DUFFING EQUATION FOR THE STEADY-STATE OSCILLATION OF A SIMPLE PENDULUM DRIVEN BY A HARMONIC MOMENT

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Introduction

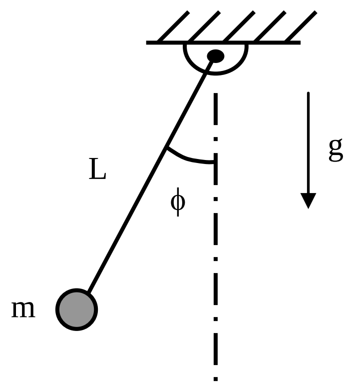

Figure 1.

Consider a simple pendulum consisting of a point mass connected by a rod to a pivot point, where the rod is mass-less.

The derivation of the equation of motion is given in Reference 1.

The angular displacement ϕ of the pendulum driven by a harmonic moment is governed by the nonlinear differential equation.

$$mL^2 \ddot{\phi} + mgL \sin \phi = M \cos \omega t \tag{1}$$

where

m	=	Mass
L	=	Length
g	=	Gravity acceleration
M	=	Moment magnitude
ω	=	Excitation frequency (rad/sec)

Divide through by mL^2.

$$\ddot{\phi} + \frac{g}{L}\sin\phi = \frac{M}{mL^2}\cos\omega t \tag{2}$$

Let

$$F = \frac{M}{mL^2} \tag{3}$$

and

$$\omega_n = \sqrt{\frac{g}{L}} \tag{4}$$

The F term is the scaled moment.

The term ω_n is the natural frequency (rad/sec) for linear motion.

By substitution,

$$\ddot{\phi} + \omega_n^2 \sin\phi = F\cos\omega t \tag{5}$$

A two-term approximation for the sine term is

$$\sin\phi \approx \left(\phi - \frac{1}{6}\phi^3\right) \tag{6}$$

$$\ddot{\phi} + \omega_n^2\left(\phi - \frac{1}{6}\phi^3\right) = F\cos\omega t \tag{7}$$

Equation (7) is Duffing's equation. It is nonlinear.

Assume a displacement function.

$$\phi(t) = U\cos\omega t \quad (8)$$

$$\phi^3(t) = U^3\cos^3\omega t = U^3\left[\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right] \quad (9)$$

$$\dot{\phi}(t) = -\omega U\sin\omega t \quad (10)$$

$$\ddot{\phi}(t) = -\omega^2 U\cos\omega t \quad (11)$$

$$-\omega^2 U\cos\omega t + \omega_n^2\left(U\cos\omega t - \frac{1}{6}U^3\left[\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right]\right) = F\cos\omega t \quad (12)$$

Omit the higher harmonic $\cos 3\omega t$.

$$-\omega^2 U\cos\omega t + \omega_n^2\left(U\cos\omega t - \frac{1}{8}U^3\cos\omega t\right) = F\cos\omega t \quad (13)$$

$$-\omega^2 U + \omega_n^2\left(U - \frac{1}{8}U^3\right) = F \quad (14)$$

$$\left(\omega_n^2 - \omega^2\right)U - \frac{1}{8}\omega_n^2 U^3 = F \quad (15)$$

$$\left(-\omega_n^2 + \omega^2\right)U + \frac{1}{8}\omega_n^2 U^3 = -F \quad (16)$$

$$\frac{1}{8}\omega_n^2 U^3 + \left(-\omega_n^2 + \omega^2\right)U + F = 0 \quad (17)$$

Equation (17) may be solved using the method in Reference 2. It can also be solved using Matlab's root function. Three roots are obtained. All three roots may be real, or there may be one real root and two complex conjugate pairs. Only the real roots are retained.

There is only one real root if $\omega > \omega_n$.

There are either one or three roots if $\omega < \omega_n$ depending on the value of F, which is the scaled moment.

References

1. T. Irvine, Pendulum Oscillation, Revision C, Vibrationdata, 1999.
2. T. Irvine, Roots of a Cubic Polynomial, Vibrationdata, 1999.

APPENDIX A

Example 1

Let

$$\omega_n = 1 \text{ rad/sec} \quad \text{(A-1)}$$

$$\omega = 0.9 \text{ rad/sec} \quad \text{(A-2)}$$

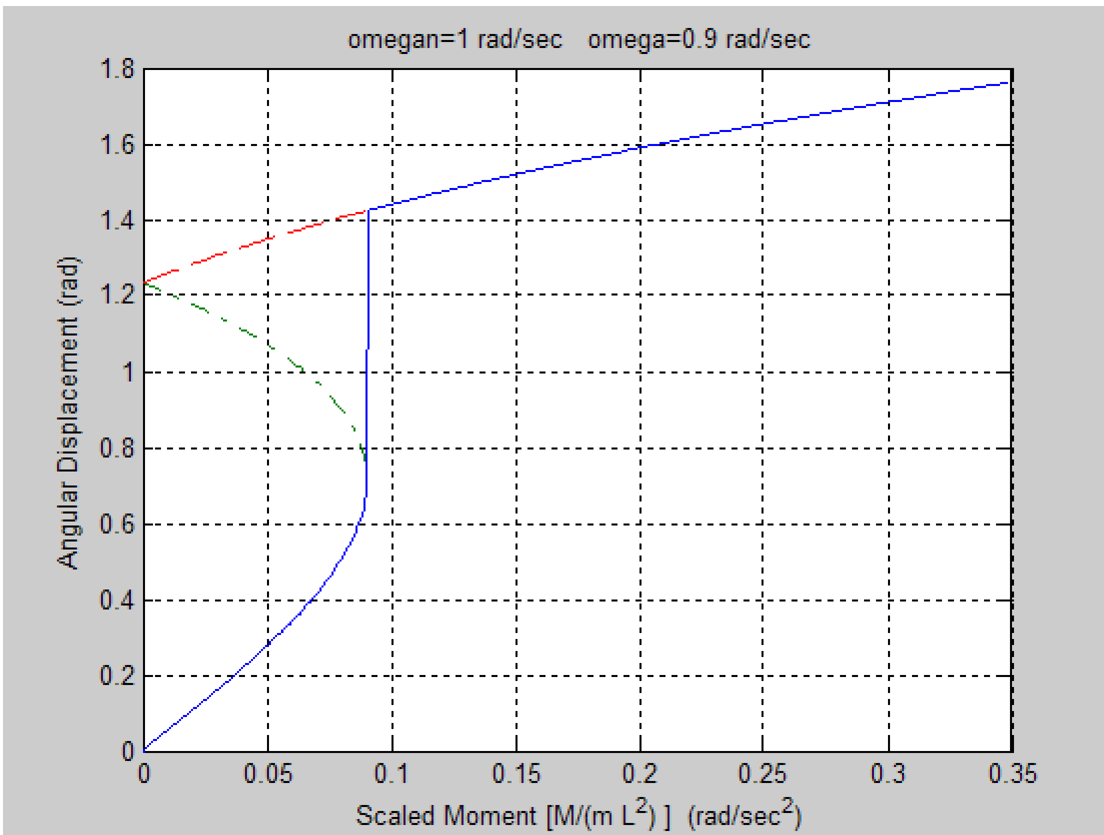

Figure A-1.

The roots are found using a Matlab program. A plot of ϕ versus F is given in Figure A-1.

APPENDIX B

Example 2

Let

$$\omega_n = 1 \text{ rad/sec} \quad \text{(B-1)}$$

$$\omega = 1.1 \text{ rad/sec} \quad \text{(B-2)}$$

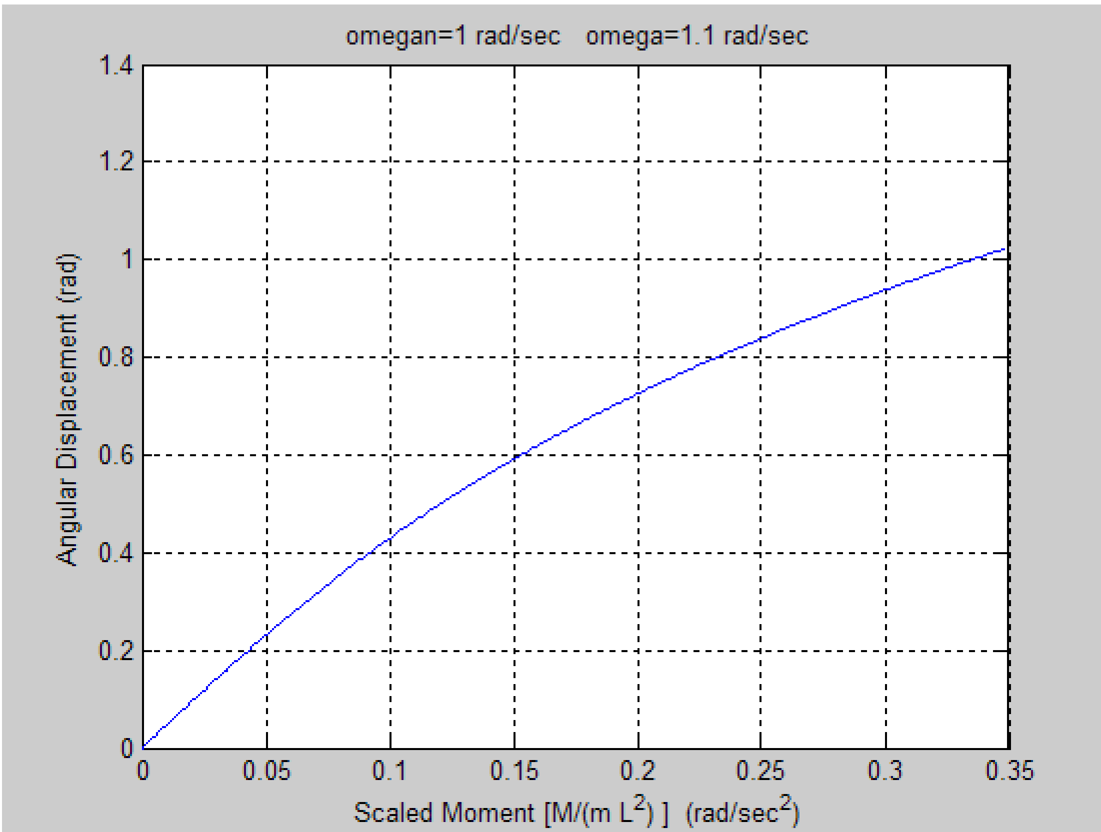

Figure B-1.