JACOBIAN MATRIX AND DETERMINANT
Revision E

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Introduction

The Jacobian of a function describes the orientation of a tangent plane to the function at a given point.

Consider a function F given by m real-valued component functions:

$$y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n)$$

The Jacobian matrix J of F is

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \tag{1}$$

Example

Consider a single-degree-of-freedom system with the following differential equation.

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2}$$

$$\ddot{x} + (c/m)\dot{x} + (k/m)x = (F/m) \tag{3}$$

$$\ddot{x} = -(c/m)\dot{x} - (k/m)x - (F/m) \tag{4}$$

Let

$$y = \dot{x}$$

$$\dot{y} = \ddot{x}$$

The second-order equation is thus transformed into two first-order equations.

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{5}$$

The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{6}$$

Other Applications

The Jacobian is also used in geometrical transformation applications, as shown in Appendix A

Reference

1. T. Irvine, The State Space Method for Solving Shock and Vibration Problems, Revision A, Vibrationdata, 2005.

APPENDIX A

Four-Node, Two-Dimensional Isoparametric Plate Element

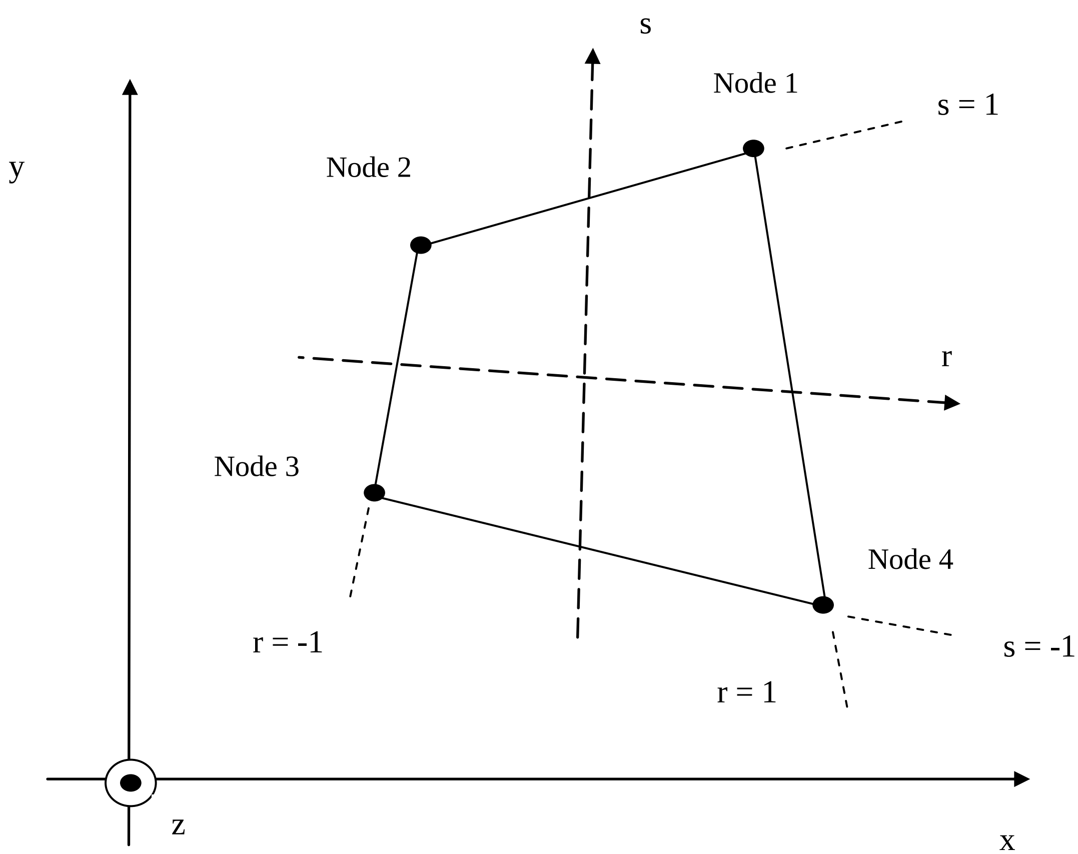

Figure A-1.

Note that

$-1 \leq r \leq +1$

$-1 \leq s \leq +1$

The coordinate interpolation is

$$x = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \tag{A-1}$$

$$y = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4 \tag{A-2}$$

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4 \tag{A-3}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4 \tag{A-4}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1+s)y_1 - \frac{1}{4}(1+s)y_2 - \frac{1}{4}(1-s)y_3 + \frac{1}{4}(1-s)y_4 \tag{A-5}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(1+r)y_1 + \frac{1}{4}(1-r)y_2 - \frac{1}{4}(1-r)y_3 - \frac{1}{4}(1+r)y_4 \tag{A-6}$$

The transformation equation is

$$\begin{bmatrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial r} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\[2ex] \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\[2ex] \dfrac{\partial}{\partial y} \end{bmatrix} \tag{A-7}$$

The Jacobian matrix J is

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \qquad \text{(A-8)}$$

By substitution,

$J =$

$$\begin{bmatrix} [(1+s)x_1 - (1+s)x_2 - (1-s)x_3 + (1-s)x_4]/4 & [(1+s)y_1 - (1+s)y_2 - (1-s)y_3 + (1-s)y_4]/4 \\ [(1+r)x_1 + (1-r)x_2 - (1-r)x_3 - (1+r)x_4]/4 & [(1+r)y_1 + (1-r)y_2 - (1-r)y_3 - (1+r)y_4]/4 \end{bmatrix}$$

(A-9)

The inverse of the Jacobian is evaluated symbolically using the software tool wxMaxima 0.8.7.

Let

$$\hat{J} = J^{-1} \tag{A-10}$$

$$\hat{J} =$$

$$\frac{1}{den}\begin{bmatrix} (2r+2)y_4 + (2-2r)y_3 + (2r-2)y_2 + (-2r-2)y_1 & -\left[(2s-2)y_4 + (2-2s)y_3 + (2s+2)y_2 + (-2s-2)y_1\right] \\ -\left[(2s-2)x_4 + (2-2s)x_3 + (2s+2)x_2 + (-2s-2)x_1\right] & (2s-2)x_4 + (2-2s)x_3 + (2s+2)x_2 + (-2s-2)x_1 \end{bmatrix} \tag{A-11}$$

$$\begin{aligned} den = \; & ((s-1)\, x3+(-s-r)\, x2+(r+1)\, x1)\, y4+((1-s)\, x4+(r-1)\, x2+(s-r)\, x1)\, y3 \\ & +((s+r)\, x4+(1-r)\, x3+(-s-1)\, x1)\, y2+((-r-1)\, x4+(r-s)\, x3+(s+1)\, x2)\, y1 \end{aligned} \tag{A-12}$$

The wxMaxima format is

$\hat{J}_{11}$ =

((2*r+2)*y4+(2-2*r)*y3+(2*r-2)*y2+(-2*r-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-13)

$\hat{J}_{12}$ =

-((2*s-2)*y4+(2-2*s)*y3+(2*s+2)*y2+(-2*s-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-14)

$\hat{J}_{21}$ =

-((2*r+2)*x4+(2-2*r)*x3+(2*r-2)*x2+(-2*r-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-15)

$\hat{J}_{22}$ =

((2*s-2)*x4+(2-2*s)*x3+(2*s+2)*x2+(-2*s-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1) (A-16)

The inverse Jacobian at each of the four nodes is

$$\hat{J}_1 =$$

$$\frac{1}{(x_2 - x_1)y_4 + (x_1 - x_4)y_2 + (x_4 - x_2)y_1} \begin{bmatrix} -(2y_4 - 2y_1) & 2y_2 - 2y_1 \\ 2x_4 - 2x_1 & -(2x_2 - 2x_1) \end{bmatrix}$$

(A-17)

$$\hat{J}_2 =$$

$$\frac{1}{(x_2 - x_1)y_3 + (x_1 - x_3)y_2 + (x_3 - x_2)y_1} \begin{bmatrix} -(2y_4 - 2y_1) & 2y_2 - 2y_1 \\ 2x_4 - 2x_1 & -(2x_2 - 2x_1) \end{bmatrix}$$

(A-18)

$$\hat{J}_3 =$$

$$\frac{1}{(x_3 - x_2)y_4 + (x_2 - x_4)y_3 + (x_4 - x_3)y_2} \begin{bmatrix} -(2y_3 - 2y_2) & -(2y_4 - 2y_3) \\ 2x_3 - 2x_2 & 2x_4 - 2x_3 \end{bmatrix}$$

(A-19)

$$\hat{J}_4 =$$

$$\frac{1}{(x_3 - x_1)y_4 + (x_1 - x_4)y_3 + (x_4 - x_3)y_1} \begin{bmatrix} -(2y_4 - 2y_1) & -(2y_4 - 2y_3) \\ 2x_4 - 2x_1 & 2x_4 - 2x_3 \end{bmatrix}$$

(A-20)

Note that the Jacobian and its inverse should be evaluated at each node. Furthermore, at node i,

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = J_i^{-1} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix}, \quad \text{at } r = r_i \text{ and } s = s_j$$

(A-21)

The determinant of the Jacobian is found via wxMaxima.

```
j11(r,s):=((1+s)*x1-(1+s)*x2+(-(1-s))*x3+(1-s)*x4)/4;
j12(r,s):=((1+s)*y1-(1+s)*y2+(-(1-s))*y3+(1-s)*y4)/4;
j21(r,s):=((1+r)*x1+(1-r)*x2+(-(1-r))*x3+(-(1+r))*x4)/4;
j22(r,s):=((1+r)*y1+(1-r)*y2+(-(1-r))*y3+(-(1+r))*y4)/4;
detJ(r,s):=ratsimp(determinant(matrix([j11(r,s),j12(r,s)],[j21(r,s),j22(r,s)])));.
```

(A-22)

The explicit form is

```
ratsimp(detJ(r,s));

-(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+
((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+
((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1)/8
```

(A-23)

Equation (A-23) is left in wxMaxima format because it is convenient to copy and paste into Matlab.

Other useful wxMaxima commands:

```
J(r,s):=matrix([j11(r,s), j12(r,s)],[ j21(r,s), j22(r,s)]);

Jinv(r,s):=invert(matrix([j11(r,s),j12(r,s)],[j21(r,s),j22(r,s)]));
```

```
Jinv11(r,s):=((2*r+2)*y4+(2-2*r)*y3+(2*r-2)*y2+(-2*r-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1);

Jinv12(r,s):=-((2*s-2)*y4+(2-2*s)*y3+(2*s+2)*y2+(-2*s-2)*y1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1);

Jinv21(r,s):=-((2*r+2)*x4+(2-2*r)*x3+(2*r-2)*x2+(-2*r-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1);

Jinv22(r,s):=((2*s-2)*x4+(2-2*s)*x3+(2*s+2)*x2+(-2*s-2)*x1)/(((s-1)*x3+(-s-r)*x2+(r+1)*x1)*y4+((1-s)*x4+(r-1)*x2+(s-r)*x1)*y3+((s+r)*x4+(1-r)*x3+(-s-1)*x1)*y2+((-r-1)*x4+(r-s)*x3+(s+1)*x2)*y1);
```

APPENDIX B

Four-Node, Two-Dimensional Isoparametric Plate Element Alternate Form

The partial derivatives can be rearranged as

$$\frac{\partial x}{\partial r}=\frac{1}{4}\{x_1-x_2-x_3+x_4\}+\frac{1}{4}\{x_1-x_2+x_3-x_4\}s \tag{B-1}$$

$$\frac{\partial x}{\partial s}=\frac{1}{4}\{x_1+x_2-x_3-x_4\}+\frac{1}{4}\{x_1-x_2+x_3-x_4\}r \tag{B-2}$$

$$\frac{\partial y}{\partial r}=\frac{1}{4}\{y_1-y_2-y_3+y_4\}+\frac{1}{4}\{y_1-y_2+y_3-y_4\}s \tag{B-3}$$

$$\frac{\partial y}{\partial s}=\frac{1}{4}\{y_1+y_2-y_3-y_4\}+\frac{1}{4}\{y_1-y_2+y_3-y_4\}r \tag{B-4}$$

Rectangle

Consider the special case of a rectangle where the local coordinate system may be rotated with respect to the global system.

$$\frac{\partial x}{\partial r}=\frac{1}{4}\{x_1-x_2-x_3+x_4\} \tag{B-5}$$

$$\frac{\partial x}{\partial s}=\frac{1}{4}\{x_1+x_2-x_3-x_4\} \tag{B-6}$$

$$\frac{\partial y}{\partial r}=\frac{1}{4}\{y_1-y_2-y_3+y_4\} \tag{B-7}$$

$$\frac{\partial y}{\partial s}=\frac{1}{4}\{y_1+y_2-y_3-y_4\} \tag{B-8}$$