TRANSVERSE VIBRATION OF A ROTATING BEAM
VIA THE FINITE ELEMENT METHOD

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Introduction

The purpose of this tutorial is to derive for a method for analyzing rotating beam vibration using the finite element method. The method is based on Reference 1.

Theory

Consider a beam, such as the cantilever beam in Figure 1. Assume that the hub radius is negligible compared with the length L.

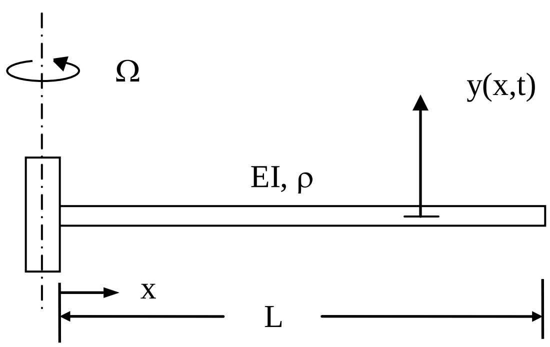

Figure 1.

The variables are

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is mass per length
Ω	is the hub rotational frequency (radians/sec)

The product EI is the bending stiffness.

Let y(x,t) represent the displacement of the beam as a function of space and time.

The free, transverse vibration of the beam is governed by the equation, as taken from Reference 1.

$$EI\frac{d^4}{\partial x^4}y(x,t)-\frac{1}{2}\rho\,\Omega^2L^2\frac{\partial}{\partial x}\left[\left(1-\frac{x^2}{L^2}\right)\frac{\partial\,y(x,t)}{\partial x}\right]+\rho\,\frac{\partial^2}{\partial t^2}y(x,t)=0 \qquad (1)$$

Equation (1) neglects rotary inertia and shear deformation. Note that it is also independent of the boundary conditions, which are applied as constraint equations.

The following derivation is an extension of the one given in Reference 2.

Assume that the solution of equation (1) is separable in time and space.

$$y(x,t) = Y(x)f(t) \qquad (2)$$

$$EI\frac{d^4}{\partial x^4}Y(x)f(t)-\frac{1}{2}\rho\,\Omega^2L^2\frac{\partial}{\partial x}\left[\left(1-\frac{x^2}{L^2}\right)\frac{\partial}{\partial x}Y(x)f(t)\right]+\rho\frac{\partial^2}{\partial t^2}Y(x)f(t)=0 \qquad (3)$$

$$f(t)\left\{EI\frac{\partial^4}{\partial x^4}Y(x)\right\}-\frac{1}{2}f(t)\rho\,\Omega^2L^2\frac{\partial}{\partial x}\left[\left(1-\frac{x^2}{L^2}\right)\frac{\partial}{\partial x}Y(x)\right]+Y(x)\,\rho\,\frac{\partial^2 f(t)}{\partial t^2}=0$$

(4)

The partial derivatives change to ordinary derivatives.

$$f(t)\left\{EI\frac{d^4}{dx^4}Y(x)\right\}-\frac{1}{2}f(t)\rho\,\Omega^2L^2\frac{d}{dx}\left[\left(1-\frac{x^2}{L^2}\right)\frac{d}{dx}Y(x)\right]+Y(x)\,\rho\,\frac{d^2 f(t)}{dt^2}=0 \qquad (5)$$

$$\frac{1}{Y(x)}\left\{\frac{EI}{\rho}\frac{d^4}{dx^4}Y(x)\right\}-\frac{1}{2}\frac{1}{Y(x)}\,\Omega^2L^2\frac{d}{dx}\left[\left(1-\frac{x^2}{L^2}\right)\frac{d}{dx}Y(x)\right]=-\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2}$$

(6)

The left-hand side of equation (6) depends on x only. The right hand side depends on t only. Both x and t are independent variables. Thus equation (6) only has a solution if both sides are constant. Let ω^2 be the constant.

$$\frac{1}{Y(x)}\left\{\frac{EI}{\rho}\frac{d^4}{dx^4}Y(x)\right\}-\frac{1}{2}\frac{1}{Y(x)}\,\Omega^2L^2\,\frac{d}{dx}\left[\left(1-\frac{x^2}{L^2}\right)\frac{d}{dx}Y(x)\right]=-\frac{1}{f(t)}\frac{d^2\,f(t)}{dt^2}=\omega^2 \tag{7}$$

Equation (7) yields two independent equations.

$$EI\frac{d^4}{dx^4}Y(x)-\frac{1}{2}\rho\Omega^2L^2\frac{d}{dx}\left[\left(1-\frac{x^2}{L^2}\right)\frac{d}{dx}Y(x)\right]-\rho\,\omega^2\,Y(x)=0 \tag{8}$$

$$EI\frac{d^4}{dx^4}Y(x)-\frac{1}{2}\rho\Omega^2L^2\left[\left(-\frac{2x}{L^2}\right)\frac{d}{dx}Y(x)\right]$$
$$-\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{d^2}{dx^2}Y(x)\right]-\rho\,\omega^2\,Y(x)=0 \tag{9}$$

$$EI\frac{d^4}{dx^4}Y(x)-\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{d^2}{dx^2}Y(x)\right]+\rho\Omega^2\left[x\frac{d}{dx}Y(x)\right]-\rho\,\omega^2\,Y(x)=0 \tag{10}$$

$$\frac{d^2}{dt^2}\,f(t)+\omega^2 f(t)=0 \tag{11}$$

Equation (10) is a homogeneous, forth order, ordinary differential equation. The weighted residual method is applied to equation (10). This method is suitable for boundary value problems.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (10) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of Y and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x)\left\{ EI\frac{d^4Y(x)}{dx^4} - \frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{d^2Y(x)}{dx^2}\right] + \rho\Omega^2\left[x\frac{dY(x)}{dx}\right] - \rho\omega^2\,Y(x)\right\}dx = 0 \tag{10}$$

The test function $\phi(x)$ can be regarded as a virtual displacement. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire beam.

$$\int_a^b \phi(x)\left\{ EI\frac{d^4Y(x)}{dx^4} - \frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{d^2Y(x)}{dx^2}\right] + \rho\Omega^2\left[x\frac{dY(x)}{dx}\right] - \rho\omega^2\,Y(x)\right\}dx = 0 \tag{11}$$

$$\int_a^b \phi(x)\left\{ EI\frac{d^4}{dx^4}Y(x)\right\}dx - \int_a^b \phi(x)\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{d^2Y(x)}{dx^2}\right]\right\}dx$$
$$+\int_a^b \phi(x)\left\{\rho\Omega^2\left[x\frac{dY(x)}{dx}\right]\right\}dx - \int_a^b \phi(x)\left\{\rho\,\omega^2\,Y(x)\right\}dx = 0 \tag{12}$$

Integrate the first integral by parts.

$$\int_a^b \frac{d}{dx}\left\{\phi(x)\left[EI\frac{d^3}{dx^3}Y(x)\right]\right\}dx - \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{EI\frac{d^3}{dx^3}Y(x)\right\}dx$$

$$-\int_a^b \frac{d}{dx}\left\{\phi(x)\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$+\int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$+\int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left[\frac{d}{dx}\left(1-\frac{x^2}{L^2}\right)\right]\left[\frac{dY(x)}{dx}\right]\right]\right\}dx$$

$$+\int_a^b\phi(x)\left\{\rho\Omega^2\left[x\frac{dY(x)}{dx}\right]\right\}dx$$

$$-\int_a^b\phi(x)\left\{\rho\,\omega^2\,Y(x)\right\}dx = 0 \tag{13}$$

Note that

$$\int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left[\frac{d}{dx}\left(1-\frac{x^2}{L^2}\right)\right]\left[\frac{dY(x)}{dx}\right]\right]\right\}dx + \int_a^b\phi(x)\left\{\rho\Omega^2\left[x\frac{dY(x)}{dx}\right]\right\}dx = 0 \tag{14}$$

Thus,

$$\int_a^b \frac{d}{dx}\left\{\phi(x)\left[EI\frac{d^3}{dx^3}Y(x)\right]\right\}dx - \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{EI\frac{d^3}{dx^3}Y(x)\right\}dx$$

$$-\int_a^b \frac{d}{dx}\left\{\phi(x)\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx + \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$-\int_a^b\phi(x)\left\{\rho\ \omega^2\ Y(x)\right\}dx = 0$$

(15)

$$\left\{\phi(x)\left[EI\frac{d^3}{dx^3}Y(x)\right]\right\}\Bigg|_a^b - \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{EI\frac{d^3}{dx^3}Y(x)\right\}dx$$

$$-\left\{\phi(x)\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}\Bigg|_a^b + \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$-\int_a^b\phi(x)\left\{\rho\ \omega^2\ Y(x)\right\}dx = 0$$

(16)

Integrate by parts again.

$$\left\{ \phi(x) \left[EI \frac{d^3}{dx^3} Y(x) \right] \right\} \Bigg|_a^b - \int_a^b \frac{d}{dx} \left\{ \left[\frac{d}{dx} \phi(x) \right] \left[EI \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$+ \int_a^b \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[EI \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$- \left\{ \phi(x) \frac{1}{2} \rho \Omega^2 L^2 \left[\left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right] \right\} \Bigg|_a^b + \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{1}{2} \rho \Omega^2 L^2 \left[\left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right] \right\} dx$$

$$- \int_a^b \phi(x) \left\{ \rho \, \omega^2 \, Y(x) \right\} dx = 0$$

(17)

$$\left\{ \phi(x) \frac{d}{dx} \left[EI \frac{d^2}{dx^2} Y(x) \right] \right\} \Bigg|_a^b - \left\{ \left[\frac{d}{dx} \phi(x) \right] \left[EI \frac{d^2}{dx^2} Y(x) \right] \right\} \Bigg|_a^b$$

$$+ \int_a^b \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[EI \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$- \left\{ \phi(x) \frac{1}{2} \rho \Omega^2 L^2 \left[\left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right] \right\} \Bigg|_a^b + \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{1}{2} \rho \Omega^2 L^2 \left[\left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right] \right\} dx$$

$$- \int_a^b \phi(x) \left\{ \rho \, \omega^2 \, Y(x) \right\} dx = 0$$

(18)

The essential boundary conditions for a cantilever beam are

$$Y(a) = 0 \tag{19}$$

$$\left.\frac{dY}{dx}\right|_{x=a} = 0 \tag{20}$$

Thus, the test functions must satisfy

$$\phi(a) = 0 \tag{21}$$

$$\left.\frac{d\phi}{dx}\right|_{x=a} = 0 \tag{22}$$

The natural boundary conditions are

$$\left.\frac{d}{dx}\left[EI\frac{d^2}{dx^2}Y(x)\right]\right|_{x=b} = 0 \tag{23}$$

and

$$\left.\left[EI\frac{d^2}{dx^2}Y(x)\right]\right|_{x=b} = 0 \tag{24}$$

Note that b=L.

$$-\left\{\phi(x)\frac{1}{2}\rho\Omega^2L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}\Bigg|_a^b = 0 \tag{25}$$

Apply the boundary conditions to equation (18). The result is

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2}\phi(x)\right]\left[EI\frac{d^2}{dx^2}Y(x)\right] \right\} dx + \int_a^b \left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\,\Omega^2 L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$-\int_a^b \phi(x)\left\{\rho\,\omega^2\,Y(x)\right\}dx = 0 \tag{26}$$

Note that equation (26) would also be obtained for other simple boundary condition cases.

Now consider that the beam consists of number of segments, or elements. The elements are arranged geometrically in series form.

Furthermore, the endpoints of each element are called nodes.

The following equation must be satisfied for each element.

$$\int \left\{ \left[\frac{d^2}{dx^2}\phi(x)\right]\left[EI\frac{d^2}{dx^2}Y(x)\right] \right\} dx + \int \left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{1}{2}\rho\,\Omega^2 L^2\left[\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right]\right\}dx$$

$$-\int \phi(x)\left\{\rho\,\omega^2\,Y(x)\right\}dx = 0 \tag{27}$$

$$EI\int \left\{ \left[\frac{d^2\phi(x)}{dx^2}\right]\left[\frac{d^2Y(x)}{dx^2}\right] \right\} dx + \frac{1}{2}\rho\,\Omega^2 L^2\int \left\{\frac{d\,\phi(x)}{dx}\right\}\left\{\left(1-\frac{x^2}{L^2}\right)\frac{dY(x)}{dx}\right\}dx$$

$$-\rho\,\omega^2\int \phi(x)\,Y(x)\,dx = 0 \tag{28}$$

Now express the displacement function Y(x) in terms of nodal displacements y_{j-1} and y_j as well as the rotations θ_{j-1} and θ_j .

$$Y(x) = L_1\, y_{j-1} + L_2\, y_j + L_3\, h\theta_{j-1} + L_4\, h\theta_{j-1}\,, \qquad (j-1)h \le x \le jh \tag{29}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ .

$$\xi = j - x/h \tag{30}$$

Note that h is the segment length.

The displacement function becomes.

$$Y(\xi) = L_1\, y_{j-1} + L_2\, h\theta_{j-1} + L_3\, y_j + L_4\, h\theta_{j-1}\,, \qquad 0 \le \xi \le 1 \tag{31}$$

The slope equation is

$$Y\,'(\xi) = L_1'\, y_{j-1} + L_2'\, h\theta_{j-1} + L_3'\, y_j + L_4'\, h\theta_{j-1}\,, \qquad 0 \le \xi \le 1 \tag{32}$$

The displacement function is represented terms of natural coordinates in Figure 2.

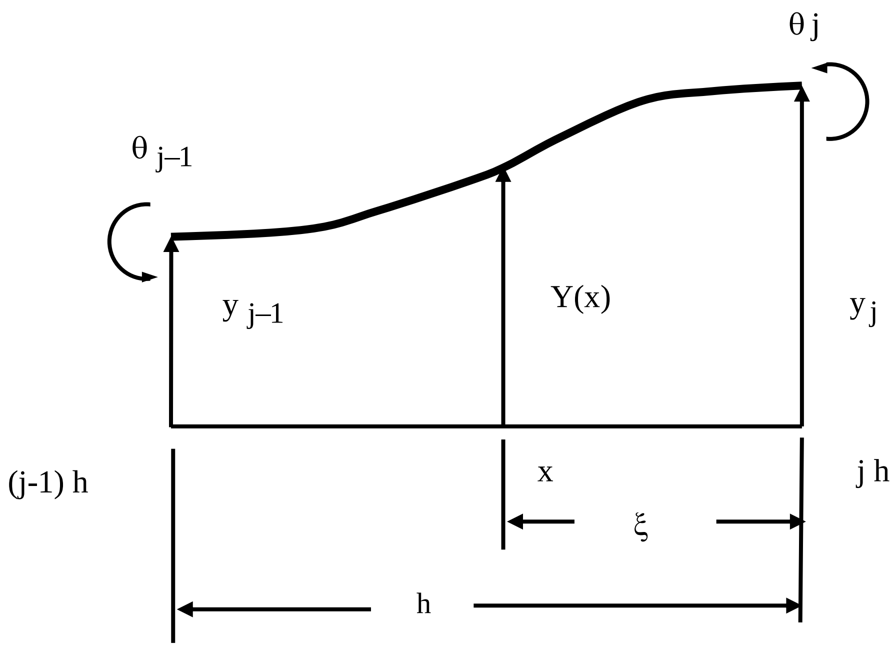

Figure 2.

Represent each L coefficient in terms of a cubic polynomial.

$$L_i = c_{i1} + c_{i2}\,\xi + c_{i3}\,\xi^2 + c_{i4}\,\xi^3, \qquad i = 1, 2, 3, 4 \tag{33}$$

$$\begin{aligned} Y(\xi) = &\left\{c_{11} + c_{12}\,\xi + c_{13}\,\xi^2 + c_{14}\,\xi^3\right\} y_{j-1} \\ &+ \left\{c_{21} + c_{22}\,\xi + c_{23}\,\xi^2 + c_{24}\,\xi^3\right\} h\,\theta_{j-1} \\ &+ \left\{c_{31} + c_{32}\,\xi + c_{33}\,\xi^2 + c_{34}\,\xi^3\right\} y_j \\ &+ \left\{c_{41} + c_{42}\,\xi + c_{43}\,\xi^2 + c_{44}\,\xi^3\right\} h\,\theta_j\,, \qquad 0 \le \xi \le 1 \end{aligned} \tag{34}$$

$$
\begin{aligned}
Y'(\xi) = & \left\{ c_{12} + 2c_{13}\xi + 3c_{14}\xi^2 \right\} y_{j-1} \\
& + \left\{ c_{22} + 2c_{23}\xi + 3c_{24}\xi^2 \right\} h\theta_{j-1} \\
& + \left\{ c_{32} + 2c_{33}\xi + 3c_{34}\xi^2 \right\} y_j \\
& + \left\{ c_{42} + 2c_{43}\xi + 3c_{44}\xi^2 \right\} h\theta_j , \qquad 0 \le \xi \le 1
\end{aligned}
\tag{35}
$$

Solve for the coefficients c_{ij}. The constraint equations are

$$Y(0) = y_j \tag{36}$$

$$Y(1) = y_{j-1} \tag{37}$$

$$Y'(0) = -h\theta_j \tag{38}$$

$$Y'(1) = -h\theta_{j-1} \tag{39}$$

The coefficients were determined in Reference 1. The resulting displacement equation is

$$
\begin{aligned}
Y(\xi) = & + \left\{ 3\xi^2 - 2\xi^3 \right\} y_{j-1} + \left\{ \xi^2 - \xi^3 \right\} h\theta_{j-1} \\
& + \left\{ 1 - 3\xi^2 + 2\xi^3 \right\} y_j + \left\{ -\xi + 2\xi^2 - \xi^3 \right\} h\theta_j , \qquad 0 \le \xi \le 1
\end{aligned}
\tag{40}
$$

Recall

$$\xi = j - x/h \tag{41}$$

Thus

$$d\xi = -dx/h \tag{42}$$

$$-h\,d\xi = dx \tag{43}$$

$$\frac{d\xi}{dx} = -1/h \tag{44}$$

Note

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} \tag{45}$$

$$\frac{d}{dx}Y(x) = -1/h\frac{d}{d\xi}Y(\xi) =$$

$$\{-1/h\}\left\{\left[6\ \xi - 6\xi^2\right]y_{j-1} + \left[2\xi - 3\xi^2\right]h\,\theta_{j-1} + \left[1-6\ \xi + 6\xi^2\right]y_j + \left[-1+4\xi - 3\xi^2\right]h\,\theta_j\right\}$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0 \le \xi \le 1$$

$$\tag{46}$$

$$\frac{d^2}{dx^2}Y(x) = 1/h^2\frac{d^2}{d\xi^2}Y(\xi) =$$

$$\left\{1/h^2\right\}\left\{[\ 6-12\xi]\,y_{j-1} + [2-6\xi]\,h\,\theta_{j-1} + [\ -6\ +12\xi]\,y_j + [\ 4-6\ \xi]\,h\,\theta_j\right\}$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0 \le \xi \le 1$$

$$\tag{47}$$

Now Let

$$Y(x) = \underline{L}^T\ \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{48}$$

where

$$L_1 = 3\,\xi^2 - 2\xi^3 \tag{49}$$

$$L_2 = \xi^2 - \xi^3 \tag{50}$$

$$L_3 = 1 - 3\ \xi^2 + 2\xi^3 \tag{51}$$

$$L_4 = -\xi + 2\xi^2 - \xi^3 \quad (52)$$

$$\bar{a} = \left[\; y_{j-1} \quad h\theta_{j-1} \quad y_j \quad h\theta_j \;\right]^T \quad (53)$$

The derivative terms are

$$\frac{d}{dx} Y(x) = \left(\frac{-1}{h}\right) \underline{L}'^{\,T} \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \quad (54)$$

$$\frac{d^2}{dx^2} Y(x) = \left(\frac{1}{h^2}\right) \underline{L}''^{\,T} \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \quad (55)$$

Note that primes indicate derivatives with respect to ξ.

In summary.

$$\underline{L} = \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \quad (56)$$

$$\underline{L}' = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \quad (57)$$

$$\underline{L}'' = \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix} \quad (58)$$

Recall

$$EI \int \left\{ \left[\frac{d^2\phi(x)}{dx^2} \right] \left[\frac{d^2Y(x)}{dx^2} \right] \right\} dx + \frac{1}{2}\rho\Omega^2L^2 \int \left\{ \frac{d\,\phi(x)}{dx} \right\} \left\{ \left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right\} dx$$

$$-\rho\,\omega^2 \int \phi(x)\; Y(x)\; dx = 0 \tag{59}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = Y(x) \tag{60}$$

Thus

$$EI \int \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx + \frac{1}{2}\rho\Omega^2L^2 \int \left\{ \frac{d\,Y(x)}{dx} \right\} \left\{ \left(1 - \frac{x^2}{L^2} \right) \frac{dY(x)}{dx} \right\} dx$$

$$-\rho\,\omega^2 \int \left[Y(x) \right]^2 \; dx = 0 \tag{61}$$

Change the integration variable. Also, apply the integration limits.

$$h\; EI \int_0^1 \left\{ \left[\frac{d^2}{d\xi^2} Y(\xi) \right] \left[\frac{d^2}{d\xi^2} Y(\xi) \right] \right\} d\xi$$

$$+ \frac{1}{2}\rho\, h\, \Omega^2L^2 \int_0^1 \left\{ \frac{d\,Y(\xi)}{d\xi} \right\} \left\{ \left(1 - \frac{(j-\xi)^2 h^2}{L^2} \right) \frac{dY(\xi)}{d\xi} \right\} d\xi - h\rho\,\omega^2 \int_0^1 \left[Y(\xi) \right]^2 \; d\xi = 0 \tag{62}$$

$$d\xi = -dx / h \tag{63}$$

$$\xi = j - x / h \tag{64}$$

$$\xi - j = -x / h \tag{65}$$

$$j - \xi = x / h \tag{66}$$

$$(j - \xi) h = x \tag{67}$$

$$h\,EI \int_0^1 \left\{ \left[\left(\frac{1}{h^2} \right) \underline{L''}^{\,T} \quad \bar{a} \right] \left[\left(\frac{1}{h^2} \right) \underline{L''}^{\,T} \quad \bar{a} \right] \right\} d\xi$$

$$+ \frac{1}{2} \rho\, h\, \Omega^2 L^2 \int_0^1 \left[\underline{\left(\frac{1}{h} \right) L'^{\,T}} \quad \bar{a} \right] \left\{ \left(1 - \frac{(j-\xi)^2 h^2}{L^2} \right) \left[\underline{\left(\frac{1}{h} \right) L'^{\,T}} \quad \bar{a} \right] \right\} d\xi$$

$$- h\, \rho\, \omega^2 \int_0^1 \left[\underline{L^{\,T}} \quad \bar{a} \right] \left[\underline{L^{\,T}} \quad \bar{a} \right] d\xi = 0$$

(68)

$$\frac{EI}{h^3} \int_0^1 \left\{ \left[\underline{L''}^{\,T} \quad \bar{a} \right] \left[\underline{L''}^{\,T} \quad \bar{a} \right] \right\} d\xi$$

$$+ \frac{1}{2} \frac{\rho}{h} \Omega^2 L^2 \int_0^1 \left\{ \left(1 - \frac{(j-\xi)^2 h^2}{L^2} \right) \left[\underline{L'^{\,T}} \quad \bar{a} \right] \left[\underline{L'^{\,T}} \quad \bar{a} \right] \right\} d\xi$$

$$- h\, \rho\, \omega^2 \int_0^1 \left[\underline{L^{\,T}} \quad \bar{a} \right] \left[\underline{L^{\,T}} \quad \bar{a} \right] d\xi = 0$$

(69)

$$\left(\frac{1}{h^3}\right) EI \int_0^1 \left\{ \left[\bar{a}^{\,T} \underline{L''}\right]\left[\underline{L''}^{\,T} \bar{a}\right]\right\} d\xi$$

$$+\frac{1}{2}\frac{\rho}{h}\Omega^2 L^2 \int_0^1 \left\{\left(1-\frac{(j-\xi)^2 h^2}{L^2}\right)\left[\bar{a}^T \underline{L'}\right]\left[\underline{L'}^{\,T} \bar{a}\right]\right\} d\xi$$

$$-h\rho\omega^2 \int_0^1 \left[\bar{a}^T \underline{L}\right]\left[\underline{L}^{\,T} \bar{a}\right] d\xi = 0 \tag{70}$$

$$\left(\frac{1}{h^3}\right) EI \int_0^1 \left\{ \bar{a}^{\,T} \underline{L''}\ \underline{L''}^{\,T} \bar{a} \right\} d\xi$$

$$+\frac{1}{2}\frac{\rho}{h}\Omega^2 L^2 \int_0^1 \left\{\left(1-\frac{(j-\xi)^2 h^2}{L^2}\right)\left[\bar{a}^T \underline{L'}\ \underline{L'}^{\,T} \bar{a}\right]\right\} d\xi$$

$$-\ h\rho\omega^2 \int_0^1 \left\{ \bar{a}^{\,T} \underline{L}\ \underline{L}^T \bar{a} \right\} d\xi = 0 \tag{71}$$

$$\bar{a}^{\,T}\left\{\left(\frac{EI}{h^3}\right) \int_0^1 \left\{ \underline{L''}\ \underline{L''}^{\,T} \right\} d\xi\right\}\bar{a}$$

$$+\bar{a}^{\,T}\left\{\frac{1}{2}\frac{\rho}{h}\Omega^2 L^2 \int_0^1 \left\{\left(1-\frac{(j-\xi)^2 h^2}{L^2}\right)\left[\underline{L'}\ \underline{L'}^{\,T}\right]\right\} d\xi\right\}\bar{a}$$

$$-\bar{a}^{\,T}\left\{h\rho\omega^2 \int_0^1 \left\{\underline{L}\ \underline{L}^T\right\} d\xi\right\}\bar{a} = 0 \tag{72}$$

$$\left(\frac{EI}{h^3}\right)\int_0^1\left\{\underline{L}''\ \underline{L}''^{\,T}\right\}d\xi+\frac{1}{2}\frac{\rho}{h}\Omega^2L^2\int_0^1\left\{\left(1-\frac{(j-\xi)^2h^2}{L^2}\right)\left[\underline{L}'\ \underline{L}'^{\,T}\right]\right\}d\xi$$

$$-h\rho\omega^2\int_0^1\left\{\underline{L}\ \underline{L}^T\right\}d\xi=0 \tag{73}$$

For a system of n elements,

$$K_j-\omega^2 M_j=0\,,\qquad j=1,2,\ldots,n \tag{74}$$

where

$$K_j=\left(\frac{EI}{h^3}\right)\int_0^1\left\{\underline{L}''\ \underline{L}''^{\,T}\right\}d\xi+\frac{1}{2}\frac{\rho}{h}\Omega^2L^2\int_0^1\left\{\left(1-\frac{(j-\xi)^2h^2}{L^2}\right)\left[\underline{L}'\ \underline{L}'^{\,T}\right]\right\}d\xi \tag{75}$$

$$M_j=h\rho\int_0^1\left\{\underline{L}\ \underline{L}^T\right\}d\xi \tag{76}$$

An element stiffness matrix for the first term on the right-hand-side of equation (75) was derived in Reference 2. The elemental mass matrix was also derived in Reference 2.

By substitution,

$$\underline{L}'\ \underline{L}'^{T}=\begin{bmatrix}6\xi-6\xi^2\\2\xi-3\xi^2\\-6\xi+6\xi^2\\-1+4\xi-3\xi^2\end{bmatrix}\begin{bmatrix}6\xi-6\xi^2 & 2\xi-3\xi^2 & -6\xi+6\xi^2 & -1+4\xi-3\xi^2\end{bmatrix} \tag{77}$$

The following matrix multiplication and integration were performed using wxMaxima 12.01.0.

$\underline{L}' \, \underline{L}'^T =$

$$\begin{bmatrix} 36\xi^4 - 72\xi^3 + 36\xi^2 & 18\xi^4 - 30\xi^3 + 12\xi^2 & -36\xi^4 + 72\xi^3 - 36\xi^2 & 18\xi^4 - 42\xi^3 + 30\xi^2 - 6\xi \\ & 9\xi^4 - 12\xi^3 + 4\xi^2 & -18\xi^4 + 30\xi^3 - 12\xi^2 & 9\xi^4 - 18\xi^3 + 11\xi^2 - 2\xi \\ & & 36\xi^4 - 72\xi^3 + 36\xi^2 & -18\xi^4 + 42\xi^3 - 30\xi^2 + 6\xi \\ & & & 9\xi^4 - 24\xi^3 + 22\xi^2 - 8\xi + 1 \end{bmatrix}$$

(78)

$$\int_0^1 \left\{ \left(1 - \frac{(j-\xi)^2 h^2}{L^2} \right) \left[\underline{L}' \, \underline{L}'^T \right] \right\} d\xi = \begin{bmatrix} T11 & T12 & T13 & T14 \\ & T22 & T23 & T24 \\ & & T33 & T34 \\ & & & T44 \end{bmatrix}$$

(79)

T11=(42*L^2-42*h^2*j^2+42*h^2*j-12*h^2)/(35*L^2) (80)

T12=(7*L^2-7*h^2*j^2+2*h^2)/(70*L^2) (81)

T13=-(42*L^2-42*h^2*j^2+42*h^2*j-12*h^2)/(35*L^2) (82)

T14=(7*L^2-7*h^2*j^2+14*h^2*j-5*h^2)/(70*L^2) (83)

T22=(14*L^2-14*h^2*j^2+21*h^2*j-9*h^2)/(105*L^2) (84)

T23=-T12 (85)

T24=-(7*L^2-7*h^2*j^2+7*h^2*j-3*h^2)/(210*L^2) (86)

T33=T11 (87)

T34=-T14 (88)

T44=(14*L^2-14*h^2*j^2+7*h^2*j-2*h^2)/(105*L^2) (89)

The displacement vector for beam is

$$\begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix} \tag{90}$$

The elemental stiffness matrix for beam bending is

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6h & -12 & 6h \\ & 4h^2 & -6h & 2h^2 \\ & & 12 & -6h \\ & & & 4h^2 \end{bmatrix} + \frac{1}{2}\frac{\rho}{h}\Omega^2 L^2 \begin{bmatrix} T11 & T12 & T13 & T14 \\ & T22 & T23 & T24 \\ & & T33 & T34 \\ & & & T44 \end{bmatrix} \tag{91}$$

The elemental mass matrix for beam bending is

$$M_j = \left(\frac{h\rho}{420}\right)\begin{bmatrix} 156 & 22h & 54 & -13h \\ & 4h^2 & 13 & -3h^2 \\ & & 156 & -22h \\ & & & 4h^2 \end{bmatrix} \tag{92}$$

Note that h is the element length. Also, j is the node number in the following formulas.

The following limits apply to the next set of equations.

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0 \le \xi \le 1$$

References

1. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967. *See Section 10.4.*

2. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision F, Vibrationdata, 2010.