DERIVATION OF MILES EQUATION   Revision D

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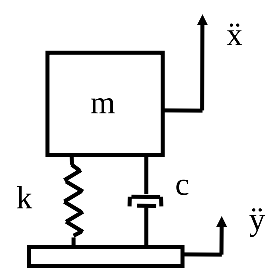
Introduction

The objective is to derive Miles equation. This equation gives the overall response of a single-degree-of-freedom system to base excitation where the excitation is in the form of a random vibration acceleration power spectral density.

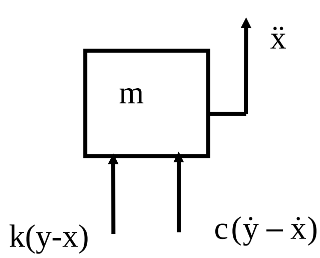
Derivation

Consider a single degree-of-freedom system



where m equals mass, c equals the viscous damping coefficient, and k equals the stiffness. The absolute displacement of the mass equals x, and the base input displacement equals y.

The free-body diagram is



Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z}+\ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n^{\,2}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (5) yields

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^{\,2}z = -\ddot{y} \tag{6}$$

Take the Fourier transform of each side.

$$\int_{-\infty}^{\infty}\left\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^{\,2}z\right\}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\{-\ddot{y}\}e^{-j\omega t}\,dt \tag{7}$$

Let

$$Y(\omega) = \int_{-\infty}^{\infty}\{y(t)\}e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{-\infty}^{\infty}\{x(t)\}e^{-j\omega t}\,dt$$

Now take the Fourier transform of the velocity term

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\left\{\frac{dz(t)}{dt}\right\}e^{-j\omega t}\,dt \tag{8}$$

Integrate by parts

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = \int_{-\infty}^{\infty} d\left\{z(t)e^{-j\omega t}\right\} - \int_{-\infty}^{\infty}[z(t)](-j\omega)e^{-j\omega t}dt \qquad (9)$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = z(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + (j\omega)\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt \qquad (10)$$

$$z(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} = 0 \text{ as t approaches the} \pm\infty \text{ limits.} \qquad (11)$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = (j\omega)\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt \qquad (12)$$

$$\int_{-\infty}^{\infty}\{\dot{z}(t)\}e^{-j\omega t}dt = (j\omega)X(\omega) \qquad (13)$$

Furthermore

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left\{\frac{d^2z(t)}{dt^2}\right\}e^{-j\omega t}dt \qquad (14)$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \int_{-\infty}^{\infty} d\left\{\frac{dz(t)}{dt}e^{-j\omega t}\right\} - \int_{-\infty}^{\infty}[\frac{dz(t)}{dt}(-j\omega)e^{-j\omega t}dt \qquad (15)$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = \frac{dz(t)}{dt}e^{-j\omega t}\Big|_{-\infty}^{\infty} + (j\omega)\int_{-\infty}^{\infty}\frac{dz(t)}{dt}e^{-j\omega t}dt \qquad (16)$$

$$\frac{dz(t)}{dt}e^{-j\omega t}\Bigg|_{-\infty}^{\infty} = 0 \text{ as t approaches the} \pm\infty \text{ limits.} \qquad (17)$$

$$\int_{-\infty}^{\infty}\{\ddot{z}(t)\}e^{-j\omega t}dt = (j\omega)\int_{-\infty}^{\infty}\frac{dz(t)}{dt}e^{-j\omega t}dt \qquad (18)$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} dt = (j\omega)(j\omega)Z(\omega) \quad (19)$$

$$\int_{-\infty}^{\infty} \{\ddot{z}(t)\} e^{-j\omega t} dt = -\omega^2 Z(\omega) \quad (20)$$

Recall

$$\int_{-\infty}^{\infty} \left\{\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z\right\} e^{-j\omega t} dt = \int_{-\infty}^{\infty} \{-\ddot{y}\} e^{-j\omega t} dt \quad (21)$$

Let the subscript A denote acceleration. By substitution,

$$-\omega^2 Z(\omega) + j\omega(2\xi\omega_n)Z(\omega) + \omega_n^2 Z(\omega) = -Y_A(\omega) \quad (22)$$

$$\left[(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\right] Z(\omega) = -Y_A(\omega) \quad (23)$$

$$Z(\omega) = \frac{-Y_A(\omega)}{\left[(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\right]} \quad (24)$$

$$Z_A(\omega) = -\omega^2 Z(\omega) \quad (25)$$

$$Z_A(\omega) = \frac{\omega^2 Y_A(\omega)}{\left[(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\right]} \quad (26)$$

The relative acceleration equation can be expressed in terms of Fourier transforms as

$$Z_A(\omega) = X_A(\omega) - Y_A(\omega) \quad (27)$$

$$X_A(\omega) = Z_A(\omega) + Y_A(\omega) \quad (28)$$

$$X_A(\omega) = \frac{\omega^2 Y_A(\omega)}{\left[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n\right]} + Y_A(\omega) \qquad (29)$$

$$X_A(\omega) = \frac{[\omega^2 + (\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n]Y_A(\omega)}{\left[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n\right]} \qquad (30)$$

$$X_A(\omega) = \frac{[\omega_n{}^2 + j2\xi\omega\omega_n]Y_A(\omega)}{\left[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n\right]} \qquad (31)$$

Multiply each side by its complex conjugate

$$X_A(\omega)X_A{}^*(\omega) = \frac{[\omega_n{}^2 + j2\xi\omega\omega_n][\omega_n{}^2 - j2\xi\omega\omega_n]Y_A(\omega)Y_A{}^*(\omega)}{\left[(\omega_n{}^2 - \omega^2) + j2\xi\omega\omega_n\right]\left[(\omega_n{}^2 - \omega^2) - j2\xi\omega\omega_n\right]} \qquad (32)$$

$$X_A(\omega)X_A{}^*(\omega) = \frac{[\omega_n{}^4 + (2\xi\omega\omega_n)^2]Y_A(\omega)Y_A{}^*(\omega)}{\left[(\omega_n{}^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2\right]} \qquad (33)$$

$$X_A(\omega)X_A{}^*(\omega) = \frac{\omega_n{}^2[\omega_n{}^2 + (2\xi\omega)^2]Y_A(\omega)Y_A{}^*(\omega)}{\left[(\omega_n{}^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2\right]} \qquad (34)$$

The Fourier transforms are converted into power spectral densities using the method shown in Reference 3, where T is the duration.

$$\lim_{T\to\infty} X_A(\omega)X_A{}^*(\omega)/T = X_{APSD}(\omega) \qquad (35)$$

$$\lim_{T\to\infty} Y_A(\omega)Y_A{}^*(\omega)/T = Y_{APSD}(\omega) \qquad (36)$$

$$X_{APSD}(\omega) = \frac{\omega_n^2[\omega_n^2 + (2\xi\omega)^2] Y_{APSD}(\omega)}{\left[(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2\right]} \tag{37}$$

Equation (A-32) can be transformed as a function of frequency f as follows

$$\hat{X}_{APSD}(f) = \frac{f_n^2[f_n^2 + (2\xi f)^2] \hat{Y}_{APSD}(f)}{\left[(f_n^2 - f^2)^2 + (2\xi f f_n)^2\right]} \tag{38}$$

Divide each side by f_n^4

$$\hat{X}_{APSD}(f) = \frac{[1 + (2\xi f / f_n)^2] \hat{Y}_{APSD}(f)}{\left[(1 - (f / f_n)^2)^2 + (2\xi f / f_n)^2\right]} \tag{39a}$$

Let $\rho = f / f_n$,

$$\hat{X}_{APSD}(f) = \frac{[1 + (2\xi\rho)^2]}{\left[(1 - \rho^2)^2 + (2\xi\rho)^2\right]} \hat{Y}_{APSD}(f), \quad \rho = f / f_n \tag{39b}$$

Define $H(\rho)$ as

$$H(\rho) = \frac{1 + j2\xi\rho}{\left(1 - \rho^2\right) + j2\xi\rho} \tag{40}$$

Multiply by the complex conjugate

$$H(\rho)H^*(\rho) = \left[\frac{1 + j2\xi\rho}{\left(1 - \rho^2\right) + j2\xi\rho}\right]\left[\frac{1 - j2\xi\rho}{\left(1 - \rho^2\right) - j2\xi\rho}\right] \tag{41}$$

Note that

$$\hat{X}_{APSD}(f) = H(\rho)H^*(\rho)\, \hat{Y}_{APSD}(f), \quad \rho = f / f_n \tag{42}$$

Rearrange equation (41) as

$$H(\rho)H^*(\rho) = \left[\frac{1+4\xi^2\rho^2}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{1}{\rho^2 + j2\xi\rho - 1}\right] \tag{43}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \tag{44}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{45}$$

$$R1, R2 = j\xi \pm \sqrt{1-\xi^2} \tag{46}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j2\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \tag{47}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{48}$$

$$R3, R4 = -j\xi \pm \sqrt{1-\xi^2} \tag{49}$$

(50)

Summary,

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{51}$$

$$R2 = +j\xi - \sqrt{1-\xi^2} \tag{52}$$

$$R3 = -j\xi + \sqrt{1-\xi^2} \tag{53}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{54}$$

Note

$$R2 = -R1^* \tag{55}$$

$$R3 = R1^* \tag{56}$$

$$R4 = -R1^* \tag{57}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)} \right] \tag{58}$$

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)} \right] \tag{59}$$

Substitute equations (55) through (57) into equation (59).

$$H(\rho)H^*(\rho) = \left[\frac{1 + 4\xi^2 \rho^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] \tag{60}$$

Expand into partial fractions.

$$\left[\frac{1+4\xi^2\rho^2}{(\rho-R1)(\rho+R1*)(\rho-R1*)(\rho+R1)}\right]=$$

$$+\frac{\alpha}{(\rho-R1)}$$
$$+\frac{\beta}{(\rho-R1*)}$$
$$+\frac{\lambda}{(\rho+R1*)}$$
$$+\frac{\sigma}{(\rho+R1)}$$

(61)

Multiply through by the denominator on the left-hand side of equation (61).

$$\left[1+4\xi^2\rho^2\right]=$$

$$+\frac{\alpha}{(\rho-R1)}(\rho-R1)(\rho+R1*)(\rho-R1*)(\rho+R1)$$
$$+\frac{\beta}{(\rho-R1*)}(\rho-R1)(\rho+R1*)(\rho-R1*)(\rho+R1)$$
$$+\frac{\lambda}{(\rho+R1*)}(\rho-R1)(\rho+R1*)(\rho-R1*)(\rho+R1)$$
$$+\frac{\sigma}{(\rho+R1)}(\rho-R1)(\rho+R1*)(\rho-R1*)(\rho+R1)$$

(62)

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$
\begin{aligned}
&+\alpha\,(\rho - R1^*)(\rho + R1^*)(\rho + R1)\\
&+\beta\,(\rho - R1)(\rho + R1^*)(\rho + R1)\\
&+\lambda\,(\rho - R1)(\rho - R1^*)(\rho + R1)\\
&+\sigma\,(\rho - R1)(\rho - R1^*)(\rho + R1^*)
\end{aligned}
\tag{63}
$$

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$
\begin{aligned}
&+\alpha\left(\rho^2 - R1^{*2}\right)(\rho + R1)\\
&+\beta\left(\rho^2 + (-R1 + R1^*)\rho - R1R1^*\right)(\rho + R1)\\
&+\lambda\left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1)\\
&+\sigma\left(\rho^2 + (-R1 - R1^*)\rho + R1R1^*\right)(\rho + R1^*)
\end{aligned}
\tag{64}
$$

$$\left\lfloor 1+4\xi^2\rho^2 \right\rfloor =$$

$$
\begin{aligned}
&+\alpha\left(\rho^3 + R1\rho^2 - R1^{*2}\rho - R1R1^{*2}\right)\\
&+\beta\left(\rho^3 + (R1 - R1 + R1^*)\rho^2 + \left(-R1R1^* - R1^2 + R1R1^*\right)\rho - R1^2R1^*\right)\\
&+\lambda\left(\rho^3 + (R1 - R1 - R1^*)\rho^2 + \left(R1R1^* - R1^2 - R1R1^*\right)\rho + R1^2R1^*\right)\\
&+\sigma\left(\rho^3 + (R1^* - R1 - R1^*)\rho^2 + \left(-R1R1^* - R1R1^* - R1^{*2}\right)\rho + R1R1^{*2}\right)
\end{aligned}
\tag{65}
$$

$$\left[1+4\xi^2\rho^2\right]=$$

$$\begin{aligned}
&+\alpha\left(\rho^3+R1\rho^2-R1*^2\rho-R1R1*^2\right)\\
&+\beta\left(\rho^3+R1*\rho^2-R1^2\rho-R1^2R1*\right)\\
&+\lambda\left(\rho^3-R1*\rho^2-R1^2\rho+R1^2R1*\right)\\
&+\sigma\left(\rho^3-R1\rho^2-R1*^2\rho+R1R1*^2\right)
\end{aligned}$$

(66)

$$\left[1+4\xi^2\rho^2\right]=$$

$$\begin{aligned}
&+\left[\alpha+\beta+\lambda+\sigma\right]\rho^3\\
&+\left[R1\alpha+R1*\beta-R1*\lambda-R1\sigma\right]\rho2\\
&+\left[-R1*^2\alpha-R1^2\beta-R1^2\lambda-R1*^2\sigma\right]\rho\\
&+\left[-R1R1*^2\alpha-R1^2R1*\beta+R1^2R1*\lambda+R1R1*^2\sigma\right]
\end{aligned}$$

(67)

$$\left[1+4\xi^2\rho^2\right]=$$

$$\begin{aligned}
&+\left[\alpha+\beta+\lambda+\sigma\right]\rho^3\\
&+\left[R1\alpha+R1*\beta-R1*\lambda-R1\sigma\right]\rho2\\
&+\left[-R1*^2\alpha-R1^2\beta-R1^2\lambda-R1*^2\sigma\right]\rho\\
&+\left[-R1*\alpha-R1\beta+R1\lambda+R1*\sigma\right]R1R1*
\end{aligned}$$

(68)

Equation (68) can be broken up into four separate equations, (69) through (72).

$$\alpha + \beta + \lambda + \sigma = 0 \tag{69}$$

$$\left[\, R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma \right] = 4\xi^2 \tag{70}$$

$$\left\lfloor - R1^{*2}\,\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\,\sigma \right\rfloor = 0 \tag{71}$$

$$\left[\, - R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma \right] R1R1^* = 1 \tag{72}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1/(R1R1^*) \end{bmatrix} \tag{73}$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{74}$$

$$R1R1^* = \left[+ j\xi + \sqrt{1-\xi^2} \right]\left[- j\xi + \sqrt{1-\xi^2} \right] \tag{75}$$

$$R1\ R1^* = 1 \tag{76}$$

Substitute equation (76) into (73).

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{77}$$

Multiply the first row by $R1*^2$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & -R1^2 + R1*^2 & -R1^2 + R1*^2 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{78}$$

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{79a}$$

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{79b}$$

Multiply the first row by -R1 and add to the second row. Also multiply the first row by R1* and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1* & -R1* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{80}$$

Multiply the third row by -R1* and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 4\xi^2 \\ 0 \\ 1 \end{bmatrix} \tag{81}$$

The first row equation yields

$$\alpha = -\sigma \tag{82}$$

The third row equation yields

$$\lambda = -\beta \tag{83}$$

Equation (81) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 4\xi^2 \\ 1 \end{bmatrix} \tag{84}$$

Complete the solution using Cramer's rule.

$$\text{determinant} \begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = 4\left[R1^2 - R1^{*2} \right] \tag{85}$$

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{86}$$

$$R1^2 = \left[+j\xi + \sqrt{1-\xi^2} \right] \left[+j\xi + \sqrt{1-\xi^2} \right] \tag{87}$$

$$R1^2 = -\xi^2 + \left(1-\xi^2\right) + j\left[2\xi \sqrt{1-\xi^2} \right] \tag{88}$$

$$R1^2 = \left(1-2\xi^2\right) + j\left[2\xi \sqrt{1-\xi^2} \right] \tag{89}$$

$$R1^* = -j\xi + \sqrt{1-\xi^2} \tag{90}$$

$$R1^{*2} = \left[-j\xi + \sqrt{1-\xi^2}\right]\left[-j\xi + \sqrt{1-\xi^2}\right] \tag{91}$$

$$R1^{*2} = -\xi^2 + \left(1-\xi^2\right) - j\left[2\xi\sqrt{1-\xi^2}\right] \tag{92}$$

$$R1^{*2} = \left(1-2\xi^2\right) - j\left[2\xi\sqrt{1-\xi^2}\right] \tag{93}$$

Thus,

$$R1^2 - R1^{*2} = j\left[4\xi\sqrt{1-\xi^2}\right] \tag{94}$$

$$4\left[R1^2 - R1^{*2}\right] = j\left[16\xi\sqrt{1-\xi^2}\right] \tag{95}$$

$$\text{determinant}\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix}\begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = j\left[16\xi\sqrt{1-\xi^2}\right] \tag{96}$$

$$\lambda = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]}\text{determinant}\begin{bmatrix} 4\xi^2 & -2R1 \\ 1 & +2R1^* \end{bmatrix} \tag{97}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(2R1^*\right) + 2R1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{98}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(R1^*\right) + R1}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{99}$$

Recall,

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{100}$$

$$\lambda = \frac{\left(4\xi^2\right)\left(-j\xi+\sqrt{1-\xi^2}\right)+j\xi+\sqrt{1-\xi^2}}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{101}$$

$$\lambda = \frac{\left(1+4\xi^2\right)\left(\sqrt{1-\xi^2}\right)+j\xi\left(1-4\xi^2\right)}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{102}$$

$$\lambda = \frac{+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\left(\sqrt{1-\xi^2}\right)}{\left[8\xi\sqrt{1-\xi^2}\right]} \tag{103}$$

$$\sigma = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]}\,\mathrm{determinant}\begin{bmatrix}-2R1^* & 4\xi^2\\ 2R1 & 1\end{bmatrix} \tag{104}$$

$$\sigma = \frac{\left\lfloor -2R1^* - \left(4\xi^2\right)\left(2R1\right)\right\rfloor}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{105}$$

$$\sigma = \frac{\left\lfloor -R1^* - \left(4\xi^2\right)\left(R1\right)\right\rfloor}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{106}$$

$$R1 = +j\xi+\sqrt{1-\xi^2} \tag{107}$$

$$\sigma = \frac{\left[-\left(-j\xi+\sqrt{1-\xi^2}\right)-\left(4\xi^2\right)\left(+j\xi+\sqrt{1-\xi^2}\right)\right]}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{108}$$

$$\sigma = \frac{\left[+ j\xi - \sqrt{1-\xi^2} - j4\xi^3 - 4\xi^2\sqrt{1-\xi^2} \right]}{j\left[8\xi\sqrt{1-\xi^2} \right]} \tag{109}$$

$$\sigma = \frac{\left[-\left(1+4\xi^2\right)\sqrt{1-\xi^2} + j\xi\left(1-4\xi^2\right) \right]}{j\left[8\xi\sqrt{1-\xi^2} \right]} \tag{110}$$

$$\sigma = \frac{\left[+\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \right]}{j\left[8\xi\sqrt{1-\xi^2} \right]} \tag{111}$$

Recall,

$$\alpha = -\sigma \tag{112}$$

$$\lambda = -\beta \tag{113}$$

The complete solution set is thus

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{8\xi\sqrt{1-\xi^2}} \begin{bmatrix} -\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ -\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) - j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \\ +\left(1-4\xi^2\right) + j\left(1+4\xi^2\right)\sqrt{1-\xi^2} \end{bmatrix} \tag{114}$$

The partial fraction expansion is thus

$$H(\rho)H^*(\rho) = + \frac{\alpha}{(\rho - R1)} + \frac{\beta}{(\rho - R1^*)} + \frac{\lambda}{(\rho + R1^*)} + \frac{\sigma}{(\rho + R1)} \tag{115}$$

The overall response $\ddot{x}_{GRMS}$ can be obtained by integrating $\hat{X}_{A\,PSD}(f)$ across the frequency spectrum and then taking the square root of the area per References 1 and 2.

$$\ddot{x}_{GRMS}(f_n, \xi) = \sqrt{\int_0^\infty H(\rho)H^*(\rho)\,\hat{Y}_{A\,PSD}(f)\,df} \tag{116}$$

$$\ddot{x}_{GRMS}(f_n, \xi) = \sqrt{f_n \int_0^\infty H(\rho)H^*(\rho)\,\hat{Y}_{A\,PSD}(f)\,d\rho} \tag{117}$$

Now assume that $\hat{Y}_{A\,PSD}(f)$ is a constant level represented by the variable A. Furthermore, assume the level is constant over an infinite frequency range, starting at zero.

$$\ddot{x}_{GRMS}(f_n, \xi) = \sqrt{A\,f_n \int_0^\infty H(\rho)H^*(\rho)\,d\rho} \tag{118}$$

$$\frac{\{\ddot{x}_{GRMS}(f_n, \xi)\}^2}{A\,f_n} = \int_0^\infty H(\rho)H^*(\rho)\,d\rho \tag{119}$$

Substituting equation (115) into (119) yields

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\,f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\int_0^\infty \frac{-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}}{\rho-\sqrt{1-\xi^2}-j\xi}\,d\rho$$

$$+\int_0^\infty \frac{-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}}{\rho+\sqrt{1-\xi^2}-j\xi}\,d\rho$$

$$+\int_0^\infty \frac{+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}}{\rho-\sqrt{1-\xi^2}+j\xi}\,d\rho$$

$$+\int_0^\infty \frac{+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}}{\rho+\sqrt{1-\xi^2}+j\xi}\,d\rho \tag{120}$$

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\int_0^{\infty}\frac{d\rho}{\rho-\sqrt{1-\xi^2}-j\xi}$$

$$+\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\int_0^{\infty}\frac{d\rho}{\rho-\sqrt{1-\xi^2}+j\xi}$$

$$+\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\int_0^{\infty}\frac{d\rho}{\rho+\sqrt{1-\xi^2}-j\xi}$$

$$+\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\int_0^{\infty}\frac{d\rho}{\rho+\sqrt{1-\xi^2}+j\xi}$$

(121)

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\ln\left[\rho-\sqrt{1-\xi^2}-j\xi\right]\Bigg|_0^{\infty}$$

$$+\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\ln\left[\rho-\sqrt{1-\xi^2}+j\xi\right]\Bigg|_0^{\infty}$$

$$+\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\ln\left[\rho+\sqrt{1-\xi^2}-j\xi\right]\Bigg|_0^{\infty}$$

$$+\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\ln\left[\rho+\sqrt{1-\xi^2}+j\xi\right]\Bigg|_0^{\infty}$$

(122)

Note that

$$\ln[x+jy]=\ln\left[\left(\sqrt{x^2+y^2}\right)\exp(\,j\,(\phi+2k\pi)\,)\right] \tag{123}$$

where

$$\phi=\arctan\left[\frac{y}{x}\right] \tag{124}$$

$$k=0,\pm 1,\pm 2,\ ..... \tag{125}$$

$$\ln[x+jy\ ]=\ln\left[\sqrt{x^2+y^2}\right]+j[\phi+2k\pi] \tag{126}$$

Take k = 0.

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\ln\sqrt{\left(\rho-\sqrt{1-\xi^2}\right)^2+\xi^2}+j\arctan\left(\frac{-\xi}{\rho-\sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\ln\sqrt{\left(\rho-\sqrt{1-\xi^2}\right)^2+\xi^2}+j\arctan\left(\frac{\xi}{\rho-\sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\ln\sqrt{\left(\rho+\sqrt{1-\xi^2}\right)^2+\xi^2}+j\arctan\left(\frac{-\xi}{\rho+\sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\ln\sqrt{\left(\rho+\sqrt{1-\xi^2}\right)^2+\xi^2}+j\arctan\left(\frac{\xi}{\rho+\sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

(127)

Both the real and imaginary components of the natural log terms cancel out at the lower integration limit.

The imaginary components of the natural log terms cancel out at the upper limit.

The sum of the real components of the natural log terms approaches zero as the upper limit approaches infinity.

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[j\left[0-\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}}\right)\right]\right]$$

$$+\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[j\left[0-\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}}\right)\right]\right]$$

$$+\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[j\left[0-\arctan\left(\frac{-\xi}{\sqrt{1-\xi^2}}\right)\right]\right]$$

$$+\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[j\left[0-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]\right]$$

(128)

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$- j\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}}\right)$$

$$- j\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}}\right)$$

$$- j\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\arctan\left(\frac{-\xi}{\sqrt{1-\xi^2}}\right)$$

$$- j\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right) \tag{129}$$

Now make trigonometric substitutions.

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} =$$

$$+\{-j\}\Bigg\{\left[-\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\pi+\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+\left[-\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+\left[+\xi\left(1-4\xi^2\right)-j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[2\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+\left[+\xi\left(1-4\xi^2\right)+j\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right]\left[\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]\Bigg\} \tag{130}$$

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n}\left\{8\xi\sqrt{1-\xi^2}\right\} = \left\{2\pi\left(1+4\xi^2\right)\sqrt{1-\xi^2}\right\} \tag{131}$$

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n} = \left\{\frac{2\pi\left(1+4\xi^2\right)\sqrt{1-\xi^2}}{8\xi\sqrt{1-\xi^2}}\right\} \tag{132}$$

$$\frac{\{\ddot{x}_{GRMS}(f_n,\xi)\}^2}{A\ f_n} = \left\{\frac{\pi\left(1+4\xi^2\right)}{4\xi}\right\} \tag{133}$$

$$\ddot{x}_{GRMS}(f_n,\xi) = \sqrt{\frac{A\ f_n\ \pi\left(1+4\xi^2\right)}{4\xi}} \tag{134}$$

$$\text{For } \xi \le 0.10, \quad \left(1+4\xi^2\right) \approx 1 \tag{135}$$

For small damping,

$$\ddot{x}_{GRMS}\left(f_n, \xi\right) = \sqrt{\frac{A\, f_n\, \pi}{4\xi}} \tag{136}$$

$$Q = \frac{1}{2\xi} \tag{137}$$

$$\ddot{x}_{GRMS}\left(f_n, Q\right) = \sqrt{\left(\frac{\pi}{2}\right) f_n \;\; Q \;\; A} \tag{138}$$

Now consider the realistic case where the acceleration power spectral density level varies with frequency. Nevertheless, constrain the acceleration power spectral density level to be constant within ± 1 octave of the natural frequency.

$$A = \hat{Y}_{APSD}(f_n) \tag{139}$$

$$\ddot{x}_{GRMS}\left(f_n, Q\right) = \sqrt{\left(\frac{\pi}{2}\right) f_n \;\; Q \;\; \hat{Y}_{APSD}(f_n)} \tag{140}$$

Note that equation (140), the Miles equation, may lead to error, particularly if the acceleration power spectral density function has significant variation with frequency. Thus, the general method in Reference 3 is the preferred method. The general method is shown as equation (141).

$$\ddot{x}_{GRMS}\left(f_n, \xi\right) = \sqrt{\sum_{i=1}^{N} \frac{\left\{1+(2\,\xi\rho_i)^2\right\}}{\left\{\left[1-\rho_i^{\,2}\right]^2+\left[2\,\xi\rho_i\right]^2\right\}} \hat{Y}_{APSD}(f_i)\Delta f_i}\,, \quad \rho_i = f_i / f_n \tag{141}$$

The general method allows the power spectral density to vary with frequency. It also allows for power spectral density inputs with finite frequency limits.

References

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