LAGRANGE'S EQUATIONS

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Introduction

Dynamic systems can be characterized in terms of one or more natural frequencies. The natural frequency is the frequency at which the system would vibrate if it were given an initial disturbance and then allowed to vibrate freely.

There are several methods available for determining the natural frequency. Some examples are

1. Newton's Law of Motion
2. Rayleigh's Method
3. Energy Method
4. Lagrange's Equation

Note that the Rayleigh, Energy, and Lagrange methods are closely related.

Some of these methods directly yield the natural frequency. Others yield a governing equation of motion, from which the natural frequency may be determined.

This tutorial focuses on Lagrange's Equation, which yields the equation of motion.

Derivation

The following derivation is taken from Reference 1.

Lagrange's equations are based on generalized coordinates.

Generalized coordinates are independent coordinates which describe the motion of the degrees-of-freedom of a system.

Consider a conservative system where the sum of the kinetic and potential energies is constant. The differential of the total energy is then zero.

$$d(T + U) = 0 \tag{1}$$

Let

q_i be a generalized displacement coordinate,

$\dot{q}_i$ be a generalized velocity coordinate.

The kinetic energy T is a function of both the generalized displacement and velocity coordinates.

$$T = T\left(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N\right) \tag{2}$$

The potential energy U is a function only of the generalized displacement coordinates.

$$U = U\left(q_1, q_2, \ldots, q_N\right) \tag{3}$$

The differential of T is

$$dT = \sum_{i=1}^{N} \frac{\partial T}{\partial q_i} dq_i + \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i \tag{4}$$

The equation for kinetic energy can be stated as

$$T = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij}\, \dot{q}_i\, \dot{q}_j \tag{5}$$

Differentiate the kinetic energy with respect to $\dot{q}_i$.

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{N} m_{ij}\, \dot{q}_j \tag{6}$$

The previous differentiation step is explained by the example in Appendix A.

Multiply equation (6) by $\dot{q}_i$ and sum over i from 1 to N.

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij}\, \dot{q}_i\, \dot{q}_j \tag{7}$$

Equation (5) can be expressed as

$$2T = \sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} \dot{q}_i \dot{q}_j \tag{8}$$

Now substitute equation (7) into (8).

$$2T = \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \tag{9}$$

Form the differential of 2T by using the calculus product rule.

$$2\,dT = \sum_{i=1}^{N} d\left(\frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \right) + \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i \tag{10}$$

Subtract equation (4) from (10). Note that the second term in equation (10) is eliminated.

$$dT = \sum_{i=1}^{N} d\left(\frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \right) - \sum_{i=1}^{N} \frac{\partial T}{\partial q_i} dq_i \tag{11}$$

$$dT = \sum_{i=1}^{N} \left\{ d\left(\frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \right) - \frac{\partial T}{\partial q_i} dq_i \right\} \tag{12}$$

The quantity dt can be shifted such that

$$d\left(\frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \right) = d\left(\frac{\partial T}{\partial \dot{q}_i} \frac{dq_i}{dt} \right) \tag{13}$$

$$d\left(\frac{\partial T}{\partial \dot{q}_i} \dot{q}_i \right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} dq_i \right) \tag{14}$$

Substitute equation (14) into (12).

$$dT = \sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} dq_i \right) - \frac{\partial T}{\partial q_i} dq_i \right\} \tag{15}$$

$$dT = \sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} \right\} dq_i \tag{16}$$

Now take the differential of the potential energy term.

$$dU = \sum_{i=1}^{N} \frac{\partial U}{\partial q_i} dq_i \tag{17}$$

Add equations (16) and (17).

$$d(T+U) = \sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} \right\} dq_i \tag{18}$$

Substitute equation (18) into (1).

$$\sum_{i=1}^{N} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} \right\} dq_i = 0 \tag{19}$$

The N generalized coordinates are independent of one another. Thus dq_i may assume arbitrary values. Thus, equation (19) requires

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \cdots, N \tag{20}$$

Equation (20) is Lagrange's equation for the free vibration of a conservative system.

Equation (1) can be modified if the system is subjected to work by external, non-potential forces.

$$d(T+U) = dW \tag{21}$$

where dW is the work of the forces when the system is subjected to an arbitrary infinitesimal displacement.

The principle of virtual work allows the work to be expressed in terms of generalized forces Q_i associated with generalized coordinates q_i.

$$dW = \sum_{i=1}^{N} Q_i \, dq_i \tag{22}$$

Lagrange's equation for a system with a nonconservative force is thus

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \,, \quad i = 1, 2, \cdots, N \tag{23}$$

Application

Examples are given in Appendices B and C.

Reference

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.

APPENDIX A

Again, the equation for kinetic energy can be stated as

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\,\dot{q}_i\,\dot{q}_j \tag{A-1}$$

Note that

$$m_{ij} = m_{ji} \tag{A-2}$$

Assume that N=3.

$$T = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}\,\dot{q}_i\,\dot{q}_j \tag{A-3}$$

$$\begin{aligned} T = \frac{1}{2}\Big\{ & m_{11}\,\dot{q}_1\,\dot{q}_1 + m_{12}\,\dot{q}_1\,\dot{q}_2 + m_{13}\,\dot{q}_1\,\dot{q}_3 \\ & + m_{21}\,\dot{q}_2\,\dot{q}_1 + m_{22}\,\dot{q}_2\,\dot{q}_2 + m_{23}\,\dot{q}_2\,\dot{q}_3 \\ & + m_{31}\,\dot{q}_3\,\dot{q}_1 + m_{32}\,\dot{q}_3\,\dot{q}_2 + m_{33}\,\dot{q}_3\,\dot{q}_3 \Big\} \end{aligned} \tag{A-4}$$

Simplify equation (A-4) in terms of mass symmetry.

$$\begin{aligned} T = \frac{1}{2}\Big\{ & m_{11}\,\dot{q}_1\,\dot{q}_1 + m_{12}\,\dot{q}_1\,\dot{q}_2 + m_{13}\,\dot{q}_1\,\dot{q}_3 \\ & + m_{12}\,\dot{q}_2\,\dot{q}_1 + m_{22}\,\dot{q}_2\,\dot{q}_2 + m_{23}\,\dot{q}_2\,\dot{q}_3 \\ & + m_{13}\,\dot{q}_3\,\dot{q}_1 + m_{23}\,\dot{q}_3\,\dot{q}_2 + m_{33}\,\dot{q}_3\,\dot{q}_3 \Big\} \end{aligned} \tag{A-5}$$

$$\begin{aligned} T = \frac{1}{2}\Big\{ & m_{11}\,\dot{q}_1^{\,2} + 2m_{12}\,\dot{q}_1\,\dot{q}_2 + 2m_{13}\,\dot{q}_1\,\dot{q}_3 \\ & + m_{22}\,\dot{q}_2^{\,2} + 2m_{23}\,\dot{q}_2\,\dot{q}_3 \\ & + m_{33}\,\dot{q}_3^{\,2} \Big\} \end{aligned} \tag{A-6}$$

$$T = \left\{ \frac{1}{2} m_{11} \dot{q}_1^{\,2} + m_{12} \dot{q}_1 \dot{q}_2 + m_{13} \dot{q}_1 \dot{q}_3 \right.$$

$$+ \frac{1}{2} m_{22} \dot{q}_2^{\,2} + m_{23} \dot{q}_2 \dot{q}_3$$

$$\left. + \frac{1}{2} m_{33} \dot{q}_3^{\,2} \right\} \tag{A-7}$$

Now take the partial derivatives.

$$\frac{\partial T}{\partial q_1} = m_{11} \dot{q}_1 + m_{12} \dot{q}_2 + m_{13} \dot{q}_3 \tag{A-8}$$

$$\frac{\partial T}{\partial q_2} = m_{12} \dot{q}_1 + m_{22} \dot{q}_2 + m_{23} \dot{q}_3 \tag{A-9}$$

$$\frac{\partial T}{\partial q_3} = m_{13} \dot{q}_1 + m_{23} \dot{q}_2 + m_{33} \dot{q}_3 \tag{A-10}$$

Equations (A-8) through (A-9) can be summarized as

$$\frac{\partial T}{\partial q_i} = m_{1i} \dot{q}_1 + m_{2i} \dot{q}_2 + m_{3i} \dot{q}_3 \tag{A-11}$$

By symmetry,

$$\frac{\partial T}{\partial q_i} = m_{i1} \dot{q}_1 + m_{i2} \dot{q}_2 + m_{i3} \dot{q}_3 \tag{A-12}$$

$$\frac{\partial T}{\partial q_i} = \sum_{j=1}^{3} m_{ij} \dot{q}_j \tag{A-13}$$

By induction, the partial derivative formula for a system of N degrees-of-freedom would be

$$\frac{\partial T}{\partial q_i} = \sum_{j=1}^{N} m_{ij} \dot{q}_j \tag{A-14}$$

APPENDIX B

Simple Pendulum Example

Consider a conservative system. An example is the pendulum shown in Figure B-1.

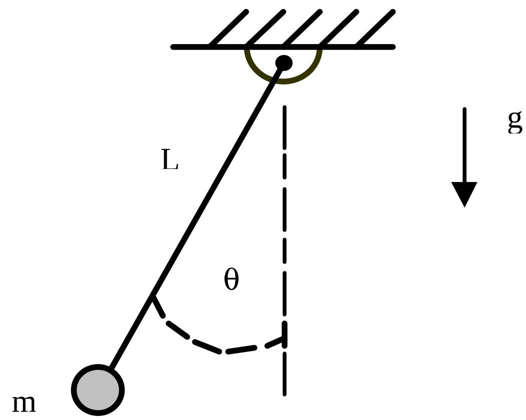


Figure B-1.

Let

m = pendulum mass,
L = length,
θ = angular displacement.

Assume a small angular displacement.

Recall Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \quad i = 1, 2, \cdots, N \tag{B-1}$$

The partial derivatives are changed to ordinary derivatives for a single-degree-of-freedom system. The appropriate form for the pendulum example is

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right) - \frac{dT}{d\theta} + \frac{dU}{d\theta} = 0 \tag{B-2}$$

The potential energy is

$$U = mgL(1 - \cos\theta) \tag{B-3}$$

$$\frac{dU}{d\theta} = mgL\sin\theta \tag{B-4}$$

The kinetic energy is

$$T = \frac{1}{2}m(L\dot{\theta})^2 \tag{B-5}$$

$$\frac{dT}{d\theta} = 0 \tag{B-6}$$

$$\frac{dT}{d\dot{\theta}} = m(L^2\dot{\theta}) \tag{B-7}$$

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}}\right) = m(L^2\,\ddot{\theta}) \tag{B-8}$$

Now substitute equations (B-8) and (B-4) into (B-2).

$$m(L^2\,\ddot{\theta}) + mgL\sin\theta = 0 \tag{B-9}$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{B-10}$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{B-11}$$

For small angular displacements,

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{B-12}$$

Equation (B-12) is the governing equation of motion.

Simple harmonic systems are known to have an equation of the form

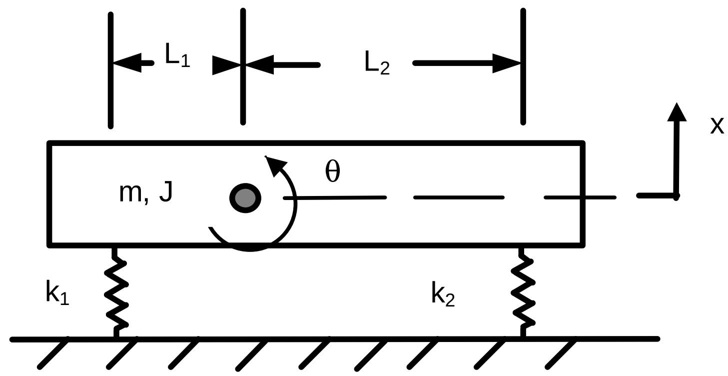
$$\ddot{\theta} + {\omega_n}^2\,\theta = 0 \tag{B-13}$$

where ω_n is the natural frequency. Thus, the natural frequency for the pendulum is

$$\omega_n = \sqrt{\frac{g}{L}} \tag{B-14}$$

APPENDIX C

Consider the two-degree-of-freedom system in Figure C-1. This system is an example of coordinate coupling.



The gray dot is the center of mass.

Figure C-1. Two-degree-of-freedom System

Let

m = mass,
J = rotational inertia,
k = spring stiffness,
x = translation of center of mass,
θ =rotation.

Lagrange's equations of motion for this system are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0 \qquad \text{(C-1)}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = 0 \qquad \text{(C-2)}$$

The kinetic energy is

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2 \tag{C-3}$$

Consider the kinetic energy with respect to the translational displacement.

$$\frac{\partial T}{\partial x} = 0 \tag{C-3}$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \tag{C-3}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\dot{x}^2 \tag{C-4}$$

Consider the kinetic energy with respect to the angular displacement.

$$\frac{\partial T}{\partial \theta} = 0 \tag{C-5}$$

$$\frac{\partial T}{\partial \dot{\theta}} = J\dot{\theta} \tag{C-6}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = J\ddot{\theta} \tag{C-7}$$

The potential energy is

$$U = \frac{1}{2} k_1 (x - L_1 \sin\theta)^2 + \frac{1}{2} k_2 (x + L_2 \sin\theta)^2 \tag{C-8}$$

Consider the potential energy with respect to the translational displacement.

$$\frac{dU}{dx} = k_1 (x - L_1 \sin\theta) + k_2 (x + L_2 \sin\theta) \tag{C-9}$$

Consider the potential energy with respect to the angular displacement.

$$\frac{dU}{dx} = -k_1 L_1 (x - L_1 \sin\theta)\cos\theta + k_2 L_2 (x + L_2 \sin\theta)\cos\theta \tag{C-10}$$

The two Lagrange equations are thus

$$m\ddot{x} + k_1 (x - L_1 \sin\theta) + k_2 (x + L_2 \sin\theta) = 0 \tag{C-11}$$

$$J\ddot{\theta} - k_1 L_1 (x - L_1 \sin\theta)\cos\theta + k_2 L_2 (x + L_2 \sin\theta)\cos\theta = 0 \tag{C-12}$$

Now assume small angular displacement.

$$m\ddot{x} + k_1 (x - L_1 \theta) + k_2 (x + L_2 \theta) = 0 \tag{C-13}$$

$$J\ddot{\theta} - k_1 L_1 (x - L_1 \theta) + k_2 L_2 (x + L_2 \theta) = 0 \tag{C-14}$$

The two equations can be expressed in matrix form.

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C-15}$$

Equation (C-15) is an example of static coupling.