

RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A CLASSICAL PULSE BASE EXCITATION Revision A

By Tom Irvine
Email: tomirvine@aol.com

August 18, 1999

Introduction

Consider the single-degree-of-freedom system in Figure A-1.

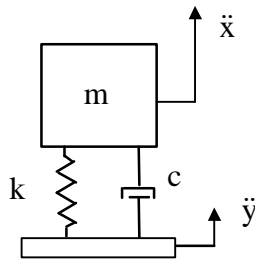


Figure A-1.

where

- m is the mass,
- c is the viscous damping coefficient,
- k is the stiffness,
- x is the absolute displacement of the mass,
- y is the base input displacement.

A free-body diagram is shown in Figure A-2.

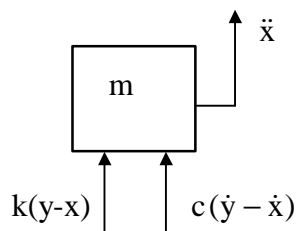


Figure A-2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \quad (\text{A-1})$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \quad (\text{A-2})$$

Let $z = x - y$ (relative displacement)

$$\dot{z} = \dot{x} - \dot{y}$$

$$\ddot{z} = \ddot{x} - \ddot{y}$$

$$\ddot{x} = \ddot{z} + \ddot{y}$$

Substituting the relative displacement terms into equation (A-2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \quad (\text{A-3})$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \quad (\text{A-4})$$

Dividing through by mass yields

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \quad (\text{A-5})$$

By convention,

$$(c/m) = 2\xi\omega_n$$

$$(k/m) = \omega_n^2$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (A-5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2z = -\ddot{y} \quad (\text{A-6})$$

Half-sine Pulse

Consider the half-sine pulse given by equation (B-1).

$$\ddot{y}(t) = \begin{cases} A \sin\left(\frac{\pi t}{T}\right), & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad (\text{B-1})$$

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -A \sin\left(\frac{\pi t}{T}\right), \quad 0 \leq t \leq T \quad (\text{B-2})$$

Let

$$\omega = \frac{\pi}{T} \quad (\text{B-3})$$

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -A \sin(\omega t), \quad 0 \leq t \leq T \quad (\text{B-4})$$

Now take the Laplace transform.

$$L\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z\} = L\{-A \sin(\omega t)\} \quad (\text{B-5})$$

$$\begin{aligned} & s^2 Z(s) - sz(0) - \dot{z}(0) \\ & + 2\xi\omega_n s Z(s) - 2\xi\omega_n z(0) \\ & + \omega_n^2 Z(s) = \frac{-A\omega}{s^2 + \omega^2} \end{aligned} \quad (\text{B-6})$$

$$\left\{ s^2 + 2\xi\omega_n s + \omega_n^2 \right\} Z(s) + \{-1\}\dot{z}(0) + \{-s - 2\xi\omega_n\}z(0) = \frac{-A\omega}{s^2 + \omega^2} \quad (\text{B-7})$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}Z(s) = \dot{z}(0) + \{s + 2\xi\omega_n\}z(0) - \frac{A\omega}{s^2 + \omega^2} \quad (\text{B-8})$$

$$Z(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} - \left\{ \frac{A\omega}{s^2 + \omega^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{B-9})$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \quad (\text{B-10})$$

where

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{B-11})$$

$$Z_f(s) = - \left\{ \frac{A\omega}{s^2 + \omega^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \quad (\text{B-12})$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \quad (\text{B-13})$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2) \quad (\text{B-14})$$

Now define the damped natural frequency,

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \quad (\text{B-15})$$

Substitute equation (B-15) into (B-14),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \quad (\text{B-16})$$

Substitute equation (B-16) into (B-12).

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\}z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (\text{B-17})$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (\text{B-18})$$

$$Z_n(s) = \left\{ \frac{(s + \xi\omega_n)z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \omega_d}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \quad (\text{B-19})$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0) \exp(-\xi\omega_n t) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad (\text{B-20})$$

$$z_n(t) = \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \quad (\text{B-21})$$

Take the first derivative to determine the relative velocity.

$$\begin{aligned} \dot{z}_n(t) = & -\xi\omega_n \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d z(0) \sin(\omega_d t) + \left\{ \dot{z}(0) + (\xi\omega_n)z(0) \right\} \cos(\omega_d t) \right\} \end{aligned} \quad (\text{B-22})$$

$$\begin{aligned} \dot{z}_n(t) = & \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) \cos(\omega_d t) - \xi\omega_n \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d z(0) \sin(\omega_d t) + \left\{ \dot{z}(0) + (\xi\omega_n)z(0) \right\} \cos(\omega_d t) \right\} \end{aligned} \quad (\text{B-23})$$

$$\begin{aligned}
\dot{z}_n(t) = & \\
& \exp(-\xi\omega_n t) \{ \{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\} \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \left[-\omega_d z(0) - \xi\omega_n \left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right] \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-24}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left[-\omega_d^2 z(0) - \xi\omega_n [\dot{z}(0) + (\xi\omega_n)z(0)] \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-25}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left[-\omega_d^2 z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-26}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left[-\omega_n^2 (1 - \xi^2) z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-27}$$

$$\begin{aligned}
\dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\
& + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left[(-\omega_n^2 + \xi^2 \omega_n^2) z(0) - \xi\omega_n \dot{z}(0) - (\xi\omega_n)^2 z(0) \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-28}$$

$$\begin{aligned} \dot{z}_n(t) = & \exp(-\xi\omega_n t) \{ \dot{z}(0) \cos(\omega_d t) \} \\ & + \exp(-\xi\omega_n t) \left\{ \frac{1}{\omega_d} \left[-\omega_n^2 z(0) - \xi\omega_n \dot{z}(0) \right] \sin(\omega_d t) \right\} \end{aligned} \quad (\text{B-29})$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{1}{\omega_d} \left[-\omega_n^2 z(0) - \xi\omega_n \dot{z}(0) \right] \sin(\omega_d t) \right\} \quad (\text{B-30})$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\omega_n z(0) - \xi \dot{z}(0) \right] \sin(\omega_d t) \right\} \quad (\text{B-31})$$

Take the second derivative to determine the acceleration.

$$\begin{aligned} \ddot{z}_n(t) = & -\xi\omega_n \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\omega_n z(0) - \xi \dot{z}(0) \right] \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d \dot{z}(0) \sin(\omega_d t) + \omega_n \left[-\omega_n z(0) - \xi \dot{z}(0) \right] \cos(\omega_d t) \right\} \end{aligned} \quad (\text{B-32})$$

$$\begin{aligned} \ddot{z}_n(t) = & \exp(-\xi\omega_n t) \left\{ -\xi\omega_n \dot{z}(0) \cos(\omega_d t) - \frac{\xi\omega_n^2}{\omega_d} \left[-\omega_n z(0) - \xi \dot{z}(0) \right] \sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t) \left\{ -\omega_d \dot{z}(0) \sin(\omega_d t) + \omega_n \left[-\omega_n z(0) - \xi \dot{z}(0) \right] \cos(\omega_d t) \right\} \end{aligned} \quad (\text{B-33})$$

$$\begin{aligned}\ddot{z}_n(t) &= \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi \dot{z}(0)\}\}\cos(\omega_d t) \\ &\quad + \exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n^2}{\omega_d}\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t)\end{aligned}\tag{B-34}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &\quad + \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_d^2 \dot{z}(0) - \xi\omega_n^2\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t)\end{aligned}\tag{B-35}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &\quad + \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_d^2 \dot{z}(0) + \xi\omega_n^3 z(0) + \xi^2\omega_n^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{B-36}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &\quad + \frac{1}{\omega_d} \exp(-\xi\omega_n t)\left\{-\omega_n^2(1-\xi^2)\dot{z}(0) + \xi\omega_n^3 z(0) + \xi^2\omega_n^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{B-37}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &\quad + \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2 \dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{B-38}$$

$$\begin{aligned}\ddot{z}_n(t) &= -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t) \\ &\quad + \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1 + 2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)\end{aligned}\tag{B-39}$$

$$\begin{aligned}\ddot{z}_n(t) = & -\omega_n \exp(-\xi\omega_n t) \{ \omega_n z(0) + 2\xi \dot{z}(0) \} \cos(\omega_d t) \\ & - \frac{\omega_n^2}{\omega_d} \exp(-\xi\omega_n t) \left\{ -\xi\omega_n z(0) + \left(1 - 2\xi^2\right) \dot{z}(0) \right\} \sin(\omega_d t)\end{aligned}\tag{B-40}$$

$$\begin{aligned}\ddot{z}_n(t) = & \\ & - \exp(-\xi\omega_n t) \left\{ \omega_n [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n^2}{\omega_d} \left[-\xi\omega_n z(0) + \left(1 - 2\xi^2\right) \dot{z}(0) \right] \sin(\omega_d t) \right\}\end{aligned}\tag{B-41}$$

$$\begin{aligned}\ddot{z}_n(t) = & \\ & - \omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\xi\omega_n z(0) + \left(1 - 2\xi^2\right) \dot{z}(0) \right] \sin(\omega_d t) \right\}\end{aligned}\tag{B-42}$$

Recall equation (B-12).

$$Z_f(s) = - \left\{ \frac{A\omega}{s^2 + \omega^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\}\tag{B-43}$$

Expand into partial fractions using Reference 2.

$$\begin{aligned}
 & \left\{ \frac{1}{s^2 + \omega^2} \right\} \left\{ \frac{A\omega}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} = \\
 & + A\omega \frac{[-2\xi\omega_n]s + [-(\omega^2 - \omega_n^2)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + \omega^2]} \\
 & + A\omega \frac{[2\xi\omega_n]s + \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + 2\xi\omega_n s + \omega_n^2]}
 \end{aligned}
 \tag{B-44}$$

$$\begin{aligned}
 Z_f(s) = & \\
 & - A\omega \frac{[-2\xi\omega_n]s + [-(\omega^2 - \omega_n^2)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + \omega^2]} \\
 & - A\omega \frac{[2\xi\omega_n]s + \left[(\omega^2 - \omega_n^2) + (2\xi\omega_n)^2 \right]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right] [s^2 + 2\xi\omega_n s + \omega_n^2]}
 \end{aligned}
 \tag{B-45}$$

$$Z_f(s) =$$

$$- \frac{-2A\xi\omega\omega_n}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\frac{s + \left(\frac{\omega^2 - \omega_n^2}{2\xi\omega_n} \right)}{s^2 + \omega^2} \right]$$

$$- \frac{2A\xi\omega\omega_n}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\frac{s + \left(\frac{\left(\omega^2 - \omega_n^2 \right) + (2\xi\omega_n)^2}{2\xi\omega_n} \right)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right]$$

(B-46)

$$Z_f(s) =$$

$$+ \frac{2A\xi\omega\omega_n}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\frac{s + \left(\frac{\omega^2 - \omega_n^2}{2\xi\omega_n} \right)}{s^2 + \omega^2} \right]$$

$$- \frac{2A\xi\omega\omega_n}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\frac{s + \left(\frac{\left(\omega^2 - \omega_n^2 \right) + (2\xi\omega_n)^2}{2\xi\omega_n} \right)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right]$$

(B-47)

Take the inverse Laplace transform using Reference 1.

$$\begin{aligned}
 z_f(t) = & \\
 & + \frac{2A\xi\omega\omega_n}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[\cos(\omega t) + \left(\frac{\omega^2 - \omega_n^2}{2\xi\omega\omega_n}\right) \sin(\omega t) \right] \\
 & - \frac{2A\xi\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[\cos(\omega_d t) + \left(\frac{\left(\frac{\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2}{2\xi\omega_n} \right) - \xi\omega_n}{\omega_d} \right) \sin(\omega_d t) \right]
 \end{aligned}
 \tag{B-48}$$

$$\begin{aligned}
 z_f(t) = & \\
 & + \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t) \right] \\
 & - \frac{2A\xi\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[\cos(\omega_d t) + \left(\frac{\left(\omega^2 - \omega_n^2\right) + (2\xi\omega_n)^2 - 2(\xi\omega_n)^2}{2\xi\omega_n\omega_d} \right) \sin(\omega_d t) \right]
 \end{aligned}
 \tag{B-49}$$

$$\begin{aligned}
z_f(t) = & \\
& + \frac{A}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) \right] \\
& - \frac{2A\xi\omega\omega_n [\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\cos(\omega_d t) + \left(\frac{\left(\omega^2 - \omega_n^2 \right) + 2(\xi\omega_n)^2}{2\xi\omega_n\omega_d} \right) \sin(\omega_d t) \right]
\end{aligned}
\tag{B-50}$$

$$\begin{aligned}
z_f(t) = & \\
& + \frac{A}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) \right] \\
& - \frac{\frac{2A\xi\omega\omega_n}{2\xi\omega_n\omega_d} [\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\left(\omega^2 - \omega_n^2 \right) + 2(\xi\omega_n)^2 \right) \sin(\omega_d t) \right]
\end{aligned}
\tag{B-51}$$

The relative displacement is thus

$$\begin{aligned}
z_f(t) = & \\
& + \frac{A}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) \right] \\
& - \frac{\frac{A\omega}{\omega_d} \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \sin(\omega_d t) \right]
\end{aligned} \tag{B-52}$$

Take the first derivative to determine the relative velocity.

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\frac{\xi\omega_n}{\omega_d} (2\xi\omega_n\omega_d) - \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \right] \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d) + \frac{\xi\omega_n}{\omega_d} \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \right] \sin(\omega_d t)
\end{aligned} \tag{B-53}$$

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\left(2\xi^2\omega_n^2 - \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \right) \cos(\omega_d t) \right. \\
& \left. + \frac{A\xi\omega_n\omega}{\omega_d} \left[\exp(-\xi\omega_n t) \right] \left[2\omega_d^2 + \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \right] \sin(\omega_d t) \right] \\
\end{aligned}
\tag{B-54}$$

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[2\xi^2\omega_n^2 - \omega^2 + \omega_n^2 (1 - 2\xi^2) \right] \cos(\omega_d t) \\
& + \frac{A\xi\omega_n\omega}{\omega_d} \left[\exp(-\xi\omega_n t) \right] \left[2\omega_n^2 (1 - \xi^2) + \omega^2 - \omega_n^2 (1 - 2\xi^2) \right] \sin(\omega_d t) \\
\end{aligned}
\tag{B-55}$$

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d t) \\
& + \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[2\omega_n^2 - 2\xi^2\omega_n^2 + \omega^2 - \omega_n^2 + 2\xi^2\omega_n^2 \right] \sin(\omega_d t)
\end{aligned}$$

(B-56)

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d t) \\
& + \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\omega_n^2 + \omega^2 \right] \sin(\omega_d t)
\end{aligned}$$

(B-57)

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[\left(\omega^2 - \omega_n^2 \right) \cos(\omega t) - (2\xi\omega\omega_n) \sin(\omega t) \right] \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} \left[\omega_n^2 + \omega^2 \right] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-58}$$

Take the second derivative to determine the relative acceleration.

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{-A\xi\omega_n\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} \left[\omega_n^2 + \omega^2 \right] \sin(\omega_d t) \right\} \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ -\omega_d \left[\omega_n^2 - \omega^2 \right] \sin(\omega_d t) + \xi\omega_n \left[\omega_n^2 + \omega^2 \right] \cos(\omega_d t) \right\}
\end{aligned} \tag{B-59}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ -\xi\omega_n \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d t) - \frac{\xi^2 \omega_n^2}{\omega_d} \left[\omega_n^2 + \omega^2 \right] \sin(\omega_d t) \right\} \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ -\omega_d \left[\omega_n^2 - \omega^2 \right] \sin(\omega_d t) + \xi\omega_n \left[\omega_n^2 + \omega^2 \right] \cos(\omega_d t) \right\}
\end{aligned} \tag{B-60}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ -\xi\omega_n \left[\omega_n^2 - \omega^2 \right] + \xi\omega_n \left[\omega_n^2 + \omega^2 \right] \right\} \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ -\frac{\xi^2 \omega_n^2}{\omega_d} \left[\omega_n^2 + \omega^2 \right] - \omega_d \left[\omega_n^2 - \omega^2 \right] \right\} \sin(\omega_d t)
\end{aligned} \tag{B-61}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{1}{\omega_d} \right\} \left\{ -\xi^2\omega_n^2 \left[\omega_n^2 + \omega^2 \right] - \omega_d^2 \left[\omega_n^2 - \omega^2 \right] \right\} \sin(\omega_d t)
\end{aligned} \tag{B-62}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{1}{\omega_d} \right\} \left\{ \xi^2\omega_n^4 - \xi^2\omega_n^2\omega^2 - \omega_d^2\omega_n^2 + \omega_d^2\omega^2 \right\} \sin(\omega_d t)
\end{aligned} \tag{B-63}$$

$$\begin{aligned}
\dot{z}_f(t) = & \\
& + \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- (2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2 \right) \cos(\omega t) \right] \\
& + \frac{\frac{A\xi\omega_n\omega}{\omega_d} \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \sin(\omega_d t) \right] \\
& - \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- (2\xi\omega_n\omega_d)\sin(\omega_d t) + \left(\omega^2 - \omega_n^2 (1 - 2\xi^2) \right) \cos(\omega_d t) \right]
\end{aligned}$$

(B-64)

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t) \right] \\
& + \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{1}{\omega_d} \right\} \left\{ -\xi^2\omega_n^4 - \xi^2\omega_n^2\omega^2 - \omega_n^4(1 - \xi^2) + \omega^2\omega_n^2(1 - \xi^2) \right\} \sin(\omega_d t)
\end{aligned}$$

(B-65)

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{\omega_n^2}{\omega_d} \right\} \left\{ -\xi^2\omega_n^2 - \xi^2\omega^2 - \omega_n^2(1-\xi^2) + \omega^2(1-\xi^2) \right\} \sin(\omega_d t)
\end{aligned}$$

(B-66)

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{\omega_n^2}{\omega_d} \right\} \left\{ -\xi^2\omega_n^2 - \xi^2\omega^2 - \omega_n^2 + \xi^2\omega_n^2 + \omega^2 - \xi^2\omega^2 \right\} \sin(\omega_d t)
\end{aligned}$$

(B-67)

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{\omega_n^2}{\omega_d} \right\} \left\{ -2\xi^2\omega^2 - \omega_n^2 + \omega^2 \right\} \sin(\omega_d t)
\end{aligned}
\tag{B-68}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ 2\xi\omega_n\omega^2 \right\} \cos(\omega_d t) \\
& + \frac{A\omega \exp(-\xi\omega_n t)}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \frac{\omega_n^2}{\omega_d} \right\} \left\{ -\omega_n^2 + \omega^2 (1 - 2\xi^2) \right\} \sin(\omega_d t)
\end{aligned}
\tag{B-69}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[2\xi\omega_n\omega^2 \right] \cos(\omega_d t) + \left[\frac{\omega_n^2}{\omega_d} \right] \left\{ -\omega_n^2 + \omega^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right\}
\end{aligned}
\tag{B-70}$$

$$\begin{aligned}
\ddot{z}_f(t) = & \\
& + \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left[- \left(\omega^2 - \omega_n^2 \right) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega\omega_n \left[\exp(-\xi\omega_n t) \right]}{\left[\left(\omega^2 - \omega_n^2 \right)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[2\xi\omega^2 \right] \cos(\omega_d t) + \left[\frac{\omega_n}{\omega_d} \right] \left\{ -\omega_n^2 + \omega^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right\}
\end{aligned}
\tag{B-71}$$

The total relative displacement for $0 \leq t \leq T$ is

$$z(t) = z_n(t) + z_f(t) \tag{B-72}$$

Substitute equations (B-21) and (B-53) into (B-73).

$$\begin{aligned}
z(t) &= \exp(-\xi\omega_n t) \left\{ z(0) \cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\} \\
&+ \frac{A}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega\omega_n) \cos(\omega t) + (\omega^2 - \omega_n^2) \sin(\omega t) \right] \\
&- \frac{\frac{A\omega}{\omega_d} [\exp(-\xi\omega_n t)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d) \cos(\omega_d t) + (\omega^2 - \omega_n^2 (1 - 2\xi^2)) \sin(\omega_d t) \right]
\end{aligned} \tag{B-73}$$

The total relative velocity for $0 \leq t \leq T$ is

$$\dot{z}(t) = \dot{z}_n(t) + \dot{z}_f(t) \tag{B-74}$$

$$\begin{aligned}
\dot{z}(t) &= \exp(-\xi\omega_n t) \left\{ \dot{z}(0) \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \{-\omega_n z(0) - \xi \dot{z}(0)\} \sin(\omega_d t) \right\} \\
&+ \frac{A\omega}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(\omega^2 - \omega_n^2) \cos(\omega t) - (2\xi\omega\omega_n) \sin(\omega t) \right] \\
&+ \frac{A\omega [\exp(-\xi\omega_n t)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ [\omega_n^2 - \omega^2] \cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d} [\omega_n^2 + \omega^2] \sin(\omega_d t) \right\}
\end{aligned} \tag{B-75}$$

The total relative acceleration for $0 \leq t \leq T$ is

$$\ddot{z}(t) = \ddot{z}_n(t) + \ddot{z}_f(t) \quad (\text{B-76})$$

$$\begin{aligned} \ddot{z}(t) = & -\omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\xi\omega_n z(0) + (1 - 2\xi^2) \dot{z}(0) \right] \sin(\omega_d t) \right\} \\ & + \frac{A\omega^2}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(\omega^2 - \omega_n^2) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\ & + \frac{A\omega\omega_n [\exp(-\xi\omega_n t)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left\{ 2\xi\omega^2 \right\} \cos(\omega_d t) + \left\{ \frac{\omega_n}{\omega_d} \right\} \left\{ -\omega_n^2 + \omega^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right\} \end{aligned}$$

(B-77)

The total absolute acceleration for $0 \leq t \leq T$ is

$$\ddot{x}(t) = \ddot{z}_n(t) + \ddot{z}_f(t) + \ddot{y}(t) \quad (\text{B-78})$$

$$\begin{aligned}
\ddot{x}(t) = & -\omega_n \exp(-\xi\omega_n t) \left\{ [\omega_n z(0) + 2\xi \dot{z}(0)] \cos(\omega_d t) + \frac{\omega_n}{\omega_d} \left[-\xi\omega_n z(0) + (1 - 2\xi^2) \dot{z}(0) \right] \sin(\omega_d t) \right\} \\
& + \frac{A\omega^2}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[-(\omega^2 - \omega_n^2) \sin(\omega t) - (2\xi\omega\omega_n) \cos(\omega t) \right] \\
& + \frac{A\omega\omega_n [\exp(-\xi\omega_n t)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[2\xi\omega^2 \right] \cos(\omega_d t) + \left[\frac{\omega_n}{\omega_d} \right] \left\{ -\omega_n^2 + \omega^2 (1 - 2\xi^2) \right\} \sin(\omega_d t) \right\} \\
& + A \sin(\omega t)
\end{aligned} \tag{B-79}$$

The relative displacement at $t = T$ is

$$\begin{aligned}
z(T) = & \exp(-\xi\omega_n T) \left\{ z(0) \cos(\omega_d T) + \left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right] \sin(\omega_d T) \right\} \\
& + \frac{A}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega\omega_n) \cos(\omega T) + (\omega^2 - \omega_n^2) \sin(\omega T) \right] \\
& - \frac{\frac{A\omega}{\omega_d} [\exp(-\xi\omega_n T)]}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(2\xi\omega_n\omega_d) \cos(\omega_d T) + (\omega^2 - \omega_n^2 (1 - 2\xi^2)) \sin(\omega_d T) \right]
\end{aligned} \tag{B-80}$$

The relative velocity at $t = T$ is

$$\begin{aligned}
\dot{z}(T) = & \exp(-\xi\omega_n T) \left\{ \dot{z}(0) \cos(\omega_d T) + \frac{\omega_n}{\omega_d} \{-\omega_n z(0) - \xi \dot{z}(0)\} \sin(\omega_d T) \right\} \\
& + \frac{A\omega}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left[(\omega^2 - \omega_n^2) \cos(\omega T) - (2\xi\omega\omega_n) \sin(\omega T) \right] \\
& + \frac{A\omega \exp(-\xi\omega_n T)}{\left[(\omega^2 - \omega_n^2)^2 + (2\xi\omega\omega_n)^2 \right]} \left\{ \left[\omega_n^2 - \omega^2 \right] \cos(\omega_d T) + \frac{\xi\omega_n}{\omega_d} \left[\omega_n^2 + \omega^2 \right] \sin(\omega_d T) \right\}
\end{aligned} \tag{B-81}$$

The relative displacement for $t > T$ is found by adding a delay into equation (B-21).

$$z(t) = \exp(-\xi\omega_n(t-T)) \left\{ z(T) \cos(\omega_d(t-T)) + \left\{ \frac{\dot{z}(T) + (\xi\omega_n)z(T)}{\omega_d} \right\} \sin(\omega_d(t-T)) \right\} \tag{B-82}$$

Note that the absolute displacement is equal to the relative displacement for $t > T$.

The relative velocity for $t > T$ is

$$\dot{z}(t) = \exp(-\xi\omega_n(t-T)) \left\{ \dot{z}(T) \cos(\omega_d(t-T)) + \frac{\omega_n}{\omega_d} \{-\omega_n z(T) - \xi \dot{z}(T)\} \sin(\omega_d(t-T)) \right\} \tag{B-83}$$

Note that the absolute velocity is equal to the relative velocity for $t > T$.

The relative acceleration for $t > T$ is

$$\begin{aligned} \ddot{z}(t) = & \\ & - \omega_n \exp(-\xi \omega_n (t - T)) \{ [\omega_n z(T) + 2\xi \dot{z}(T)] \cos(\omega_d (t - T)) \} \\ & - \omega_n \exp(-\xi \omega_n (t - T)) \left\{ \frac{\omega_n}{\omega_d} \left[-\xi \omega_n z(T) + (1 - 2\xi^2) \dot{z}(T) \right] \sin(\omega_d (t - T)) \right\} \end{aligned} \tag{B-84}$$

Note that the absolute acceleration is equal to the relative acceleration for $t > T$.

$$\begin{aligned} \ddot{x}(t) = & \\ & - \omega_n \exp(-\xi \omega_n (t - T)) \{ [\omega_n z(T) + 2\xi \dot{z}(T)] \cos(\omega_d (t - T)) \} \\ & - \omega_n \exp(-\xi \omega_n (t - T)) \left\{ \frac{\omega_n}{\omega_d} \left[-\xi \omega_n z(T) + (1 - 2\xi^2) \dot{z}(T) \right] \sin(\omega_d (t - T)) \right\} \end{aligned} \tag{B-85}$$

Example

An example is shown in Figure B-1. The input is an 10 G, 11 millisecond half-sine shock.

RESPONSE OF SDOF SYSTEM ($f_n = 100$ Hz, $\xi = 0.05$)
TO 10 G, 11 msec HALF-SINE SHOCK

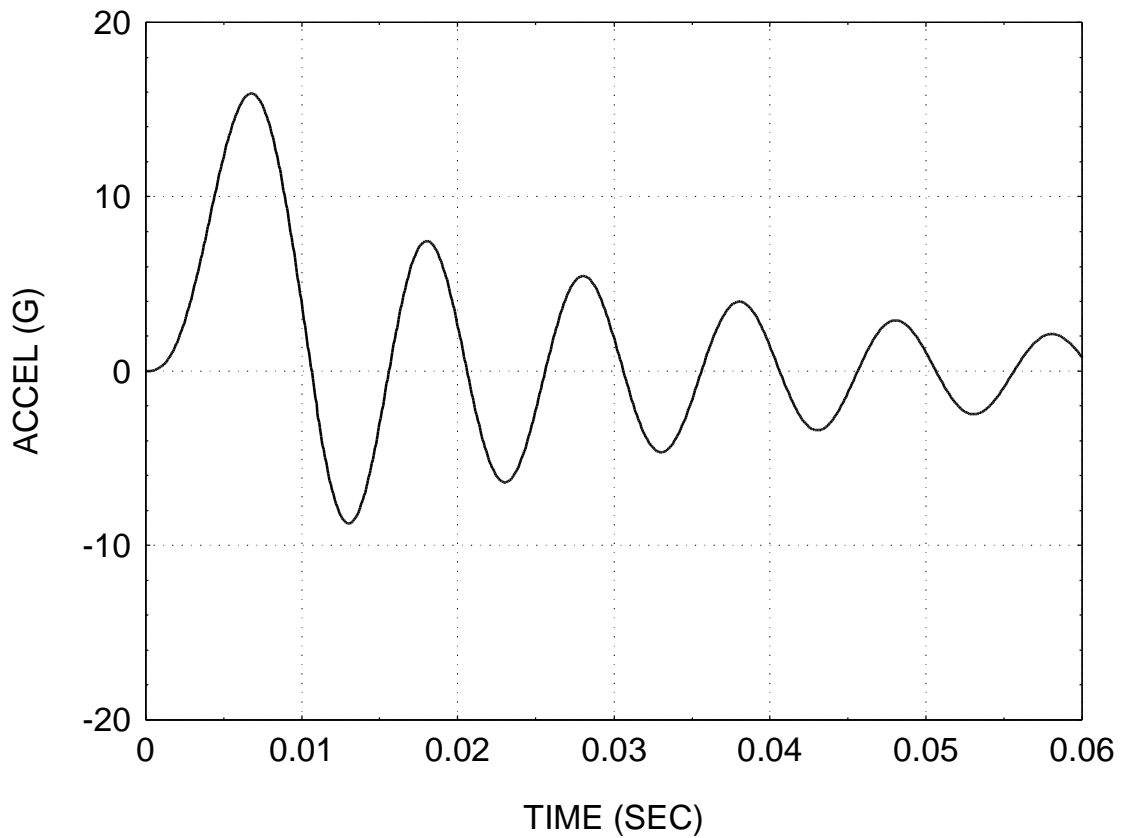


Figure B-1.

References

1. T. Irvine, Table of Laplace Transforms, Vibrationdata.com Publications, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata.com Publications, 1999.