A FOUR NODE, THICK RECTANGULAR PLATE BENDING ELEMENT

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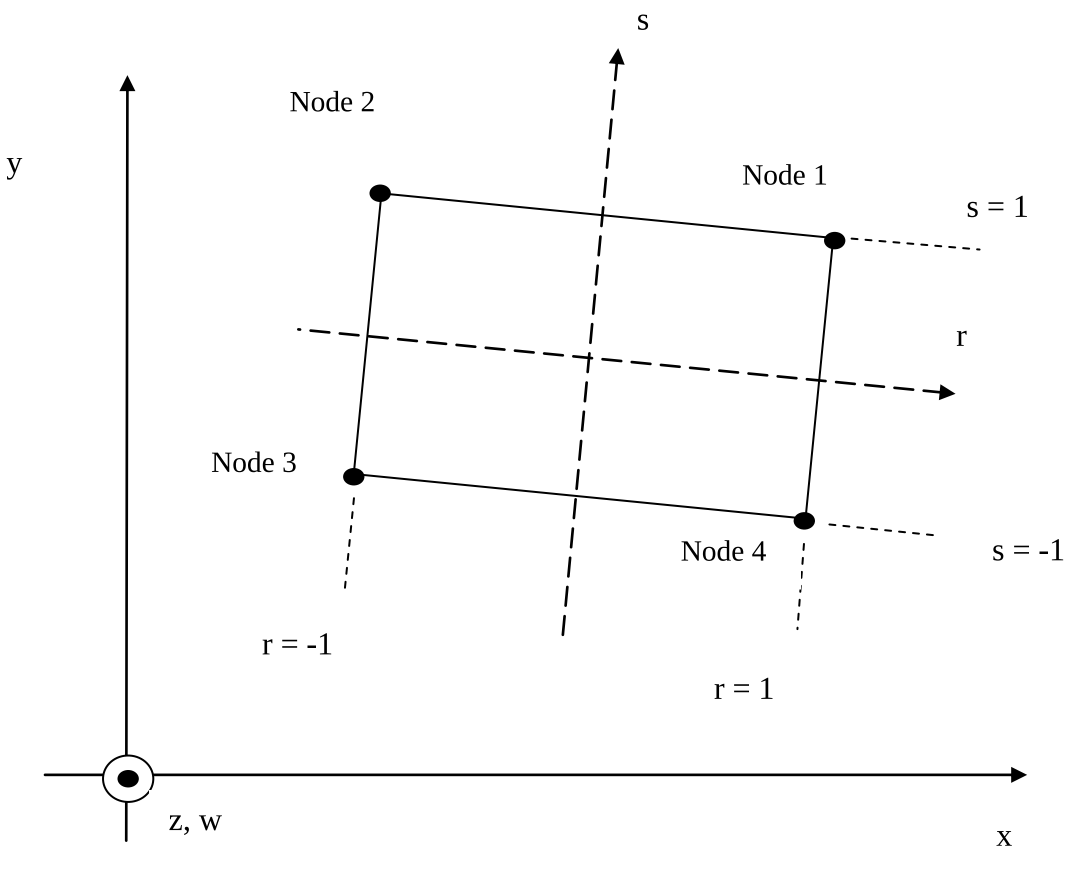
Introduction



Note that

$$-1 \le r \le +1$$

$$-1 \le s \le +1$$

Displacement variables:

u	The in-plane displacement along the x-axis
v	The in-plane displacement along the y-axis
w	The out-of-plane displacement along the z-axis
β_x	The rotation about the y-axis
β_y	The rotation about the x-axis

Stress and Strain

The following equations are taken from Bathe, pages 251-253.

The element strains are

$$\varepsilon^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} \tag{1}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{3}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4}$$

The in-plane translational displacement are related to the rotational displacements by

$$u = z\beta_x(x, y) \tag{5}$$

$$v = -z\beta_y(x, y) \tag{6}$$

This is small-displacement theory. The bending strains vary linearly throughout the thickness of the plate.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \partial\beta_x / \partial x \\ -\partial\beta_y / \partial y \\ \partial\beta_x / \partial y - \partial\beta_y / \partial x \end{bmatrix} \tag{7}$$

The transverse shear strains are assumed to be constant throughout the plate thickness.

$$\begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \tag{8}$$

Assume plane stress. The resulting stress-strain relationship is

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{zE}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \partial\beta_x / \partial x \\ -\partial\beta_y / \partial y \\ \partial\beta_x / \partial y - \partial\beta_y / \partial x \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \tag{12}$$

$$U = \frac{1}{2}\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} \right\} dz\, dA$$

$$+ \frac{\hat{k}}{2}\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \gamma_{yz} & \gamma_{zx} \end{bmatrix} \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} \right\} dz\, dA \tag{13}$$

where

h	is the plate thickness
A	is the surface area
$\hat{k}$	is the shear factor

$$U = \frac{1}{2}\left[\frac{E}{1-\mu^2}\right]\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \right\} dz\, dA$$

$$+ \frac{\hat{k}}{2}\left[\frac{E}{2(1+\mu)}\right]\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y & \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \right\} dz\, dA \tag{14}$$

U =

$$\frac{1}{2}\left[\frac{E}{1-\mu^2}\right]\int_A \int_{-h/2}^{h/2} \left\{ z^2 \begin{bmatrix} \frac{\partial \beta_x}{\partial x} & -\frac{\partial \beta_y}{\partial y} & \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \partial \beta_x / \partial x \\ -\partial \beta_y / \partial y \\ \partial \beta_x / \partial y - \partial \beta_y / \partial x \end{bmatrix} \right\} dz\, dA$$

$$+\frac{\hat{k}}{2}\left[\frac{E}{2(1+\mu)}\right]\int_A \int_{-h/2}^{h/2} \left\{ \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y & \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \right\} dz\, dA$$

(15)

U =

$$\frac{1}{2}\left[\frac{1}{12}\right]\left[\frac{Eh^3}{1-\mu^2}\right]\int_A \left\{ \begin{bmatrix} \frac{\partial \beta_x}{\partial x} & -\frac{\partial \beta_y}{\partial y} & \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \partial \beta_x / \partial x \\ -\partial \beta_y / \partial y \\ \partial \beta_x / \partial y - \partial \beta_y / \partial x \end{bmatrix} \right\} dA$$

$$+\frac{1}{2}\left[\frac{Eh\hat{k}}{2(1+\mu)}\right]\int_A \left\{ \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y & \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \right\} dA$$

(16)

Let

$$\eta = \begin{bmatrix} \partial \beta_x / \partial x \\ -\partial \beta_y / \partial y \\ \partial \beta_x / \partial y - \partial \beta_y / \partial x \end{bmatrix} \qquad (17)$$

$$C_b = \left[\frac{1}{12}\right]\left[\frac{Eh^3}{1-\mu^2}\right]\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{18}$$

$$\psi = \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix} \tag{19}$$

$$C_s = \left[\frac{Eh\hat{k}}{2(1+\mu)}\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

$$U = \frac{1}{2}\int_A \left\{ \eta^T \, C_b \, \eta \right\} dA + \frac{1}{2}\int_A \left\{ \psi^T \, C_s \, \psi \right\} dA \tag{21}$$

The elemental displacement vector is

$$\hat{u} = \begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix} \tag{22}$$

Let

$$\eta(r,s) = B\hat{u} \tag{23}$$

$$\psi(r,s) = V\hat{u} \tag{24}$$

The B and V matrices are defined via interpolation functions in the appendices.

By substitution,

$$U = \frac{1}{2}\int_A \left\{ \{B\hat{u}\}^T C_b\ B\hat{u} \right\} dA \quad + \frac{1}{2}\int_A \left\{ \{V\hat{u}\}^T C_s\ V\hat{u} \right\} dA \tag{25}$$

The stiffness matrix can thus be represented as

$$K = \int_{-1}^{1}\int_{-1}^{1} \left\{ B^T\ C_b\ B \right\} \det[J]\, dr\, ds + \int_{-1}^{1}\int_{-1}^{1} \left\{ V^T\ C_s\ V \right\} \det[J]\, dr\, ds \tag{26}$$

The Jacobian is given in Appendix A.

The first and second stiffness matrices are given in Appendices B and C, respectively.

<u>References</u>

1. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1982.

2. R. Cook, Concepts and Applications of Finite Element Analysis, Second Edition, Wiley, New York, 1981.

3. T. Irvine, A Four Node, Isoparametric Plate Bending Element Mass Matrix, Revision B, Vibrationdata, 2012.

APPENDIX A

Jacobian Matrix

The coordinate interpolation is

$$x = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \quad \text{(A-1)}$$

$$y = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4 \quad \text{(A-2)}$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4 \quad \text{(A-3)}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4 \quad \text{(A-4)}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1+r)y_1 + \frac{1}{4}(1-r)y_2 - \frac{1}{4}(1-r)y_3 - \frac{1}{4}(1+r)y_4 \quad \text{(A-5)}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(1+r)y_1 + \frac{1}{4}(1-r)y_2 - \frac{1}{4}(1-r)y_3 - \frac{1}{4}(1+r)y_4 \quad \text{(A-6)}$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}\{x_1 - x_2 - x_3 + x_4\} + \frac{1}{4}\{x_1 - x_2 + x_3 - x_4\}s \quad \text{(A-7)}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}\{x_1 + x_2 - x_3 - x_4\} + \frac{1}{4}\{x_1 - x_2 + x_3 - x_4\}r \quad \text{(A-8)}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}\{y_1 - y_2 - y_3 + y_4\} + \frac{1}{4}\{y_1 - y_2 + y_3 - y_4\}s \quad \text{(A-9)}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\{y_1 + y_2 - y_3 - y_4\} + \frac{1}{4}\{y_1 - y_2 + y_3 - y_4\}r \quad \text{(A-10)}$$

Consider the special case of a rectangle where the local coordinate system may be rotated with respect to the global system.

$$\frac{\partial x}{\partial r} = \frac{1}{4}\left\{x_1 - x_2 - x_3 + x_4\right\} \tag{A-11}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}\left\{x_1 + x_2 - x_3 - x_4\right\} \tag{A-12}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}\left\{y_1 - y_2 - y_3 + y_4\right\} \tag{A-13}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}\left\{y_1 + y_2 - y_3 - y_4\right\} \tag{A-14}$$

The Jacobian matrix J is

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \tag{A-15}$$

$$\begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial r} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \tag{A-16}$$

Let

$$\hat{J} = J^{-1} \tag{A-17}$$

\

Thus

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \hat{J} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \\ \dfrac{\partial}{\partial s} \end{bmatrix} \tag{A-18}$$

APPENDIX B

Displacement Interpolation for First Stiffness Matrix

The displacement vector is

$$\hat{u} = \begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix} \tag{B-1}$$

The rotation about the x-axis

$$\alpha = \beta_y \tag{B-2}$$

The rotation about the y-axis is

$$\beta = \beta_x \tag{B-3}$$

The displacement interpolation is

$$w = \frac{1}{4}(1+r)(1+s)w_1 + \frac{1}{4}(1-r)(1+s)w_2 + \frac{1}{4}(1-r)(1-s)w_3 + \frac{1}{4}(1+r)(1-s)w_4 \tag{B-4}$$

$$\alpha = \frac{1}{4}(1+r)(1+s)\alpha_1 + \frac{1}{4}(1-r)(1+s)\alpha_2 + \frac{1}{4}(1-r)(1-s)\alpha_3 + \frac{1}{4}(1+r)(1-s)\alpha_4 \tag{B-5}$$

$$\beta = \frac{1}{4}(1+r)(1+s)\beta_1 + \frac{1}{4}(1-r)(1+s)\beta_2 + \frac{1}{4}(1-r)(1-s)\beta_3 + \frac{1}{4}(1+r)(1-s)\beta_4 \qquad \text{(B-6)}$$

Evaluate the derivatives of displacement

$$\frac{\partial w}{\partial r} = \frac{1}{4}(1+s)w_1 - \frac{1}{4}(1+s)w_2 - \frac{1}{4}(1-s)w_3 + \frac{1}{4}(1-s)w_4 \qquad \text{(B-7)}$$

$$\frac{\partial w}{\partial s} = \frac{1}{4}(1+r)w_1 + \frac{1}{4}(1-r)w_2 - \frac{1}{4}(1-r)w_3 - \frac{1}{4}(1+r)w_4 \qquad \text{(B-8)}$$

$$\frac{\partial \alpha}{\partial r} = \frac{1}{4}(1+s)\alpha_1 - \frac{1}{4}(1+s)\alpha_2 - \frac{1}{4}(1-s)\alpha_3 + \frac{1}{4}(1-s)\alpha_4 \qquad \text{(B-9)}$$

$$\frac{\partial \alpha}{\partial s} = \frac{1}{4}(1+r)\alpha_1 + \frac{1}{4}(1-r)\alpha_2 - \frac{1}{4}(1-r)\alpha_3 - \frac{1}{4}(1+r)\varepsilon_4 \qquad \text{(B-10)}$$

$$\frac{\partial \beta}{\partial r} = \frac{1}{4}(1+s)\beta_1 - \frac{1}{4}(1+s)\beta_2 - \frac{1}{4}(1-s)\beta_3 + \frac{1}{4}(1-s)\beta_4 \qquad \text{(B-11)}$$

$$\frac{\partial \beta}{\partial s} = \frac{1}{4}(1+r)\beta_1 + \frac{1}{4}(1-r)\beta_2 - \frac{1}{4}(1-r)\beta_3 - \frac{1}{4}(1+r)\beta_4 \qquad \text{(B-12)}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \hat{J} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}$$

(B-13)

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4} \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

(B-14)

$$\begin{bmatrix} \dfrac{\partial \alpha}{\partial x} \\ \\ \dfrac{\partial \alpha}{\partial y} \end{bmatrix} = \frac{1}{4} \ \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha 4 \end{bmatrix}$$

(B-15)

$$\begin{bmatrix} \dfrac{\partial \beta}{\partial x} \\ \\ \dfrac{\partial \beta}{\partial y} \end{bmatrix} = \frac{1}{4} \ \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta 4 \end{bmatrix}$$

(B-16)

$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4} \ \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w 4 \end{bmatrix}$$

(B-17)

$$\begin{bmatrix} \dfrac{\partial \alpha}{\partial x} \\ \\ \dfrac{\partial \alpha}{\partial y} \end{bmatrix} = \frac{1}{4} \ \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha 4 \end{bmatrix}$$

(B-18)

$$\begin{bmatrix} \dfrac{\partial \beta}{\partial x} \\ \\ \dfrac{\partial \beta}{\partial y} \end{bmatrix} = \frac{1}{4} \ \hat{J} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta 4 \end{bmatrix}$$

(B-19)

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \hat{J} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \\ \dfrac{\partial}{\partial s} \end{bmatrix} \tag{B-20}$$

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \\ \dfrac{\partial}{\partial s} \end{bmatrix} \tag{B-21}$$

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \hat{J}_{11} \dfrac{\partial}{\partial r} + \hat{J}_{12} \dfrac{\partial}{\partial s} \\ \\ \hat{J}_{21} \dfrac{\partial}{\partial r} + \hat{J}_{22} \dfrac{\partial}{\partial s} \end{bmatrix} \tag{B-22}$$

$$\eta = \begin{bmatrix} \partial\beta / \partial x \\ -\partial\alpha / \partial y \\ \partial\beta / \partial y - \partial\alpha / \partial x \end{bmatrix} = \begin{bmatrix} \hat{J}_{11} \partial\beta / \partial r + \hat{J}_{12} \partial\beta / \partial s \\ -\hat{J}_{21} \partial\alpha / \partial r - \hat{J}_{22} \partial\alpha / \partial s \\ \hat{J}_{21} \partial\beta / \partial r + \hat{J}_{22} \partial\beta / \partial s - \hat{J}_{11} \partial\alpha / \partial r - \hat{J}_{12} \partial\alpha / \partial s \end{bmatrix} \tag{B-23}$$

$$\begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix} \begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w4 \end{bmatrix} \tag{B-24}$$

$$\begin{bmatrix} \dfrac{\partial\alpha}{\partial x} \\ \\ \dfrac{\partial\alpha}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha 4 \end{bmatrix} \tag{B-25}$$

$$\begin{bmatrix} \dfrac{\partial\alpha}{\partial x} \\ \\ \dfrac{\partial\alpha}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \hat{J}_{11}(1+s)+\hat{J}_{12}(1+r) & -\hat{J}_{11}(1+s)+\hat{J}_{12}(1-r) & -\hat{J}_{11}(1-s)-\hat{J}_{12}(1-r) & \hat{J}_{11}(1-s)-\hat{J}_{12}(1+r) \\ \hat{J}_{21}(1+s)+\hat{J}_{22}(1+r) & -\hat{J}_{21}(1+s)+\hat{J}_{22}(1-r) & -\hat{J}_{21}(1-s)-\hat{J}_{22}(1-r) & \hat{J}_{21}(1-s)-\hat{J}_{22}(1+r) \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha 4 \end{bmatrix} \tag{B-26}$$

$$\begin{bmatrix} \dfrac{\partial\beta}{\partial x} \\ \\ \dfrac{\partial\beta}{\partial y} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix}\begin{bmatrix} (1+s) & -(1+s) & -(1-s) & (1-s) \\ (1+r) & (1-r) & -(1-r) & -(1+r) \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta 4 \end{bmatrix} \tag{B-27}$$

$$\begin{bmatrix} \dfrac{\partial \beta}{\partial x} \\ \dfrac{\partial \beta}{\partial y} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \hat{J}_{11}(1+s)+\hat{J}_{12}(1+r) & -\hat{J}_{11}(1+s)+\hat{J}_{12}(1-r) & -\hat{J}_{11}(1-s)-\hat{J}_{12}(1-r) & \hat{J}_{11}(1-s)-\hat{J}_{12}(1+r) \\ \hat{J}_{21}(1+s)+\hat{J}_{22}(1+r) & -\hat{J}_{21}(1+s)+\hat{J}_{22}(1-r) & -\hat{J}_{21}(1-s)-\hat{J}_{22}(1-r) & \hat{J}_{21}(1-s)-\hat{J}_{22}(1+r) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta 4 \end{bmatrix} \tag{B-28}$$

$$\frac{\partial \beta}{\partial x} = \frac{1}{4} \begin{bmatrix} \hat{J}_{11}(1+s)+\hat{J}_{12}(1+r) & -\hat{J}_{11}(1+s)+\hat{J}_{12}(1-r) & -\hat{J}_{11}(1-s)-\hat{J}_{12}(1-r) & \hat{J}_{11}(1-s)-\hat{J}_{12}(1+r) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta 4 \end{bmatrix} \tag{B-29}$$

$$\frac{\partial \alpha}{\partial y} = \frac{1}{4} \begin{bmatrix} \hat{J}_{21}(1+s)+\hat{J}_{22}(1+r) & -\hat{J}_{21}(1+s)+\hat{J}_{22}(1-r) & -\hat{J}_{21}(1-s)-\hat{J}_{22}(1-r) & \hat{J}_{21}(1-s)-\hat{J}_{22}(1+r) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha 4 \end{bmatrix} \tag{B-30}$$

$$\partial\beta / \partial y - \partial\alpha / \partial x =$$

$$-\frac{1}{4}\left[\hat{J}_{11}(1+s)+\hat{J}_{12}(1+r) \quad -\hat{J}_{11}(1+s)+\hat{J}_{12}(1-r) \quad -\hat{J}_{11}(1-s)-\hat{J}_{12}(1-r) \quad \hat{J}_{11}(1-s)-\hat{J}_{12}(1+r)\right]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha 4\end{bmatrix}$$

$$+\frac{1}{4}\left[\hat{J}_{21}(1+s)+\hat{J}_{22}(1+r) \quad -\hat{J}_{21}(1+s)+\hat{J}_{22}(1-r) \quad -\hat{J}_{21}(1-s)-\hat{J}_{22}(1-r) \quad \hat{J}_{21}(1-s)-\hat{J}_{22}(1+r)\right]\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta 4\end{bmatrix}$$

(B-31)

$$\eta = \begin{bmatrix} \partial\beta/\partial x \\ -\partial\alpha/\partial y \\ \partial\beta/\partial y - \partial\alpha/\partial x \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 \\ 0 & -b_1 & 0 & 0 & -b_2 & 0 & 0 & -b_3 & 0 & 0 & -b_4 & 0 \\ 0 & -a_1 & b_1 & 0 & -a_2 & b_2 & 0 & -a_3 & b_3 & 0 & -a_4 & b_4 \end{bmatrix} \begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix}$$

(B-32)

$$B = \frac{1}{4}\begin{bmatrix} 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 \\ 0 & -b_1 & 0 & 0 & -b_2 & 0 & 0 & -b_3 & 0 & 0 & -b_4 & 0 \\ 0 & -a_1 & b_1 & 0 & -a_2 & b_2 & 0 & -a_3 & b_3 & 0 & -a_4 & b_4 \end{bmatrix}$$

(B-33)

where

$$a_1 = \hat{J}_{11}(1+s) + \hat{J}_{12}(1+r) \qquad \text{(B-34)}$$

$$a_2 = -\hat{J}_{11}(1+s) + \hat{J}_{12}(1-r) \qquad \text{(B-35)}$$

$$a_3 = -\hat{J}_{11}(1-s) - \hat{J}_{12}(1-r) \qquad \text{(B-36)}$$

$$a_4 = \hat{J}_{11}(1-s) - \hat{J}_{12}(1+r) \qquad \text{(B-37)}$$

$$b_1 = \hat{J}_{21}(1+s) + \hat{J}_{22}(1+r) \qquad \text{(B-38)}$$

$$b_2 = -\hat{J}_{21}(1+s) + \hat{J}_{22}(1-r) \qquad \text{(B-39)}$$

$$b_3 = -\hat{J}_{21}(1-s) - \hat{J}_{22}(1-r) \qquad \text{(B-40)}$$

$$b_4 = \hat{J}_{21}(1-s) - \hat{J}_{22}(1+r) \qquad \text{(B-41)}$$

Some useful wxMaxima commands for forming the matrix $\left\{ B^T \ C_b \ B \right\}$ are

```
c:matrix([1,mu,0],[mu,1,0],[0,0,e3]);

b:matrix([0,0,a1,0,0,a2,0,0,a3,0,0,a4],[0,-b1,0,0,-b2,0,0,-b3,0,0,-b4,0],[0,-a1,b1,0,-a2,b2,0,-
a3,b3,0,-a4,b4]);

d:c.b;

v:transpose(b).d;
```

The resulting matrix is shown over this page and the next.

$\left\{ B^T \ C_b \ B \right\} =$

$$\left[\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a1^2\, e3 + b1^2 & -a1\, b1\, \mu - a1\, b1\, e3 & 0 & a1\, a2\, e3 + b1\, b2 & -a2\, b1\, \mu - a1\, b2\, e3 \\
0 & -a1\, b1\, \mu - a1\, b1\, e3 & b1^2\, e3 + a1^2 & 0 & -a1\, b2\, \mu - a2\, b1\, e3 & b1\, b2\, e3 + a1\, a2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a1\, a2\, e3 + b1\, b2 & -a1\, b2\, \mu - a2\, b1\, e3 & 0 & a2^2\, e3 + b2^2 & -a2\, b2\, \mu - a2\, b2\, e3 \\
0 & -a2\, b1\, \mu - a1\, b2\, e3 & b1\, b2\, e3 + a1\, a2 & 0 & -a2\, b2\, \mu - a2\, b2\, e3 & b2^2\, e3 + a2^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a1\, a3\, e3 + b1\, b3 & -a1\, b3\, \mu - a3\, b1\, e3 & 0 & a2\, a3\, e3 + b2\, b3 & -a2\, b3\, \mu - a3\, b2\, e3 \\
0 & -a3\, b1\, \mu - a1\, b3\, e3 & b1\, b3\, e3 + a1\, a3 & 0 & -a3\, b2\, \mu - a2\, b3\, e3 & b2\, b3\, e3 + a2\, a3 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a1\, a4\, e3 + b1\, b4 & -a1\, b4\, \mu - a4\, b1\, e3 & 0 & a2\, a4\, e3 + b2\, b4 & -a2\, b4\, \mu - a4\, b2\, e3 \\
0 & -a4\, b1\, \mu - a1\, b4\, e3 & b1\, b4\, e3 + a1\, a4 & 0 & -a4\, b2\, \mu - a2\, b4\, e3 & b2\, b4\, e3 + a2\, a4
\end{matrix} \right.$$

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a1\, a3\, e3 + b1\, b3 & -a3\, b1\, \mu - a1\, b3\, e3 & 0 & a1\, a4\, e3 + b1\, b4 & -a4\, b1\, \mu - a1\, b4\, e3 \\
0 & -a1\, b3\, \mu - a3\, b1\, e3 & b1\, b3\, e3 + a1\, a3 & 0 & -a1\, b4\, \mu - a4\, b1\, e3 & b1\, b4\, e3 + a1\, a4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a2\, a3\, e3 + b2\, b3 & -a3\, b2\, \mu - a2\, b3\, e3 & 0 & a2\, a4\, e3 + b2\, b4 & -a4\, b2\, \mu - a2\, b4\, e3 \\
0 & -a2\, b3\, \mu - a3\, b2\, e3 & b2\, b3\, e3 + a2\, a3 & 0 & -a2\, b4\, \mu - a4\, b2\, e3 & b2\, b4\, e3 + a2\, a4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a3^2\, e3 + b3^2 & -a3\, b3\, \mu - a3\, b3\, e3 & 0 & a3\, a4\, e3 + b3\, b4 & -a4\, b3\, \mu - a3\, b4\, e3 \\
0 & -a3\, b3\, \mu - a3\, b3\, e3 & b3^2\, e3 + a3^2 & 0 & -a3\, b4\, \mu - a4\, b3\, e3 & b3\, b4\, e3 + a3\, a4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a3\, a4\, e3 + b3\, b4 & -a3\, b4\, \mu - a4\, b3\, e3 & 0 & a4^2\, e3 + b4^2 & -a4\, b4\, \mu - a4\, b4\, e3 \\
0 & -a4\, b3\, \mu - a3\, b4\, e3 & b3\, b4\, e3 + a3\, a4 & 0 & -a4\, b4\, \mu - a4\, b4\, e3 & b4^2\, e3 + a4^2
\end{matrix}\right]
$$

(B-42)

Note that integration is still required. This can be performed in Matlab using Gauss quadrature. Refer to Matlab script: thick_plate_mass_stiff.m

The integration can also be performed symbolically using wxMaxima as follows.

```
a1(r,s):=(( Jinv11*(1+s)+Jinv12*(1+r))/4);
a2(r,s):=((-Jinv11*(1+s)+Jinv12*(1-r))/4);
a3(r,s):=((-Jinv11*(1-s)-Jinv12*(1-r))/4);
a4(r,s):=(( Jinv11*(1-s)-Jinv12*(1+r))/4);

b1(r,s):=(( Jinv21*(1+s)+Jinv22*(1+r))/4);
b2(r,s):=((-Jinv21*(1+s)+Jinv22*(1-r))/4);
b3(r,s):=((-Jinv21*(1-s)-Jinv22*(1-r))/4);
b4(r,s):=(( Jinv21*(1-s)-Jinv22*(1+r))/4);

c:matrix([1,mu,0],[mu,1,0],[0,0,e3]);

b:matrix([0,0,a1(r,s),0,0,a2(r,s),0,0,a3(r,s),0,0,a4(r,s)],[0,-b1(r,s),0,0,-b2(r,s),0,0,-b3(r,s),0,0,-
b4(r,s),0],[0,-a1(r,s),b1(r,s),0,-a2(r,s),b2(r,s),0,-a3(r,s),b3(r,s),0,-a4(r,s),b4(r,s)]);

d:c.b;
```

```
v:transpose(b).d;

ratsimp(integrate(integrate(v(r,s),r,-1,1),s,-1,1));
```

APPENDIX C

Displacement Interpolation for Second Stiffness Matrix

$$C_S = \left[\frac{Eh\hat{k}}{2(1+\mu)} \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{C-1}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \hat{J} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} \tag{C-2}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} \tag{C-3}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \hat{J}_{11} \frac{\partial}{\partial r} + \hat{J}_{12} \frac{\partial}{\partial s} \\ \hat{J}_{21} \frac{\partial}{\partial r} + \hat{J}_{22} \frac{\partial}{\partial s} \end{bmatrix} \tag{C-4}$$

$$\frac{\partial w}{\partial r} = \frac{1}{4}(1+s)w_1 - \frac{1}{4}(1+s)w_2 - \frac{1}{4}(1-s)w_3 + \frac{1}{4}(1-s)w_4 \tag{C-5}$$

$$\frac{\partial w}{\partial s} = \frac{1}{4}(1+r)w_1 + \frac{1}{4}(1-r)w_2 - \frac{1}{4}(1-r)w_3 - \frac{1}{4}(1+r)w_4 \tag{C-6}$$

$$\alpha = \frac{1}{4}(1+r)(1+s)\alpha_1 + \frac{1}{4}(1-r)(1+s)\alpha_2 + \frac{1}{4}(1-r)(1-s)\alpha_3 + \frac{1}{4}(1+r)(1-s)\alpha_4 \quad \text{(C-7)}$$

$$\beta = \frac{1}{4}(1+r)(1+s)\beta_1 + \frac{1}{4}(1-r)(1+s)\beta_2 + \frac{1}{4}(1-r)(1-s)\beta_3 + \frac{1}{4}(1+r)(1-s)\beta_4 \quad \text{(C-8)}$$

$$\psi = \begin{bmatrix} \partial w/\partial y - \beta_y \\ \partial w/\partial x + \beta_x \end{bmatrix} = \begin{bmatrix} \partial w/\partial y - \alpha \\ \partial w/\partial x + \beta \end{bmatrix} \quad \text{(C-9)}$$

$$\psi = \begin{bmatrix} \hat{J}_{21}\,\partial w/\partial r + \hat{J}_{22}\,\partial w/\partial s - \alpha \\ \hat{J}_{11}\,\partial w/\partial r + \hat{J}_{12}\,\partial w/\partial s + \beta \end{bmatrix} \quad \text{(C-10)}$$

$$\hat{J}_{21}\partial w/\partial r + \hat{J}_{22}\,\partial w/\partial s - \alpha =$$

$$\hat{J}_{21}\left\{\frac{1}{4}(1+s)w_1 - \frac{1}{4}(1+s)w_2 - \frac{1}{4}(1-s)w_3 + \frac{1}{4}(1-s)w_4\right\}$$

$$+\hat{J}_{22}\left\{\frac{1}{4}(1+r)w_1 + \frac{1}{4}(1-r)w_2 - \frac{1}{4}(1-r)w_3 - \frac{1}{4}(1+r)w_4\right\}$$

$$-\left\{\frac{1}{4}(1+r)(1+s)\alpha_1 + \frac{1}{4}(1-r)(1+s)\alpha_2 + \frac{1}{4}(1-r)(1-s)\alpha_3 + \frac{1}{4}(1+r)(1-s)\alpha_4\right\}$$

(C-11)

$$\hat{J}_{11}\,\partial w/\partial r + \hat{J}_{12}\,\partial w/\partial s + \beta =$$

$$\hat{J}_{11}\left\{\frac{1}{4}(1+s)w_1 - \frac{1}{4}(1+s)w_2 - \frac{1}{4}(1-s)w_3 + \frac{1}{4}(1-s)w_4\right\}$$

$$+\hat{J}_{12}\left\{\frac{1}{4}(1+r)w_1 + \frac{1}{4}(1-r)w_2 - \frac{1}{4}(1-r)w_3 - \frac{1}{4}(1+r)w_4\right\}$$

$$+\left\{\frac{1}{4}(1+r)(1+s)\beta_1 + \frac{1}{4}(1-r)(1+s)\beta_2 + \frac{1}{4}(1-r)(1-s)\beta_3 + \frac{1}{4}(1+r)(1-s)\beta_4\right\}$$

(C-12)

$$\hat{J}_{21}\,\partial w/\partial r + \hat{J}_{22}\,\partial w/\partial s - \alpha =$$

$$+\frac{1}{4}\left[\hat{J}_{21}(1+s)+\hat{J}_{22}(1+r) \quad \hat{J}_{21}(1+s)+\hat{J}_{22}(1-r) \quad -\hat{J}_{21}(1-s)-\hat{J}_{22}(1-r) \quad \hat{J}_{21}(1-s)-\hat{J}_{22}(1+r)\right]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$-\frac{1}{4}\left[(1+r)(1+s) \quad (1-r)(1+s) \quad (1-r)(1-s) \quad (1+r)(1-s)\right]\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

(C-13)

$$\hat{J}_{11}\,\partial w/\partial r + \hat{J}_{12}\,\partial w/\partial s + \beta =$$

$$+\frac{1}{4}\left[\hat{J}_{11}(1+s)+\hat{J}_{12}(1+r) \quad -\hat{J}_{11}(1+s)+\hat{J}_{12}(1-r) \quad -\hat{J}_{11}(1-s)-\hat{J}_{12}(1-r) \quad \hat{J}_{11}(1-s)-\hat{J}_{12}(1+r)\right]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$+\frac{1}{4}\left[(1+r)(1+s) \quad (1-r)(1+s) \quad (1-r)(1-s) \quad (1+r)(1-s)\right]\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

(C-1)

$$\psi = \frac{1}{4}\begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} & v_{19} & v_{110} & v_{111} & v_{112} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & v_{26} & v_{27} & v_{28} & v_{29} & v_{210} & v_{211} & v_{212} \end{bmatrix}\begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix}$$

(C-15)

$$v_{11} = \hat{J}_{21}\left[\frac{1}{4}(1+s)\right] + \hat{J}_{22}\left[\frac{1}{4}(1+r)\right] \quad \text{(C-16)}$$

$$v_{12} = -\frac{1}{4}(1+r)(1+s) \quad \text{(C-17)}$$

$$v_{13} = 0 \quad \text{(C-18)}$$

$$v_{14} = \hat{J}_{21}\left[-\frac{1}{4}(1+s)\right] + \hat{J}_{22}\left[\frac{1}{4}(1-r)\right] \quad \text{(C-19)}$$

$$v_{15} = -\frac{1}{4}(1-r)(1+s) \quad \text{(C-20)}$$

$$v_{16} = 0 \quad \text{(C-21)}$$

$$v_{17} = \hat{J}_{21}\left[-\frac{1}{4}(1-s)\right] + \hat{J}_{22}\left[-\frac{1}{4}(1-r)\right] \quad \text{(C-22)}$$

$$v_{18} = -\frac{1}{4}(1-r)(1-s) \quad \text{(C-23)}$$

$$v_{19} = 0 \quad \text{(C-24)}$$

$$v_{110} = \hat{J}_{21}\left[\frac{1}{4}(1-s)\right] + \hat{J}_{22}\left[-\frac{1}{4}(1+r)\right] \quad \text{(C-25)}$$

$$v_{111} = -\frac{1}{4}(1+r)(1-s) \quad \text{(C-26)}$$

$$v_{112} = 0 \qquad \text{(C-27)}$$

$$v_{21} = \hat{J}_{11}\left[\frac{1}{4}(1+s)\right] + \hat{J}_{12}\left[\frac{1}{4}(1+r)\right] \qquad \text{(C-28)}$$

$$v_{22} = 0 \qquad \text{(C-29)}$$

$$v_{23} = \frac{1}{4}(1+r)(1+s) \qquad \text{(C-30)}$$

$$v_{24} = \hat{J}_{11}\left[-\frac{1}{4}(1+s)\right] + \hat{J}_{12}\left[+\frac{1}{4}(1-r)\right] \qquad \text{(C-31)}$$

$$v_{25} = 0 \qquad \text{(C-32)}$$

$$v_{26} = \frac{1}{4}(1-r)(1+s) \qquad \text{(C-33)}$$

$$v_{27} = \hat{J}_{11}\left[-\frac{1}{4}(1-s)\right] + \hat{J}_{12}\left[-\frac{1}{4}(1-r)\right] \qquad \text{(C-34)}$$

$$v_{28} = 0 \qquad \text{(C-35)}$$

$$v_{29} = \frac{1}{4}(1-r)(1-s) \qquad \text{(C-36)}$$

$$v_{210} = \hat{J}_{11}\left[+\frac{1}{4}(1-s)\right] + \hat{J}_{12}\left[-\frac{1}{4}(1+r)\right] \qquad \text{(C-37)}$$

$$v_{211} = 0 \qquad \text{(C-38)}$$

$$v_{212} = \frac{1}{4}(1+r)(1-s) \tag{C-39}$$

The useful wxMaxima commands are

z:matrix([v11,v12,0,v14,v15,0,v17,v18,0,v110,v111,0],[v21,0,v23,v24,0,v26,v27,0,v29,v210,
0,v212]);

transpose(z).z;

The resulting matrix is shown over this page and the next.

$\left\{ V^T\ C_s\ V \right\} =$

$$\begin{bmatrix}
v21^2+v11^2 & v11\ v12 & v21\ v23 & v21\ v24+v11\ v14 & v11\ v15 & v21\ v26 \\
v11\ v12 & v12^2 & 0 & v12\ v14 & v12\ v15 & 0 \\
v21\ v23 & 0 & v23^2 & v23\ v24 & 0 & v23\ v26 \\
v21\ v24+v11\ v14 & v12\ v14 & v23\ v24 & v24^2+v14^2 & v14\ v15 & v24\ v26 \\
v11\ v15 & v12\ v15 & 0 & v14\ v15 & v15^2 & 0 \\
v21\ v26 & 0 & v23\ v26 & v24\ v26 & 0 & v26^2 \\
v21\ v27+v11\ v17 & v12\ v17 & v23\ v27 & v24\ v27+v14\ v17 & v15\ v17 & v26\ v27 \\
v11\ v18 & v12\ v18 & 0 & v14\ v18 & v15\ v18 & 0 \\
v21\ v29 & 0 & v23\ v29 & v24\ v29 & 0 & v26\ v29 \\
v21\ v210+v11\ v110 & v110\ v12 & v210\ v23 & v210\ v24+v110\ v14 & v110\ v15 & v210\ v26 \\
v11\ v111 & v111\ v12 & 0 & v111\ v14 & v111\ v15 & 0 \\
v21\ v212 & 0 & v212\ v23 & v212\ v24 & 0 & v212\ v26
\end{bmatrix}$$

$$
\left.\begin{matrix}
v21\,v27+v11\,v17 & v11\,v18 & v21\,v29 & v21\,v210+v11\,v110 & v11\,v111 & v21\,v212 \\
v12\,v17 & v12\,v18 & 0 & v110\,v12 & v111\,v12 & 0 \\
v23\,v27 & 0 & v23\,v29 & v210\,v23 & 0 & v212\,v23 \\
v24\,v27+v14\,v17 & v14\,v18 & v24\,v29 & v210\,v24+v110\,v14 & v111\,v14 & v212\,v24 \\
v15\,v17 & v15\,v18 & 0 & v110\,v15 & v111\,v15 & 0 \\
v26\,v27 & 0 & v26\,v29 & v210\,v26 & 0 & v212\,v26 \\
v27^2+v17^2 & v17\,v18 & v27\,v29 & v210\,v27+v110\,v17 & v111\,v17 & v212\,v27 \\
v17\,v18 & v18^2 & 0 & v110\,v18 & v111\,v18 & 0 \\
v27\,v29 & 0 & v29^2 & v210\,v29 & 0 & v212\,v29 \\
v210\,v27+v110\,v17 & v110\,v18 & v210\,v29 & v210^2+v110^2 & v110\,v111 & v210\,v212 \\
v111\,v17 & v111\,v18 & 0 & v110\,v111 & v111^2 & 0 \\
v212\,v27 & 0 & v212\,v29 & v210\,v212 & 0 & v212^2
\end{matrix}\right]
$$

(C-40)

Note that integration is still required. This can be performed in Matlab using Gauss quadrature. Refer to Matlab script: thick_plate_mass_stiff.m

The integration can also be performed symbolically using wxMaxima.

APPENDIX D

Mass Matrix

The thick rectangular plate mass matrix can easily be calculated as a special case of the isoparametric mass element in Reference 3.