DERIVATION OF MILES EQUATION FOR AN APPLIED FORCE Revision C

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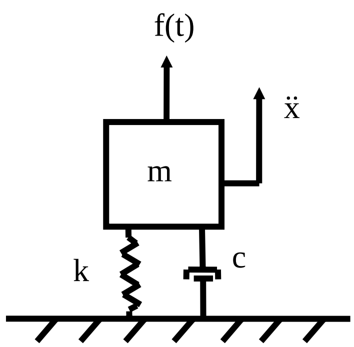
August 6, 2008

Introduction

The objective is to derive a Miles equation which gives the overall response of a single-degree-of-freedom system to an applied force, where the excitation is in the form of a random vibration acceleration power spectral density.

Derivation

Consider a single-degree-of-freedom system.

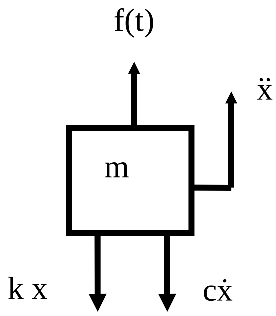


where

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
x	is the displacement of the mass
f(t)	is the applied force

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -c\dot{x} - kx + f(t) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{3}$$

$$\ddot{x} + (c/m)\dot{x} + (k/m)x = (1/m)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n^{\,2}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n^{\,2}x = (1/m)f(t) \tag{5}$$

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n^{\,2}x = (\omega_n^{\,2}/k)f(t) \tag{6}$$

Let

$$f(t) = \hat{F}\exp(j\omega t) \tag{7}$$

$$x(t) = \hat{X}\exp(j\omega t) \tag{8}$$

$$\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]\hat{X}\exp(j\omega t) = (\omega_n^2 / k)\hat{F}\exp(j\omega t) \tag{9}$$

Take the Fourier transform of each side.

$$\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]X(\omega) = (\omega_n^2 / k)F(\omega) \tag{10}$$

$$X(\omega) = \frac{(\omega_n^2 / k)F(\omega)}{\left[-\omega^2 + \omega_n^2 + j2\xi\omega\omega_n\right]} \tag{11}$$

$$X(f) = \frac{(f_n^2 / k)F(f)}{\left[-f^2 + f_n^2 + j2\xi f f_n\right]} \tag{12}$$

$$X(f) = \frac{(1/k)F(f)}{\left[1 - \rho^2 + j2\xi\rho\right]} \tag{13}$$

$$X(f) = \frac{(1/k)F(f)}{\left[1 - \rho^2 + j2\xi\rho\right]} \tag{14}$$

Multiply each side by its complex conjugate.

$$X(f)X^*(f) = \frac{(1/k)}{\left[1 - \rho^2 + j2\xi\rho\right]}\frac{(1/k)}{\left[1 - \rho^2 - j2\xi\rho\right]}F(f)F^*(f) \tag{15}$$

$$X(f)X^*(f) = \frac{(1/k)}{\left[\rho^2 - 1 + j2\xi\rho\right]} \frac{(1/k)}{\left[\rho^2 - 1 - j2\xi\rho\right]} F(f)F^*(f) \tag{16}$$

The transfer function $H(\rho)$ times its complex conjugate is

$$H(\rho)H^*(\rho) = \left[\frac{1/k}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{1/k}{\rho^2 + j2\xi\rho - 1}\right] \tag{17}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \tag{18}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{19}$$

$$R1, R2 = j\xi \pm \sqrt{1 - \xi^2} \tag{20}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \tag{21}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{22}$$

$$R3, R4 = -j\xi \pm \sqrt{1 - \xi^2} \tag{23}$$

(24)

Summary,

$$R1 = +j\xi + \sqrt{1 - \xi^2} \tag{25}$$

$$R2 = +j\xi - \sqrt{1-\xi^2} \tag{26}$$

$$R3 = -j\xi + \sqrt{1-\xi^2} \tag{27}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{28}$$

Note

$$R2 = -R1^* \tag{29}$$

$$R3 = R1^* \tag{30}$$

$$R4 = -R1^* \tag{31}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1/k^2}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)}\right] \tag{32}$$

$$H(\rho)H^*(\rho) = \left[\frac{1/k^2}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)}\right] \tag{33}$$

$$H(\rho)H^*(\rho) = \left[\frac{1/k^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] \tag{34}$$

Expand into partial fractions.

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = \begin{array}{l} + \dfrac{\alpha}{(\rho - R1)} \\ + \dfrac{\beta}{(\rho - R1^*)} \\ + \dfrac{\lambda}{(\rho + R1^*)} \\ + \dfrac{\sigma}{(\rho + R1)} \end{array} \tag{35}$$

Multiply through by the denominator on the left-hand side of equation (35).

$$1 = \begin{array}{l} + \dfrac{\alpha}{(\rho - R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ + \dfrac{\beta}{(\rho - R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ + \dfrac{\lambda}{(\rho + R1^*)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \\ + \dfrac{\sigma}{(\rho + R1)}(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1) \end{array} \tag{36}$$

$$1 = \begin{array}{l} + \alpha\,(\rho - R1^*)(\rho + R1^*)(\rho + R1) \\ + \beta\,(\rho - R1)(\rho + R1^*)(\rho + R1) \\ + \lambda\,(\rho - R1)(\rho - R1^*)(\rho + R1) \\ + \sigma\,(\rho - R1)(\rho - R1^*)(\rho + R1^*) \end{array} \tag{37}$$

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^2 - R1*^2\right)(\rho + R1) \\
& +\beta\left(\rho^2 + (-R1 + R1*)\rho - R1R1*\right)(\rho + R1) \\
& +\lambda\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1) \\
& +\sigma\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1*)
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + (R1 - R1 + R1*)\rho^2 + \left(-R1R1* - R1^2 + R1R1*\right)\rho - R1^2 R1*\right) \\
& +\lambda\left(\rho^3 + (R1 - R1 - R1*)\rho^2 + \left(R1R1* - R1^2 - R1R1*\right)\rho + R1^2 R1*\right) \\
& +\sigma\left(\rho^3 + (R1* - R1 - R1*)\rho^2 + \left(-R1R1* - R1R1* - R1*^2\right)\rho + R1R1*^2\right)
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
1 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + R1*\rho^2 - R1^2\rho - R1^2 R1*\right) \\
& +\lambda\left(\rho^3 - R1*\rho^2 - R1^2\rho + R1^2 R1*\right) \\
& +\sigma\left(\rho^3 - R1\rho^2 - R1*^2\rho + R1R1*^2\right)
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
1 = \quad & +[\alpha + \beta + \lambda + \sigma]\rho^3 \\
& +[R1\alpha + R1*\beta - R1*\lambda - R1\sigma]\rho 2 \\
& +\left[-R1*^2\alpha - R1^2\beta - R1^2\lambda - R1*^2\sigma\right]\rho \\
& +\left[-R1R1*^2\alpha - R1^2 R1*\beta + R1^2 R1*\lambda + R1R1*^2\sigma\right]
\end{aligned}
\tag{41}
$$

$$1 = \quad +\left[\alpha+\beta+\lambda+\sigma\right]\rho^3$$

$$+\left[R1\alpha+R1^*\beta-R1^*\lambda-R1\sigma\right]\rho 2$$

$$+\left[-R1^{*2}\,\alpha-R1^2\,\beta-R1^2\lambda-R1^{*2}\,\sigma\right]\rho$$

$$+\left[-R1^*\alpha-R1\beta+R1\lambda+R1^*\sigma\right]R1R1^*$$

(42)

Equation (42) can be broken up into four separate equations,

$$\alpha+\beta+\lambda+\sigma=0 \tag{43}$$

$$\left[R1\alpha+R1^*\beta-R1^*\lambda-R1\sigma\right]=0 \tag{44}$$

$$\left[-R1^{*2}\,\alpha-R1^2\beta-R1^2\lambda-R1^{*2}\,\sigma\right]=0 \tag{45}$$

$$\left[-R1^*\alpha-R1\beta+R1\lambda+R1^*\sigma\right]R1R1^* \quad =1 \tag{46}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/R1R1^* \end{bmatrix}$$

(47)

Recall

$$R1=+j\xi+\sqrt{1-\xi^2} \tag{48}$$

$$R1R1^*=\left[+j\xi+\sqrt{1-\xi^2}\right]\left[-j\xi+\sqrt{1-\xi^2}\right] \tag{49}$$

$$R1\ R1^* = 1 \tag{50}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ -R1^{*2} & -R1^2 & -R1^2 & -R1^{*2} \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{51}$$

Multiply the first row by $R1^{*2}$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & -R1^2 + R1^{*2} & -R1^2 + R1^{*2} & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{52}$$

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{53a}$$

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{54b}$$

Multiply the first row by -R1 and add to the second row. Also multiply the first row by R1* and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1^* & -R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{55}$$

Multiply the third row by -R1* and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{56}$$

The first row equation yields

$$\alpha = -\sigma \tag{57}$$

The third row equation yields

$$\beta = -\lambda \tag{58}$$

Equation (56) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{59}$$

$$-2R1^*\lambda - 2R1\sigma = 0 \tag{60}$$

$$-R1^*\lambda = R1\sigma \tag{61}$$

$$\lambda = \frac{-R1}{R1^*}\sigma \tag{62}$$

$$2R1\lambda + 2R1^*\sigma = 1 \tag{63}$$

$$2R1\left[\frac{-R1}{R1*}\ \sigma\right] + 2R1*\sigma = 1 \tag{64}$$

$$\left\{R1\left[\frac{-R1}{R1*}\ \right] + R1*\right\}\sigma = \frac{1}{2} \tag{65}$$

$$\sigma = \frac{1}{2}\frac{1}{\left\{R1\left[\frac{-R1}{R1*}\ \right] + R1*\right\}} \tag{66}$$

$$\sigma = \frac{1}{2}\frac{R1*}{[-R1^2 + R1*^2]} \tag{67}$$

$$\lambda = \frac{-R1}{R1*}\ \sigma \tag{68}$$

$$\lambda = -\frac{1}{2}\frac{R1}{[-R1^2 + R1*^2]} \tag{69}$$

Recall,

$$\alpha = -\sigma \tag{70}$$

$$\beta = -\lambda \tag{71}$$

The complete solution set is thus

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{2}\frac{1}{[-R1^2 + R1*^2]}\begin{bmatrix}-R1*\\ R1\\ -R1\\ R1*\end{bmatrix} \tag{72}$$

Note that

$$-R1^2 + R1^{*2} = -j\left[4\xi\sqrt{1-\xi^2}\right] \tag{73}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{2}\left[\frac{1}{-j\left[4\xi\sqrt{1-\xi^2}\right]}\right]\begin{bmatrix}-\sqrt{1-\xi^2}+j\xi\\ +\sqrt{1-\xi^2}+j\xi\\ -\sqrt{1-\xi^2}-j\xi\\ +\sqrt{1-\xi^2}-j\xi\end{bmatrix} \tag{74}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{j}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}-\sqrt{1-\xi^2}+j\xi\\ +\sqrt{1-\xi^2}+j\xi\\ -\sqrt{1-\xi^2}-j\xi\\ +\sqrt{1-\xi^2}-j\xi\end{bmatrix} \tag{75}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}-\xi-j\sqrt{1-\xi^2}\\ -\xi+j\sqrt{1-\xi^2}\\ +\xi-j\sqrt{1-\xi^2}\\ +\xi+j\sqrt{1-\xi^2}\end{bmatrix} \tag{76}$$

Let

$$\psi = \frac{\xi}{\sqrt{1-\xi^2}} \tag{77}$$

$$\begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \frac{1}{8\xi} \begin{bmatrix} -\psi - j \\ -\psi + j \\ +\psi - j \\ +\psi + j \end{bmatrix} \tag{78}$$

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{79}$$

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} \right] = \begin{aligned} &+ \frac{-\psi - j}{\left(\rho - \sqrt{1-\xi^2} - j\xi\right)} \left[\frac{1}{8\xi} \right] \\ &+ \frac{-\psi + j}{\left(\rho - \sqrt{1-\xi^2} + j\xi\right)} \left[\frac{1}{8\xi} \right] \\ &+ \frac{+\psi - j}{\left(\rho + \sqrt{1-\xi^2} - j\xi\right)} \left[\frac{1}{8\xi} \right] \\ &+ \frac{+\psi + j}{\left(\rho + \sqrt{1-\xi^2} + j\xi\right)} \left[\frac{1}{8\xi} \right] \end{aligned} \tag{80}$$

The overall displacement is found by integration.

$$[x_{RMS}(f_n, \xi)]^2 = \int_0^\infty H(\rho) H^*(\rho)\, \hat{F}_{PSD}(f)\, df \tag{81}$$

$$[x_{RMS}(f_n, \xi)]^2 = f_n \int_0^\infty H(\rho) H^*(\rho)\, \hat{F}_{PSD}(f)\, d\rho \tag{82}$$

Assume that the force PSD is constant

$$\hat{F}_{PSD}(f) = A \tag{83}$$

$$[x_{RMS}(f_n,\xi)]^2 = A\, f_n \int_0^\infty H(\rho) H^*(\rho)\, d\rho \tag{84}$$

$$\{ x_{RMS}(f_n,\xi) \}^2 \left[\frac{8\xi k^2}{A\, f_n} \right] = + \int_0^\infty \frac{-\psi - j}{\left(\rho - \sqrt{1-\xi^2} - j\xi\right)} d\rho + \int_0^\infty \frac{-\psi + j}{\left(\rho - \sqrt{1-\xi^2} + j\xi\right)} d\rho + \int_0^\infty \frac{+\psi - j}{\left(\rho + \sqrt{1-\xi^2} - j\xi\right)} d\rho + \int_0^\infty \frac{+\psi + j}{\left(\rho + \sqrt{1-\xi^2} + j\xi\right)} d\rho \tag{85}$$

$$\{ x_{RMS}(f_n,\xi) \}^2 \left[\frac{8\xi k^2}{A\, f_n} \right] = + (-\psi - j) \ln\left(\rho - \sqrt{1-\xi^2} - j\xi\right)\Big|_0^\infty + (-\psi + j) \ln\left(\rho - \sqrt{1-\xi^2} + j\xi\right)\Big|_0^\infty + (+\psi - j) \ln\left(\rho + \sqrt{1-\xi^2} - j\xi\right)\Big|_0^\infty + (+\psi + j) \ln\left(\rho + \sqrt{1-\xi^2} + j\xi\right)\Big|_0^\infty \tag{86}$$

Note that

$$\ln\left[x + jy\right] = \ln\left[\sqrt{x^2 + y^2}\right] + j\arctan\left[\frac{y}{x}\right] \tag{87}$$

$$\{x_{RMS}(f_n, \xi)\}^2 \left[\frac{8\xi k^2}{A\, f_n}\right] =$$

$$+(-\psi - j)\left[\ln\sqrt{\left(\rho - \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{-\xi}{\rho - \sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+(-\psi + j)\left[\ln\sqrt{\left(\rho - \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{\xi}{\rho - \sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+(+\psi - j)\left[\ln\sqrt{\left(\rho + \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{-\xi}{\rho + \sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty$$

$$+(+\psi + j)\left[\ln\sqrt{\left(\rho + \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{\xi}{\rho + \sqrt{1-\xi^2}}\right)\right]\Bigg|_0^\infty \tag{88}$$

Both the real and imaginary components of the natural log terms cancel out at the lower integration limit.

The imaginary components of the natural log terms cancel out at the upper limit.

The sum of the real components of the natural log terms approaches zero as the upper limit approaches infinity.

$$\{ x_{RMS}(f_n, \xi) \}^2 \left[\frac{8\xi k^2}{A\, f_n} \right] =$$

$$+ (-\psi - j) \left[-j \arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (-\psi + j) \left[-j \arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+\psi - j) \left[-j \arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+\psi + j) \left[-j \arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

(89)

$$\{ x_{RMS}(f_n, \xi) \}^2 \left[\frac{8\xi k^2}{A\, f_n} \right] =$$

$$+ (-\psi - j) \left[0 - j \arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (-\psi + j) \left[0 - j \arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+\psi - j) \left[0 - j \arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+\psi + j) \left[0 - j \arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

(90)

$$\{ x_{RMS}(f_n, \xi) \}^2 \left[\frac{8 \xi k^2}{A f_n} \right] =$$

$$+ (-1 + j\psi) \left[\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+1 + j\psi) \left[\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}} \right) \right]$$

$$+ (-1 - j\psi) \left[\arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

$$+ (+1 - j\psi) \left[\arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}} \right) \right]$$

(91)

$$\{x_{RMS}(f_n,\xi)\}^2\left[\frac{8\xi k^2}{A\ f_n}\right] =$$

$$+(-1+j\psi)\left[-\pi+\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+(+1+j\psi)\left[-\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+(-1-j\psi)\left[-2\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$+(+1-j\psi)\left[\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

(92)

$$\{x_{RMS}(f_n,\xi)\}^2\left[\frac{8\xi k^2}{A\ f_n}\right] = 2\pi \tag{93}$$

$$\{x_{RMS}(f_n,\xi)\}^2 = \left[\frac{\pi}{4\xi}\right]\frac{A\ f_n}{k^2} \tag{94}$$

$$x_{RMS}(f_n, \xi) = \sqrt{\frac{\pi A \; f_n}{4 \xi k^2}} \tag{95}$$

$$x_{RMS} = \sqrt{\frac{\pi A \; \frac{1}{2\pi}\sqrt{\frac{k}{m}}}{4 \xi k^2}} \tag{96}$$

The RMS displacement is finally

$$x_{RMS} = \left[\frac{A}{8\xi}\right]^{1/2} \left[\frac{1}{m}\right]^{1/4} \left[\frac{1}{k}\right]^{3/4} \tag{97}$$

As a review,

m	is the mass
k	is the stiffness
ξ	is viscous damping ratio
A	is the amplitude of the force PSD in dimensions of [force^2 / Hz] at the natural frequency

APPENDIX A

Velocity Response

Recall

$$X(f) = \frac{(1/k)F(f)}{\left[1-\rho^2 + j2\xi\rho\right]} \tag{A-14}$$

The corresponding velocity V(f) is

$$V(f) = \frac{j(2\pi f / k)F(f)}{\left[1-\rho^2 + j2\xi\rho\right]} \tag{A-14}$$

$$V(f) = f_n \frac{j(2\pi f / f_n k)F(f)}{\left[1-\rho^2 + j2\xi\rho\right]} \tag{A-14}$$

$$V(f) = j\left[\frac{2\pi f_n}{k}\right]\frac{\rho F(f)}{\left[1-\rho^2 + j2\xi\rho\right]} \tag{A-14}$$

Multiply each side by its complex conjugate.

$$V(f)V^*(f) = -\left[\frac{2\pi f_n}{k}\right]^2 \frac{\rho}{\left[1-\rho^2 + j2\xi\rho\right]}\frac{\rho}{\left[1-\rho^2 - j2\xi\rho\right]}F(f)F^*(f) \tag{A-15}$$

$$V(f)V^*(f) = -\left[\frac{2\pi f_n}{k}\right]^2 \frac{\rho}{\left[\rho^2 - 1 + j2\xi\rho\right]} \frac{\rho}{\left[\rho^2 - 1 - j2\xi\rho\right]} F(f)F^*(f) \tag{A-16}$$

The transfer function $H(\rho)$ times its complex conjugate is

$$H(\rho)H^*(\rho) = -\left[\frac{2\pi f_n}{k}\right]^2 \left[\frac{\rho}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{\rho}{\rho^2 + j2\xi\rho - 1}\right] \tag{A-17}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \tag{A-18}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{A-19}$$

$$R1, R2 = j\xi \pm \sqrt{1 - \xi^2} \tag{A-20}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \tag{A-21}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{A-22}$$

$$R3, R4 = -j\xi \pm \sqrt{1 - \xi^2} \tag{A-23}$$

(A-24)

Summary,

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{A-25}$$

$$R2 = +j\xi - \sqrt{1-\xi^2} \tag{A-26}$$

$$R3 = -j\xi + \sqrt{1-\xi^2} \tag{A-27}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{A-28}$$

Note

$$R2 = -R1^* \tag{A-29}$$

$$R3 = R1^* \tag{A-30}$$

$$R4 = -R1^* \tag{A-31}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho)$$

$$= \left[\frac{2\pi f_n}{k}\right]^2 \left[\frac{\rho^2}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)}\right] \tag{A-32}$$

$$H(\rho)H^*(\rho) = \left[\frac{2\pi f_n}{k}\right]^2 \left[\frac{\rho^2}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)}\right] \tag{A-33}$$

$$H(\rho)H^*(\rho) = \left[\frac{2\pi f_n}{k}\right]^2 \left[\frac{\rho^2}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] \quad \text{(A-34)}$$

Expand into partial fractions.

$$\left[\frac{\rho^2}{(\rho - R1)(\rho + R1*)(\rho - R1*)(\rho + R1)}\right] = \quad + \frac{\alpha}{(\rho - R1)} + \frac{\beta}{(\rho - R1*)} + \frac{\lambda}{(\rho + R1*)} + \frac{\sigma}{(\rho + R1)}$$

(A-35)

Multiply through by the denominator on the left-hand side of equation (A-35).

$$\begin{aligned}\rho^2 = \quad & + \frac{\alpha}{(\rho - R1)}(\rho - R1)(\rho + R1*)(\rho - R1*)(\rho + R1) \\ & + \frac{\beta}{(\rho - R1*)}(\rho - R1)(\rho + R1*)(\rho - R1*)(\rho + R1) \\ & + \frac{\lambda}{(\rho + R1*)}(\rho - R1)(\rho + R1*)(\rho - R1*)(\rho + R1) \\ & + \frac{\sigma}{(\rho + R1)}(\rho - R1)(\rho + R1*)(\rho - R1*)(\rho + R1)\end{aligned}$$

(A-36)

$$\begin{aligned}\rho^2 = \quad & + \alpha\,(\rho - R1*)(\rho + R1*)(\rho + R1) \\ & + \beta\,(\rho - R1)(\rho + R1*)(\rho + R1) \\ & + \lambda\,(\rho - R1)(\rho - R1*)(\rho + R1) \\ & + \sigma\,(\rho - R1)(\rho - R1*)(\rho + R1*)\end{aligned}$$

(A-37)

$$
\begin{aligned}
\rho^2 = \quad & +\alpha\left(\rho^2 - R1*^2\right)(\rho + R1) \\
& +\beta\left(\rho^2 + (-R1 + R1*)\rho - R1R1*\right)(\rho + R1) \\
& +\lambda\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1) \\
& +\sigma\left(\rho^2 + (-R1 - R1*)\rho + R1R1*\right)(\rho + R1*)
\end{aligned}
$$

(A-38)

$$
\begin{aligned}
\rho^2 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + (R1 - R1 + R1*)\rho^2 + \left(-R1R1* - R1^2 + R1R1*\right)\rho - R1^2R1*\right) \\
& +\lambda\left(\rho^3 + (R1 - R1 - R1*)\rho^2 + \left(R1R1* - R1^2 - R1R1*\right)\rho + R1^2R1*\right) \\
& +\sigma\left(\rho^3 + (R1* - R1 - R1*)\rho^2 + \left(-R1R1* - R1R1* - R1*^2\right)\rho + R1R1*^2\right)
\end{aligned}
$$

(A-39)

$$
\begin{aligned}
\rho^2 = \quad & +\alpha\left(\rho^3 + R1\rho^2 - R1*^2\rho - R1R1*^2\right) \\
& +\beta\left(\rho^3 + R1*\rho^2 - R1^2\rho - R1^2R1*\right) \\
& +\lambda\left(\rho^3 - R1*\rho^2 - R1^2\rho + R1^2R1*\right) \\
& +\sigma\left(\rho^3 - R1\rho^2 - R1*^2\rho + R1R1*^2\right)
\end{aligned}
$$

(A-40)

$$\rho^2 = +[\alpha+\beta+\lambda+\sigma]\rho^3$$
$$+[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma]\rho^2$$
$$+\left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma\right]\rho$$
$$+\left[-R1R1^{*2}\alpha - R1^2R1^*\beta + R1^2R1^*\lambda + R1R1^{*2}\sigma\right]$$

(A-41)

$$\rho^2 = +[\alpha+\beta+\lambda+\sigma]\rho^3$$
$$+[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma]\rho^2$$
$$+\left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma\right]\rho$$
$$+[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma]R1R1^*$$

(A-42)

Equation (42) can be broken up into four separate equations,

$$\alpha+\beta+\lambda+\sigma = 0 \quad \text{(A-43)}$$

$$[R1\alpha + R1^*\beta - R1^*\lambda - R1\sigma] = 1 \quad \text{(A-44)}$$

$$\left[-R1^{*2}\alpha - R1^2\beta - R1^2\lambda - R1^{*2}\sigma\right] = 0 \quad \text{(A-45)}$$

$$[-R1^*\alpha - R1\beta + R1\lambda + R1^*\sigma]R1R1^* = 0 \quad \text{(A-46)}$$

The four equations are assembled into matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ -R1*^2 & -R1^2 & -R1^2 & -R1*^2 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-47)

Recall

$$R1 = +j\xi + \sqrt{1-\xi^2} \qquad \text{(A-48)}$$

$$R1R1* = \left[+j\xi + \sqrt{1-\xi^2}\right]\left[-j\xi + \sqrt{1-\xi^2}\right] \qquad \text{(A-49)}$$

$$R1\ R1* = 1 \qquad \text{(A-50)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ -R1*^2 & -R1^2 & -R1^2 & -R1*^2 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-51)

Multiply the first row by $R1*^2$ and add to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1* & -R1* & -R1 \\ 0 & -R1^2 + R1*^2 & -R1^2 + R1*^2 & 0 \\ -R1* & -R1 & +R1 & +R1* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-52)

Scale the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-53a)

Multiply the third row by -1 and add to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ R1 & R1^* & -R1^* & -R1 \\ 0 & 1 & 1 & 0 \\ -R1^* & -R1 & +R1 & +R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-54b)

Multiply the first row by -R1 and add to the second row. Also multiply the first row by R1* and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & R1^* & -R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & -R1 & +R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-55)

Multiply the third row by -R1* and add to the second row. Also, multiply the third row by R1 and add to the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -2R1^* & -2R1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(A-56)

The first row equation yields

$$\alpha = -\sigma \qquad \text{(A-57)}$$

The third row equation yields

$$\beta = -\lambda \tag{A-58}$$

Equation (56) thus reduces to

$$\begin{bmatrix} -2R1^* & -2R1 \\ +2R1 & +2R1^* \end{bmatrix} \begin{bmatrix} \lambda \\ \sigma \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{A-59}$$

Note that the determinant of the coefficient matrix from Reference 2 is

$$-4R1^{*2} + 4R1^2 = j\left[16\xi\sqrt{1-\xi^2}\right] \tag{A-94}$$

Apply Cramer's rule.

$$\lambda = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \det\begin{bmatrix} 1 & -2R1 \\ 0 & +2R1^* \end{bmatrix} \tag{A-67}$$

$$\lambda = \frac{2R1^*}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{A-67}$$

$$\lambda = \frac{2\left[-j\xi + \sqrt{1-\xi^2}\right]}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{A-67}$$

$$\lambda = \frac{\left[\xi + j\sqrt{1-\xi^2}\right]}{-\left[8\xi\sqrt{1-\xi^2}\right]} \tag{A-67}$$

$$\lambda = \frac{-\xi - j\sqrt{1-\xi^2}}{8\xi\sqrt{1-\xi^2}} \tag{A-67}$$

$$\sigma = \frac{1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \det\begin{bmatrix} -2R1^* & 1 \\ +2R1 & 0 \end{bmatrix} \tag{A-67}$$

$$\sigma = \frac{-2R1}{j\left[16\xi\sqrt{1-\xi^2}\right]} \tag{A-67}$$

$$\sigma = \frac{-j\xi - \sqrt{1-\xi^2}}{j\left[8\xi\sqrt{1-\xi^2}\right]} \tag{A-67}$$

$$\sigma = \frac{\xi - j\sqrt{1-\xi^2}}{-8\xi\sqrt{1-\xi^2}} \tag{A-67}$$

$$\sigma = \frac{-\xi + j\sqrt{1-\xi^2}}{8\xi\sqrt{1-\xi^2}} \tag{A-67}$$

Recall,

$$\alpha = -\sigma \tag{A-70}$$

$$\beta = -\lambda \tag{A-71}$$

The complete solution set is thus

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}+\xi - j\sqrt{1-\xi^2}\\ +\xi + j\sqrt{1-\xi^2}\\ -\xi - j\sqrt{1-\xi^2}\\ -\xi + j\sqrt{1-\xi^2}\end{bmatrix}$$

(A-74)

Let

$$\psi = \frac{\xi}{\sqrt{1-\xi^2}} \qquad \text{(A-77)}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{8\xi}\begin{bmatrix}+\psi - j\\ +\psi + j\\ -\psi - j\\ -\psi + j\end{bmatrix}$$

(A-78)

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = + \frac{+\psi - j}{\left(\rho - \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{+\psi + j}{\left(\rho - \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{-\psi - j}{\left(\rho + \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{-\psi + j}{\left(\rho + \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right]$$

(A-80)

The overall velocity is found by integration.

$$\left[\dot{x}_{RMS}(f_n,\xi)\right]^2 = \int_0^\infty H(\rho)H^*(\rho)\,\hat{F}_{PSD}(f)\,df \tag{A-81}$$

$$\left[\dot{x}_{RMS}(f_n,\xi)\right]^2 = f_n \int_0^\infty H(\rho)H^*(\rho)\,\hat{F}_{PSD}(f)\,d\rho \tag{A-82}$$

Assume that the force PSD is constant

$$\hat{F}_{PSD}(f) = A \tag{A-83}$$

$$\left[\dot{x}_{RMS}(f_n,\xi)\right]^2 = A\,f_n \int_0^\infty H(\rho)H^*(\rho)\,d\rho \tag{A-84}$$

$$\begin{aligned}\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\,f_n}\right] = & -\int_0^\infty \frac{+\psi - j}{\left(\rho-\sqrt{1-\xi^2}-j\xi\right)}d\rho \\ & -\int_0^\infty \frac{+\psi + j}{\left(\rho-\sqrt{1-\xi^2}+j\xi\right)}d\rho \\ & -\int_0^\infty \frac{-\psi - j}{\left(\rho+\sqrt{1-\xi^2}-j\xi\right)}d\rho \\ & -\int_0^\infty \frac{-\psi + j}{\left(\rho+\sqrt{1-\xi^2}+j\xi\right)}d\rho\end{aligned} \tag{A-85}$$

$$
\{ \dot{x}_{RMS}(f_n, \xi) \}^2 \left[\frac{k}{2\pi f_n} \right]^2 \left[\frac{8\xi}{A\ f_n} \right] = -(+\psi - j) \ln\left(\rho - \sqrt{1-\xi^2} - j\xi \right)\Big|_0^\infty
$$

$$
-(+\psi + j) \ln\left(\rho - \sqrt{1-\xi^2} + j\xi \right)\Big|_0^\infty
$$

$$
-(-\psi - j) \ln\left(\rho + \sqrt{1-\xi^2} - j\xi \right)\Big|_0^\infty
$$

$$
-(-\psi + j) \ln\left(\rho + \sqrt{1-\xi^2} + j\xi \right)\Big|_0^\infty
$$

(A-86)

Note that

$$
\ln[x + jy] = \ln\left[\sqrt{x^2 + y^2} \right] + j \arctan\left[\frac{y}{x} \right] \qquad \text{(A-87)}
$$

$$\left\{ \dot{x}_{RMS}(f_n, \xi) \right\}^2 \left[\frac{k}{2\pi f_n} \right]^2 \left[\frac{8\xi}{A\, f_n} \right] =$$

$$-(+\psi - j)\left[\ln\sqrt{\left(\rho - \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{-\xi}{\rho - \sqrt{1-\xi^2}} \right) \right] \Bigg|_0^{\infty}$$

$$-(+\psi + j)\left[\ln\sqrt{\left(\rho - \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{\xi}{\rho - \sqrt{1-\xi^2}} \right) \right] \Bigg|_0^{\infty}$$

$$-(-\psi - j)\left[\ln\sqrt{\left(\rho + \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{-\xi}{\rho + \sqrt{1-\xi^2}} \right) \right] \Bigg|_0^{\infty}$$

$$-(-\psi + j)\left[\ln\sqrt{\left(\rho + \sqrt{1-\xi^2}\right)^2 + \xi^2} + j\arctan\left(\frac{\xi}{\rho + \sqrt{1-\xi^2}} \right) \right] \Bigg|_0^{\infty}$$

(A-88)

Both the real and imaginary components of the natural log terms cancel out at the lower integration limit.

The imaginary components of the natural log terms cancel out at the upper limit.

The sum of the real components of the natural log terms approaches zero as the upper limit approaches infinity.

$$\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\ f_n}\right]=$$

$$-(+\psi-j)\left[-j\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(+\psi+j)\left[-j\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(-\psi-j)\left[-j\arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

$$-(-\psi+j)\left[-j\arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

(A-89)

$$\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\ f_n}\right]=$$

$$-(-\psi-j)\left[0-j\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(-\psi+j)\left[0-j\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(+\psi-j)\left[0-j\arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

$$-(+\psi+j)\left[0-j\arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

(A-90)

$$\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\ f_n}\right] =$$

$$-(-1+j\psi)\left[\arctan\left(\frac{-\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(+1+j\psi)\left[\arctan\left(\frac{\xi}{-\sqrt{1-\xi^2}}\right)\right]$$

$$-(-1-j\psi)\left[\arctan\left(\frac{-\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

$$-(+1-j\psi)\left[\arctan\left(\frac{\xi}{+\sqrt{1-\xi^2}}\right)\right]$$

(A-91)

$$\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\ f_n}\right] =$$

$$-(-1+j\psi)\left[\pi+\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$-(+1+j\psi)\left[\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$-(-1-j\psi)\left[2\pi-\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

$$-(+1-j\psi)\left[\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right]$$

(A-92)

$$\{\dot{x}_{RMS}(f_n,\xi)\}^2\left[\frac{k}{2\pi f_n}\right]^2\left[\frac{8\xi}{A\ f_n}\right] = 2\pi \quad \text{(A-93)}$$

$$\left\{ \dot{x}_{RMS}\left(f_n, \xi\right) \right\}^2 = \frac{2\pi}{\left[\dfrac{k}{2\pi f_n}\right]^2 \left[\dfrac{8\xi}{A\, f_n}\right]} \tag{A-94}$$

$$\left\{ \dot{x}_{RMS}\left(f_n, \xi\right) \right\}^2 = \frac{\pi^3 A f_n{}^3}{k^2 \xi} \tag{A-94}$$

$$\left\{ \dot{x}_{RMS}\left(f_n, \xi\right) \right\}^2 = \frac{A}{8\xi} \frac{k^{3/2}}{k^2 m^{3/2}} \tag{A-94}$$

$$\left\{ \dot{x}_{RMS}\left(f_n, \xi\right) \right\}^2 = \frac{A}{8\xi} \frac{1}{k^{1/2} m^{3/2}} \tag{A-94}$$

The RMS velocity is finally

$$\dot{x}_{RMS} = \left[\frac{A}{8\xi}\right]^{1/2} \frac{1}{k^{1/4}\, m^{3/4}} \tag{A-94}$$

Recall that the RMS displacement is

$$x_{RMS} = \left[\frac{1}{m}\right]^{1/4} \left[\frac{1}{k}\right]^{3/4} \left[\frac{A}{8\xi}\right]^{1/2} \tag{A-97}$$

$$\frac{\dot{x}_{RMS}}{x_{RMS}} = \frac{\dfrac{1}{k^{1/4} m^{3/4}} \left[\dfrac{A}{8\xi}\right]^{1/2}}{\left[\dfrac{1}{m}\right]^{1/4} \left[\dfrac{1}{k}\right]^{3/4} \left[\dfrac{A}{8\xi}\right]^{1/2}} \tag{A-94}$$

$$\frac{\dot{x}_{RMS}}{x_{RMS}} = \frac{k^{1/2}}{m^{1/2}} \tag{A-94}$$

$$\frac{\dot{x}_{RMS}}{x_{RMS}} = \omega_n \tag{A-94}$$

APPENDIX B

Acceleration Response

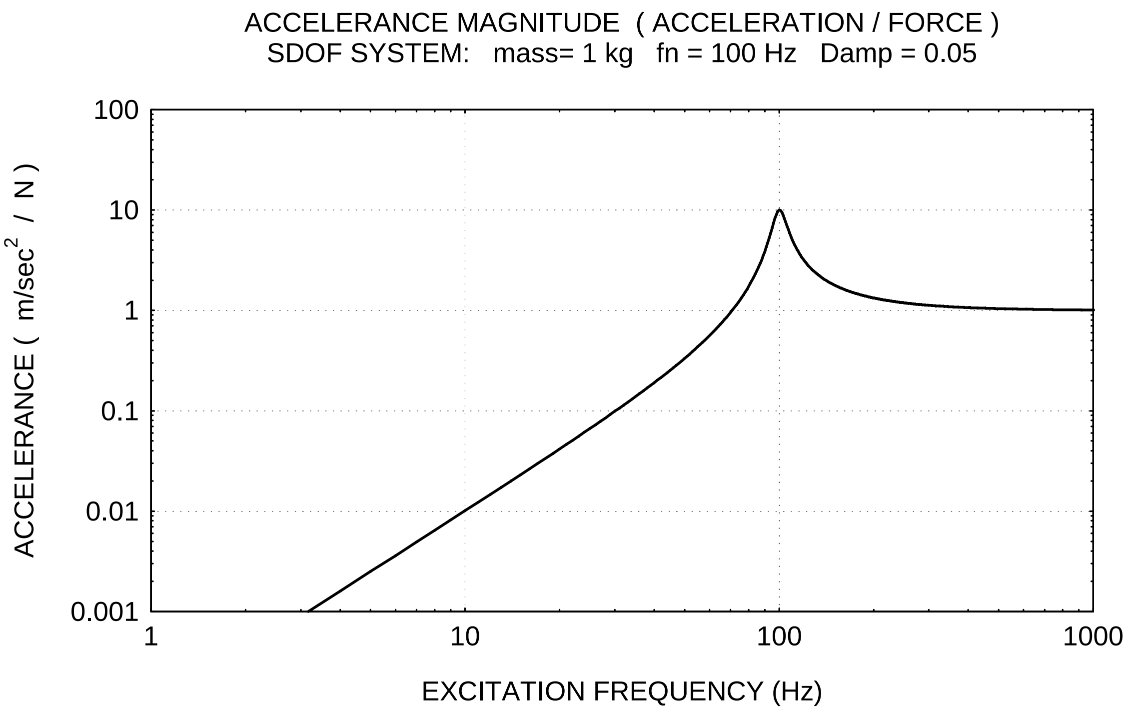


Figure B-1.

An accelerance FRF curve is shown for a sample system in Figure B-1. Note that the normalized accelerance converges to 1 as the excitation frequency becomes much larger than the natural frequency.

As a result, the acceleration response would be infinitely high for a white noise force excitation which extended up to an infinitely high frequency.

Thus, a Miles equation for the acceleration response to a white noise applied force cannot be derived.