\xi\omega_n t)\right]\left[\sin \omega_d t\right] \tag{11}$$

Further details regarding this derivation are given in Reference 2.

The corresponding Laplace transform is

$$H_d(s) = \left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] \tag{12}$$

The Z-transform for the ramp invariant simulation is

$$\hat{H}_d(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{1}{s^2\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right]\right\}$$

where T is the time step

(13)

Evaluate the inverse Laplace transform.

$$L^{-1}\left[\frac{1}{s^2\left(s^2+2\xi\omega_n s+\omega_n{}^2\right)}\right]=\frac{1}{\omega_n^4}\left\{-2\xi\omega_n+\omega_n^2 t+\left[2\xi\omega_n\right]\cos(\omega_d t)+\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d t)\right\}$$

(14)

The Z-transform becomes

$$\hat{H}_d(z)=$$

$$\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{Tz}\right]Z\left\{-2\xi\omega_n+\omega_n^2 t+\exp(-\xi\omega_n t)\left\{\left[2\xi\omega_n\right]\cos(\omega_d t)+\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d t)\right\}\right\}$$

(15)

Let

$$\alpha=\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right] \tag{16}$$

$$\beta=2\xi\omega_n \tag{17}$$

Evaluate the Z-transform.

$$\hat{H}_d(z) = \frac{-\beta}{\omega_n^4}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{z}{z-1}\right]$$

$$+\frac{\omega_n^2}{\omega_n^4}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{Tz}{(z-1)^2}\right]$$

$$+\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{\beta z\{z-\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}}{z^2-2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}}\right]$$

$$+\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{\alpha z\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2-2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}}\right]$$

(18)

$$\hat{H}_d(z) = \frac{1}{\omega_n^4}\left[(-\beta)\left(\frac{z-1}{T}\right)+\omega_n^2\right]$$

$$+\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{T}\right]\left[\frac{\beta\{z-\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}}{z^2-2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}}\right]$$

$$+\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{T}\right]\left[\frac{\alpha\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2-2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}}\right]$$

(19)

$$\hat{H}_d(z) = \frac{1}{\omega_n^4}\left[(-\beta)\left(\frac{z-1}{T}\right) + \omega_n^2\right]$$

$$+\frac{1}{\omega_n^4}\left[\frac{(z-1)^2}{T}\right]\left[\frac{\beta\{z-\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \alpha\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}\right] \tag{20}$$

$$\hat{H}_d(z) = \frac{1}{\omega_n^4 T}\left[-\beta z + \beta + \omega_n^2 T\right]$$

$$+\frac{1}{\omega_n^4 T}\left[(z-1)^2\right]\left[\frac{\beta z + \exp(-\xi\omega_n T)\{\alpha\sin(\omega_d T) - \beta\cos(\omega_d T)\}}{z^2 - 2z\{\exp(-\xi\omega_n T)\}\{\cos(\omega_d T)\} + \{\exp(-2\xi\omega_n T)\}}\right] \tag{21}$$

Let

$$\psi = \exp(-\xi\omega_n T)\{\alpha\sin(\omega_d T) - \beta\cos(\omega_d T)\} \tag{22}$$

$$\rho = -2\{\exp(-\xi\omega_n T)\}\{\cos(\omega_d T)\} \tag{23}$$

$$\lambda = \exp(-2\xi\omega_n T) \tag{24}$$

$$\eta = \beta + \omega_n^2 T \tag{25}$$

By substitution,

$$\hat{H}_d(z) = \frac{1}{\omega_n^4 T}[-\beta z + \eta] + \frac{1}{\omega_n^4 T}\left[(z-1)^2\right]\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right] \tag{26}$$

$$\hat{H}_d(z) = \frac{1}{\omega_n^4 T}[-\beta z + \eta]\left[\frac{z^2 + z\rho + \lambda}{z^2 + z\rho + \lambda}\right] + \frac{1}{\omega_n^4 T}\left[z^2 - 2z + 1\right]\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right] \tag{27}$$

$$\hat{H}_d(z) = \frac{1}{\omega_n^4 T}\left[-\beta z\right]\left[\frac{z^2 + z\rho + \lambda}{z^2 + z\rho + \lambda}\right]$$

$$+\frac{1}{\omega_n^4 T}\left[\eta\right]\left[\frac{z^2 + z\rho + \lambda}{z^2 + z\rho + \lambda}\right]$$

$$+\frac{1}{\omega_n^4 T}\left[z^2\right]\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right]$$

$$+\frac{1}{\omega_n^4 T}\left[-2z\right]\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right]$$

$$+\frac{1}{\omega_n^4 T}\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right] \tag{28}$$

$$\hat{H}_d(z) = \frac{1}{\omega_n^4 T}\left[\frac{-\beta z^3 - \beta\rho z^2 - \beta\lambda z}{z^2 + z\rho + \lambda}\right] + \frac{1}{\omega_n^4 T}\left[\frac{\eta z^2 + \eta\rho z + \eta\lambda}{z^2 + z\rho + \lambda}\right]$$

$$+\frac{1}{\omega_n^4 T}\left[\frac{\beta z^3 + \psi z^2}{z^2 + z\rho + \lambda}\right] + \frac{1}{\omega_n^4 T}\left[\frac{-2\beta z^2 - 2\psi z}{z^2 + z\rho + \lambda}\right] + \frac{1}{\omega_n^4 T}\left[\frac{\beta z + \psi}{z^2 + z\rho + \lambda}\right] \tag{29}$$

$$\hat{H}_d(z) = \frac{-\beta z^3 - \beta\rho z^2 - \beta\lambda z + \eta z^2 + \eta\rho z + \eta\lambda + \beta z^3 + \psi z^2 - 2\beta z^2 - 2\psi z + \beta z + \psi}{\omega_n^4 T\left(z^2 + z\rho + \lambda\right)} \tag{30}$$

$$\hat{H}_d(z) = \frac{(-\beta\rho + \psi + \eta - 2\beta)z^2 + (-\beta\lambda + \eta\rho - 2\psi + \beta)z + (\eta\lambda + \psi)}{\omega_n^4 T\left(z^2 + z\rho + \lambda\right)} \tag{31}$$

Solve for the filter coefficients using the method in Reference 1.

$$\frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{\left[(-\beta\rho + \psi + \eta - 2\beta)z^2 + (-\beta\lambda + \eta\rho - 2\psi + \beta)z + (\eta\lambda + \psi)\right] / \left[m\omega_n^4 T\right]}{z^2 + z\rho + \lambda} \tag{32}$$

Solve for a_1.

$$a_1 = \rho = -2\exp(-\xi\omega_n T)\cos(\omega_d T) \tag{33}$$

Solve for a_2.

$$a_2 = \lambda = \exp(-2\xi\omega_n T) \tag{34}$$

Note that the a_1 and a_2 coefficients are common for displacement, velocity and acceleration.

Solve for b_0.

$$b_0 = [-\beta\rho + \psi + \eta - 2\beta] / \left[\omega_n^4 T\right] \tag{35}$$

$$b_0 = [\psi + \eta - \beta(\rho + 2)] / \left[\omega_n^4 T\right] \tag{36}$$

$$b_0 = \frac{\exp(-\xi\omega_n T)[\alpha\sin(\omega_d T) - \beta\cos(\omega_d T)] + \beta + \omega_n^2 T - \beta[-2\exp(-\xi\omega_n T)\cos(\omega_d T) + 2]}{\omega_n^4 T} \tag{37}$$

$$b_0 = \frac{\exp(-\xi\omega_n T)[\alpha\sin(\omega_d T) - \beta\cos(\omega_d T)] + \beta + \omega_n^2 T + 2\beta[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1]}{\omega_n^4 T} \tag{38}$$

$$b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d T) - 2\xi\omega_n\cos(\omega_d T)\right]}{\omega_n^4 T} + \frac{2\xi\omega_n + \omega_n^2 T + 4\xi\omega_n[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1]}{\omega_n^4 T} \tag{39}$$

$$b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d T)\right] + \omega_n^2 T + 2\xi\omega_n[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1]}{\omega_n^4 T} \tag{40}$$

$$b_0 = \frac{\exp(-\xi\omega_n T)\left[\frac{\omega_n}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d T)\right] + \omega_n T + 2\xi[\exp(-\xi\omega_n T)\cos(\omega_d T) - 1]}{\omega_n^3 T} \tag{41}$$

$$b_0 = \frac{2\xi\left[\exp(-\xi\omega_n T)\cos(\omega_d T)-1\right]+\exp(-\xi\omega_n T)\left[\frac{\omega_n}{\omega_d}\left[2\xi^2-1\right]\sin(\omega_d T)\right]+\omega_n T}{\omega_n^3 T} \tag{42}$$

Solve for b_1.

$$b_1 = \frac{-\beta\lambda+\eta\rho-2\psi+\beta}{\omega_n^4 T} \tag{43}$$

$$b_1 = \frac{\eta\rho-2\psi+\beta(1-\lambda)}{\omega_n^4 T} \tag{44}$$

$b_1 =$

$$\frac{-2\left(\beta+\omega_n^2 T\right)\exp(-\xi\omega_n T)\cos(\omega_d T)-2\exp(-\xi\omega_n T)\left[\alpha\sin(\omega_d T)-\beta\cos(\omega_d T)\right]}{\omega_n^4 T}$$

$$+\frac{\beta\left[1-\exp(-2\xi\omega_n T)\right]}{\omega_n^4 T} \tag{45}$$

$$b_1 = \frac{-2\omega_n^2 T\exp(-\xi\omega_n T)\cos(\omega_d T)-2\exp(-\xi\omega_n T)\alpha\sin(\omega_d T)+\beta\left[1-\exp(-2\xi\omega_n T)\right]}{\omega_n^4 T} \tag{46}$$

$$b_1 = \frac{-2\omega_n^2 T\exp(-\xi\omega_n T)\cos(\omega_d T)-2\exp(-\xi\omega_n T)\alpha\sin(\omega_d T)+\beta\left[1-\exp(-2\xi\omega_n T)\right]}{\omega_n^4 T} \tag{47}$$

$b_1 =$

$$\frac{-2\omega_n^2 T\exp(-\xi\omega_n T)\cos(\omega_d T) - 2\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_n^4 T}$$

$$+\frac{2\xi\omega_n\left[1-\exp(-2\xi\omega_n T)\right]}{\omega_n^4 T}$$

(48)

$b_1 =$

$$\frac{-2\omega_n T\exp(-\xi\omega_n T)\cos(\omega_d T) + 2\xi\left[1-\exp(-2\xi\omega_n T)\right] - 2\frac{\omega_n}{\omega_d}\left[2\xi^2-1\right]\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_n^3 T}$$

(49)

Solve for b_2.

$$b_2 = \frac{\eta\lambda+\psi}{\omega_n^4 T} \tag{50}$$

$$b_2 = \frac{\left(\beta+\omega_n^2 T\right)\exp(-2\xi\omega_n T) + \exp(-\xi\omega_n T)\{\alpha\sin(\omega_d T) - \beta\cos(\omega_d T)\}}{\omega_n^4 T} \tag{51}$$

$b_2 =$

$$\frac{\left(2\xi\omega_n+\omega_n^2 T\right)\exp(-2\xi\omega_n T) + \exp(-\xi\omega_n T)\left\{\frac{\omega_n^2}{\omega_d}\left[-1+2\xi^2\right]\sin(\omega_d T) - 2\xi\omega_n\cos(\omega_d T)\right\}}{\omega_n^4 T}$$

(52)

$$b_2 = \frac{\left(2\xi + \omega_n T\right)\exp(-2\xi\omega_n T) + \exp(-\xi\omega_n T)\left\{\frac{\omega_n}{\omega_d}\left[2\xi^2 - 1\right]\sin(\omega_d T) - 2\xi\cos(\omega_d T)\right\}}{\omega_n^3 T}$$

(53)

The digital recursive filtering relationship for the relative displacement is

$$u_i = -a_1 u_{i-1} - a_2 u_{i-2}$$
$$- b_0 \ddot{y}_i - b_1 \ddot{y}_{i-1} - b_2 \ddot{y}_{i-2}$$

(54)

The digital recursive relationship for the relative displacement is thus

$$u_i =$$

$$+2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t]u_{i-1}$$

$$-\exp[-2\xi\omega_n \Delta t]u_{i-2}$$

$$-\frac{1}{\omega_n^{3}T}\left\{2\xi[\exp(-\xi\omega_n T)\cos(\omega_d T)-1]+\exp(-\xi\omega_n T)\left[\frac{\omega_n}{\omega_d}\left[2\xi^2-1\right]\sin(\omega_d T)\right]+\omega_n T\right\}\ddot{y}_i$$

$$-\frac{1}{\omega_n^{3}T}\left\{-2\omega_n T\exp(-\xi\omega_n T)\cos(\omega_d T)+2\xi[1-\exp(-2\xi\omega_n T)]-2\frac{\omega_n}{\omega_d}\left[2\xi^2-1\right]\exp(-\xi\omega_n T)\sin(\omega_d T)\right\}\ddot{y}_{i-1}$$

$$-\frac{1}{\omega_n^{3}T}\left\{\left(2\xi+\omega_n T\right)\exp(-2\xi\omega_n T)+\exp(-\xi\omega_n T)\left\{\frac{\omega_n}{\omega_d}\left[2\xi^2-1\right]\sin(\omega_d T)-2\xi\cos(\omega_d T)\right\}\right\}\ddot{y}_{i-2}$$

(55)

Relative Velocity

The impulse response function for the relative velocity response is

$$\hat{h}_v(t) = \exp(-\xi\omega_n t)\left[\left(\frac{-\xi\omega_n}{\omega_d}\right)\sin\omega_d t + \cos\omega_d t\right] \tag{56}$$

The corresponding Laplace transform is

$$H_v(s) = \frac{s}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \tag{57}$$

The Z-transform for the ramp invariant simulation is

$$\hat{H}_v(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{s}{s^2\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right]\right\} \tag{58}$$

$$\hat{H}_v(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{1}{s\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right]\right\} \tag{59}$$

Evaluate the inverse Laplace transform per References 3 and 4.

$$L^{-1}\left[\frac{1}{s\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right] = \frac{1}{\omega_n^2} - \frac{1}{\omega_n^2}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{60}$$

The Z-transform for the ramp invariant simulation is

$$\hat{H}_v(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ \frac{1}{\omega_n^2} - \frac{1}{\omega_n^2}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\} \tag{61}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2}\left[\frac{(z-1)^2}{Tz}\right] Z\left\{1-\exp(-\xi\omega_n t)\left[\cos(\omega_d t)+\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]\right\} \tag{62}$$

The Z-transform is evaluated using the method in Reference 5.

$$\hat{H}_v(z) = \frac{1}{\omega_n^2}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{z}{z-1}-\frac{z\left[z-\exp(-\xi\omega_n T)\cos(\omega_d T)\right]+z\dfrac{\xi\omega_n}{\omega_d}\left[\exp(-\xi\omega_n t)\sin(\omega_d t)\right]}{z^2-2z\exp(-\xi\omega_n T)\cos(\omega_d T)+\exp(-2\xi\omega_n T)}\right] \tag{63}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{(z-1)^2}{z}\right]\left[\frac{z}{z-1}-\frac{z\left[z-\exp(-\xi\omega_n T)\cos(\omega_d T)\right]+z\dfrac{\xi\omega_n}{\omega_d}\left[\exp(-\xi\omega_n t)\sin(\omega_d t)\right]}{z^2-2z\exp(-\xi\omega_n T)\cos(\omega_d T)+\exp(-2\xi\omega_n T)}\right] \tag{64}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-(z-1)^2\,\frac{\left[z-\exp(-\xi\omega_n T)\cos(\omega_d T)\right]+\dfrac{\xi\omega_n}{\omega_d}\left[\exp(-\xi\omega_n t)\sin(\omega_d t)\right]}{z^2-2z\exp(-\xi\omega_n T)\cos(\omega_d T)+\exp(-2\xi\omega_n T)}\right] \tag{65}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-(z-1)^2 \frac{z-\exp(-\xi\omega_n T)\left[\cos(\omega_d T)-\dfrac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]}{z^2-2\,z\exp(-\xi\omega_n T)\cos(\omega_d T)+\exp(-2\xi\omega_n T)}\right]$$

(66)

Let

$$\psi = \exp(-\xi\omega_n T)\left[\cos(\omega_d T)-\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{67}$$

$$\rho = -2\{\exp(-\xi\omega_n T)\}\{\cos(\omega_d T)\} \tag{68}$$

$$\lambda = \exp(-2\xi\omega_n T) \tag{69}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-(z-1)^2 \frac{z-\psi}{z^2+\rho\,z+\lambda}\right] \tag{70}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-\left(z^2-2z+1\right)\frac{z-\psi}{z^2+\rho\,z+\lambda}\right] \tag{71}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-\frac{z\left(z^2-2z+1\right)-\psi\left(z^2-2z+1\right)}{z^2+\rho\,z+\lambda}\right] \tag{72}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-\frac{z^3-2z^2+z-\left(\psi z^2-2\psi z+\psi\right)}{z^2+\rho\,z+\lambda}\right] \tag{72}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)-\frac{z^3-2z^2+z-\psi z^2+2\psi z-\psi}{z^2+\rho\,z+\lambda}\right] \tag{74}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1) - \frac{z^3 + (-2-\psi)z^2 + (1+2\psi)z - \psi}{z^2 + \rho z + \lambda}\right] \tag{75}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1) + \frac{-z^3 + (2+\psi)z^2 - (1+2\psi)z + \psi}{z^2 + \rho z + \lambda}\right] \tag{76}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[(z-1)\frac{z^2 + \rho z + \lambda}{z^2 + \rho z + \lambda} + \frac{-z^3 + (2+\psi)z^2 - (1+2\psi)z + \psi}{z^2 + \rho z + \lambda}\right] \tag{77}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{z^3 + \rho z^2 + \lambda z - z^2 - \rho z - \lambda}{z^2 + \rho z + \lambda} + \frac{-z^3 + (2+\psi)z^2 - (1+2\psi)z + \psi}{z^2 + \rho z + \lambda}\right] \tag{78}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{z^3 + (\rho-1)z^2 + (\lambda-\rho)z - \lambda}{z^2 + \rho z + \lambda} + \frac{-z^3 + (2+\psi)z^2 - (1+2\psi)z + \psi}{z^2 + \rho z + \lambda}\right] \tag{79}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{(2+\psi+\rho-1)z^2 + (\lambda-\rho-1-2\psi)z + \psi - \lambda}{z^2 + \rho z + \lambda}\right] \tag{80}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{(\psi+\rho+1)z^2 + (\lambda-\rho-2\psi-1)z + \psi - \lambda}{z^2 + \rho z + \lambda}\right] \tag{81}$$

$$\hat{H}_v(z) = \frac{1}{\omega_n^2 T}\left[\frac{(\psi+\rho+1)z^2 + (\lambda-\rho-2\psi-1)z + \psi - \lambda}{z^2 + \rho z + \lambda}\right] \tag{82}$$

Solve for the filter coefficients using the method in Reference 1.

$$\frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = \frac{1}{\omega_n^2 T}\left[\frac{(\psi+\rho+1)z^2 + (\lambda-\rho-2\psi-1)z + \psi - \lambda}{z^2 + \rho z + \lambda}\right] \tag{83}$$

Solve for a_1.

$$a_1 = \rho = -2\exp(-\xi\omega_n T)\cos(\omega_d T) \tag{84}$$

Solve for a_2.

$$a_2 = \lambda = \exp(-2\xi\omega_n T) \tag{85}$$

Solve for c_0.

$$c_0 = \frac{\psi+\rho+1}{\omega_n^2 T} \tag{86}$$

$$c_0 = \frac{\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] - 2\exp(-\xi\omega_n T)\cos(\omega_d T) + 1}{\omega_n^2 T} \tag{87}$$

$$c_0 = \frac{\exp(-\xi\omega_n T)\left[-\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] + 1}{\omega_n^2 T} \tag{88}$$

Solve for c_1.

$$c_1 = \frac{\lambda - \rho - 2\psi - 1}{\omega_n^2 T} \tag{89}$$

$$c_1 = \frac{\exp(-2\xi\omega_n T) + 2\exp(-\xi\omega_n T)\cos(\omega_d T) - 2\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \dfrac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] - 1}{\omega_n^2 T} \tag{90}$$

$$c_1 = \frac{\exp(-2\xi\omega_n T) + 2\exp(-\xi\omega_n T)\left[\dfrac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] - 1}{\omega_n^2 T} \tag{91}$$

Solve for c_2.

$$c_2 = \frac{\psi - \lambda}{\omega_n^2 T} \tag{92}$$

$$c_2 = \frac{\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \dfrac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] - \exp(-2\xi\omega_n T)}{\omega_n^2 T} \tag{93}$$

The digital recursive filtering relationship for the relative velocity is

$$
\begin{aligned}
\dot{u}_{i} = \; & -a_1 \dot{u}_{i-1} \quad - a_2 \dot{u}_{i-2} \\
& -c_0 \ddot{y}_{i} \quad - c_1 \ddot{y}_{i-1} \quad - c_2 \ddot{y}_{i-2}
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
\dot{u}_{i} = & \\
& + 2\exp\left[-\xi\omega_n \Delta t\right]\cos\left[\omega_d \Delta t\right]\dot{u}_{i-1} \\
& - \exp\left[-2\xi\omega_n \Delta t\right]\dot{u}_{i-2} \\
& + \frac{1}{\omega_n^2 T}\left\{\exp(-\xi\omega_n T)\left[-\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d}\sin\left(\omega_d T\right)\right] + 1\right\}\ddot{y}_{i} \\
& + \frac{1}{\omega_n^2 T}\left\{\exp\left(-2\xi\omega_n T\right) + 2\exp(-\xi\omega_n T)\left[\frac{\xi\omega_n}{\omega_d}\sin\left(\omega_d T\right)\right] - 1\right\}\ddot{y}_{i-1} \\
& + \frac{1}{\omega_n^2 T}\left\{\exp(-\xi\omega_n T)\left[\cos(\omega_d T) - \frac{\xi\omega_n}{\omega_d}\sin\left(\omega_d T\right)\right] - \exp\left(-2\xi\omega_n T\right)\right\}\ddot{y}_{i-2}
\end{aligned}
\tag{95}
$$

Absolute Acceleration

The impulse response function for the absolute acceleration response is

$$\hat{h}_a(t) = \exp(-\xi\omega_n t)\left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n^{\,2}}{\omega_d}\left(1-2\xi^2\right)\sin(\omega_d t)\right] \tag{96}$$

The corresponding Laplace transform is

$$H_a(s) = \left[\frac{2\xi\omega_n s + \omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right] \tag{97}$$

The Z-transform is

$$\hat{H}_a(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{1}{s^2}\left(\frac{2\xi\omega_n s + \omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right)\right]\right\} \tag{98}$$

$$L^{-1}\left[\frac{1}{s^2}\left(\frac{2\xi\omega_n s + \omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right)\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right] \tag{99}$$

$$L^{-1}\left[\frac{1}{s^2}\left(\frac{2\xi\omega_n s + \omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right)\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{(s+\xi\omega_n)^2 + \omega_d^{\,2}}\right] \tag{100}$$

$$L^{-1}\left[\frac{1}{s^2}\left(\frac{2\xi\omega_n s+\omega_n{}^2}{s^2+2\xi\omega_n s+\omega_n{}^2}\right)\right]=L^{-1}\left[t-\left(\frac{1}{\omega_d}\right)\exp(-\xi\omega_n t)\sin(\omega_d t)\right] \tag{101}$$

Evaluate the Z-transform.

$$\hat{H}_a(z)=\left[\frac{(z-1)^2}{Tz}\right]Z\left\{t-\left(\frac{1}{\omega_d}\right)\exp(-\xi\omega_n t)\sin(\omega_d t)\right\} \tag{102}$$

$$\hat{H}_a(z)=\left[\frac{(z-1)^2}{Tz}\right]\left\{\frac{zT}{(z-1)^2}-\frac{\left(\frac{1}{\omega_d}\right)\exp[-\xi\omega_n T]\sin[\omega_d T]}{z^2-2z\{\exp[-\xi\omega_n T]\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}}\right\} \tag{103}$$

$$\hat{H}_a(z)=1-(z-1)^2\frac{\left(\frac{1}{\omega_d T}\right)\exp[-\xi\omega_n T]\sin[\omega_d T]}{z^2-2z\{\exp[-\xi\omega_n T]\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}} \tag{104}$$

$$\hat{H}_a(z)=1-\frac{\left(z^2-2z+1\right)\left(\frac{1}{\omega_d T}\right)\exp[-\xi\omega_n T]\sin[\omega_d T]}{z^2-2z\{\exp[-\xi\omega_n T]\cos[\omega_d T]\}+\{\exp[-2\xi\omega_n T]\}} \tag{105}$$

$$\hat{H}_a(z)=\frac{z^2-2z\exp[-\xi\omega_n T]\cos[\omega_d T]+\exp[-2\xi\omega_n T]-\left(z^2-2z+1\right)\left(\frac{1}{\omega_d T}\right)\exp[-\xi\omega_n T]\sin[\omega_d T]}{z^2-2z\exp[-\xi\omega_n T]\cos[\omega_d T]+\exp[-2\xi\omega_n T]}$$

(106)

$$\hat{H}_a(z) =$$

$$\frac{z^2 - 2z\exp[-\xi\omega_n T]\cos[\omega_d T] + \exp[-2\xi\omega_n T] + \left(-z^2 + 2z - 1\right)\left(\frac{1}{\omega_d T}\right)\exp[-\xi\omega_n T]\sin[\omega_d T]}{z^2 - 2z\exp[-\xi\omega_n T]\cos[\omega_d T] + \exp[-2\xi\omega_n T]} \tag{107}$$

$$\hat{H}_a(z) =$$

$$z^2\left\{\frac{1 - \left(\frac{1}{\omega_d T}\right)\exp(-\xi\omega_n T)\sin(\omega_d T)}{z^2 - 2z\exp[-\xi\omega_n T]\cos(\omega_d T) + \exp[-2\xi\omega_n T]}\right\}$$

$$+ 2z\left\{\frac{\exp(-\xi\omega_n T)\left(-\cos(\omega_d T) + \left(\frac{1}{\omega_d T}\right)\sin(\omega_d T)\right)}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right\}$$

$$+\left\{\frac{\exp(-2\xi\omega_n T) - \left(\frac{1}{\omega_d T}\right)\exp(-\xi\omega_n T)\sin(\omega_d T)}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right\} \tag{108}$$

$$\frac{c_0 z^2 + c_1 z + c_2}{z^2 + a_1 z + a_2} = \quad z^2 \left\{ \frac{1 - \left(\frac{1}{\omega_d T}\right) \exp(-\xi\omega_n T)\sin(\omega_d T)}{z^2 - 2z\exp[-\xi\omega_n T]\cos(\omega_d T) + \exp[-2\xi\omega_n T]} \right\}$$

$$+ 2z \left\{ \frac{\exp(-\xi\omega_n T)\left(-\cos(\omega_d T) + \left(\frac{1}{\omega_d T}\right)\sin(\omega_d T)\right)}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)} \right\}$$

$$+ \left\{ \frac{\exp(-2\xi\omega_n T) - \left(\frac{1}{\omega_d T}\right)\exp(-\xi\omega_n T)\sin(\omega_d T)}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)} \right\} \tag{109}$$

Solve for a_1.

$$a_1 - 2\exp(-\xi\omega_n T)\cos(\omega_d T) \tag{110}$$

Solve for a_2.

$$a_2 = \exp(-2\xi\omega_n T) \tag{111}$$

Solve for c_0.

$$c_0 = \frac{1}{T}\left\{ 1 - \left(\frac{1}{\omega_d T}\right)\exp(-\xi\omega_n T)\sin(\omega_d T) \right\} \tag{112}$$

Solve for c_1.

$$c_1 = 2\left\{ \exp(-\xi\omega_n T)\left(-\cos(\omega_d T) + \left(\frac{1}{\omega_d T}\right)\sin(\omega_d T)\right) \right\} \tag{113}$$

Solve for c_2.

$$c_2 = \left\{ \exp(-2\xi\omega_n T) - \left(\frac{1}{\omega_d T}\right) \exp(-\xi\omega_n T) \sin(\omega_d T) \right\} \tag{114}$$

The digital recursive filtering relationship for absolute acceleration is

$$\begin{aligned} \ddot{x}_i = \; & -a_1 \ddot{x}_{i-1} - a_2 \ddot{x}_{i-2} \\ & -c_0 \ddot{y}_i - c_1 \ddot{y}_{i-1} - c_2 \ddot{y}_{i-2} \end{aligned} \tag{115}$$

$$\begin{aligned} \ddot{x}_i = \; & +2\exp[-\xi\omega_n \Delta t] \cos[\omega_d \Delta t] \ddot{x}_{i-1} \\ & - \exp[-2\xi\omega_n \Delta t] \ddot{x}_{i-2} \\ & + \left\{ 1 - \left(\frac{1}{\omega_d T}\right) \exp(-\xi\omega_n T) \sin(\omega_d T) \right\} \ddot{y}_i \\ & + \left\{ 2\exp(-\xi\omega_n T) \left(-\cos(\omega_d T) + \left(\frac{1}{\omega_d T}\right) \sin(\omega_d T) \right) \right\} \ddot{y}_{i-1} \\ & + \left\{ \exp(-2\xi\omega_n T) - \left(\frac{1}{\omega_d T}\right) \exp(-\xi\omega_n T) \sin(\omega_d T) \right\} \ddot{y}_{i-2} \end{aligned} \tag{116}$$

<u>Relative Acceleration</u>

The impulse response function for the relative acceleration response is

$$\hat{h}_{ra}(t) = \left\{ -\delta(t) + \exp(-\xi\omega_n t)\left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left(1-2\xi^2\right)\sin(\omega_d t)\right]\right\} \tag{117}$$

The corresponding Laplace transform is

$$H_{ra}(s) = -\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \tag{118}$$

The Z-transform is

$$\hat{H}_a(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{1}{s^2}\left(\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right)\right]\right\} \tag{119}$$

$$\hat{H}_a(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ L^{-1}\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right]\right\} \tag{120}$$

Evaluate the inverse Laplace transform.

$$L^{-1}\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = L^{-1}\left[\frac{1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] \tag{121}$$

$$L^{-1}\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = \frac{1}{\omega_d}\exp(-\xi\omega_n t)\sin\left(\omega_d t\right) \tag{122}$$

The Z-transform for the ramp invariant simulation is

$$\hat{H}_{ra}(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{ \frac{1}{\omega_d}\exp(-\xi\omega_n T)\sin\left(\omega_d T\right)\right\} \tag{123}$$

$$\hat{H}_{ra}(z) = \left[\frac{(z-1)^2}{Tz}\right] Z\left\{\frac{1}{\omega_d}\exp(-\xi\omega_n T)\sin\left(\omega_d T\right)\right\} \tag{124}$$

The Z-transform is evaluated as.

$$\hat{H}_{ra}(z) = \frac{1}{\omega_d}\left[\frac{(z-1)^2}{Tz}\right]\left[\frac{z[\exp(-\xi\omega_n T)\sin(\omega_d T)]}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right] \tag{125}$$

$$\hat{H}_{ra}(z) = \frac{1}{\omega_d T}\left[\frac{(z-1)^2[\exp(-\xi\omega_n T)\sin(\omega_d T)]}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right] \tag{126}$$

$$\hat{H}_{ra}(z) = \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_d T}\left[\frac{(z-1)^2}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right] \tag{127}$$

$$\hat{H}_{ra}(z) = \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_d T}\left[\frac{z^2 - 2z + 1}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right] \tag{128}$$

Solve for the filter coefficients.

$$\frac{d_0 z^2 + d_1 z + d_2}{z^2 + a_1 z + a_2} = \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_d T}\left[\frac{z^2 - 2z + 1}{z^2 - 2z\exp(-\xi\omega_n T)\cos(\omega_d T) + \exp(-2\xi\omega_n T)}\right]$$

(129)

Solve for a_1.

$$a_1 = \rho = -2\exp(-\xi\omega_n T)\cos(\omega_d T) \tag{130}$$

Solve for a_2.

$$a_2 = \lambda = \exp(-2\xi\omega_n T) \tag{131}$$

Solve for d_0.

$$d_0 = \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_d T} \tag{132}$$

Solve for d_1.

$$d_1 = -2d_0 \tag{133}$$

Solve for d_2.

$$d_2 = d_0 \tag{134}$$

The digital recursive filtering relationship for the relative acceleration is

$$\ddot{u}_i = -a_1\ddot{u}_{i-1} - a_2\ddot{u}_{i-2} + d_0\ddot{y}_i + d_1\ddot{y}_{i-1} + d_2\ddot{y}_{i-2} \tag{135}$$

$$\ddot{u}_i = +2\exp[-\xi\omega_n T]\cos[\omega_d T]\ddot{u}_{i-1} - \exp[-2\xi\omega_n T]\ddot{u}_{i-2} + \frac{\exp(-\xi\omega_n T)\sin(\omega_d T)}{\omega_d T}\{\ddot{y}_i - 2\ddot{y}_{i-1} + \ddot{y}_{i-2}\} \tag{136}$$

References

1. David O. Smallwood, An Improved Recursive Formula for Calculating Shock Response Spectra, Shock and Vibration Bulletin, No. 51, May 1981.

2. T. Irvine, The Impulse Response Function for Base Excitation, Vibrationdata, 2012.

3. T. Irvine, Table of Laplace Transforms, Revision J, Vibrationdata, 2011.

4. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Revision I, Vibrationdata, 2012.

5. R. Dorf, Modern Control Systems, Addison-Wesley, Reading, Massachusetts, 1980.