DERIVATION OF MILES EQUATION FOR
THE RELATIVE DISPLACEMENT RESPONSE TO BASE EXCITATION

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July 28, 2008

Introduction

The objective is to derive a Miles equation which gives the overall relative displacement response of a single-degree-of-freedom system to an applied force, where the excitation is in the form of a random vibration acceleration power spectral density.

Derivation

Consider a single degree-of-freedom system

![Free-body diagram]

where m equals mass, c equals the viscous damping coefficient, and k equals the stiffness. The absolute displacement of the mass equals x, and the base input displacement equals y.

The free-body diagram is

![Free-body diagram]

Summation of forces in the vertical direction,

\[ \sum F = m \ddot{x} \]  \hspace{1cm} (1)
\[
m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}
\]

Substituting the relative displacement terms into equation (2) yields

\[
m(\ddot{z} + \ddot{y}) = -c\ddot{z} - kz \tag{3}
\]

\[
m\ddot{z} + c\ddot{z} + kz = -m\ddot{y} \tag{4}
\]

Dividing through by mass yields,

\[
\ddot{z} + (c/m)\ddot{z} + (k/m)z = -\ddot{y} \tag{5}
\]

By convention,

\[
\frac{c}{m} = 2\xi\omega_n
\]
\[
\frac{k}{m} = \omega_n^2
\]

where \(\omega_n\) is the natural frequency in (radians/sec), and \(\xi\) is the damping ratio.

\[
\ddot{z} + 2\xi\omega_n \ddot{z} + \omega_n^2 z = -\ddot{y} \tag{6}
\]

Let

\[
z(t) = \hat{Z}\exp(j\omega t) \tag{7}
\]
\[
y(t) = \hat{Y}\exp(j\omega t) \tag{8}
\]

\[
\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]\hat{Z}\exp(j\omega t) = -\hat{Y}\exp(j\omega t) \tag{9}
\]

Take the Fourier transform of each side.

\[
\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]Z(\omega) = -Y(\omega) \tag{10}
\]
\[ Z(\omega) = \frac{-Y(\omega)}{-\omega^2 + \omega_n^2 + j2\xi_\omega \omega_n} \]  
\[ Z(f) = \frac{-\left(1/2\pi\right)Y(f)}{-f^2 + f_n^2 + j2\xi f f_n} \]

Divide through by \( f_n^2 \).

\[ Z(f) = \frac{-\left(1/(2\pi f_n)^2\right)Y(f)}{-\rho^2 + 1 + j2\xi \rho} \]

where \( \rho = f/f_n \)

\[ Z(f) = \frac{-\left(1/(2\pi f_n)^2\right)Y(f)}{1 - \rho^2 + j2\xi \rho} \]

Multiply each side by its complex conjugate.

\[ Z(f)Z^*(f) = \frac{\left(1/(2\pi f_n)^2\right)}{1 - \rho^2 + j2\xi \rho} \frac{\left(1/(2\pi f_n)^2\right)}{1 - \rho^2 - j2\xi \rho} Y(f)Y^*(f) \]

\[ Z(f)Z^*(f) = \frac{\left(1/(2\pi f_n)^2\right)}{\rho^2 - 1 + j2\xi \rho} \frac{\left(1/(2\pi f_n)^2\right)}{\rho^2 - 1 - j2\xi \rho} Y(f)Y^*(f) \]
The transfer function \( H(\rho) \) times its complex conjugate is

\[
H(\rho)H^*(\rho) = \left[ \begin{array}{c}
\frac{1/(2\pi f_n)^2}{\rho^2 - j2\xi\rho - 1} \\
\frac{1/(2\pi f_n)^2}{\rho^2 + j2\xi\rho - 1}
\end{array} \right]
\]

(17)

Solve for the roots \( R_1 \) and \( R_2 \) of the first denominator.

\[
R_1, R_2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2}
\]

(18)

\[
R_1, R_2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2}
\]

(19)

\[
R_1, R_2 = j\xi \pm \sqrt{1 - \xi^2}
\]

(20)

Solve for the roots \( R_3 \) and \( R_4 \) of the second denominator.

\[
R_3, R_4 = -j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}
\]

(21)

\[
R_3, R_4 = -j2\xi \pm \sqrt{-4\xi^2 + 4}
\]

(22)

\[
R_3, R_4 = -j\xi \pm \sqrt{1 - \xi^2}
\]

(23)

Summary,

\[
R_1 = +j\xi + \sqrt{1 - \xi^2}
\]

(25)

\[
R_2 = +j\xi - \sqrt{1 - \xi^2}
\]

(26)

\[
R_3 = -j\xi + \sqrt{1 - \xi^2}
\]

(27)
\[ R_4 = -j\xi - \sqrt{1 - \xi^2} \]  

Note

\[ R_2 = -R_1^* \]  

\[ R_3 = R_1^* \]  

\[ R_4 = -R_1^* \]  

Now substitute into the denominators.

\[
H(\rho)H^*(\rho) = \left[ \frac{1/(2\pi f_n)^4}{(\rho - R_1)(\rho - R_2)(\rho - R_3)(\rho - R_4)} \right]
\]  

\[
H(\rho)H^*(\rho) = \left[ \frac{1/(2\pi f_n)^4}{(\rho - R_1)(\rho + R_1^*)(\rho - R_1^*)(\rho + R_1)} \right]
\]
Expand into partial fractions.

\[
\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)} = \frac{\alpha}{(\rho - R1)} + \frac{\beta}{(\rho - R1^*)} + \frac{\lambda}{(\rho + R1^*)} + \frac{\sigma}{(\rho + R1)}
\]  

Equation (35) is solved using the method in Reference 1.

\[
\begin{bmatrix}
\alpha \\
\beta \\
\lambda \\
\sigma \\
\end{bmatrix} = \frac{1}{8\xi\sqrt{1 - \xi^2}} \begin{bmatrix}
-\xi - j\sqrt{1 - \xi^2} \\
-\xi + j\sqrt{1 - \xi^2} \\
+\xi - j\sqrt{1 - \xi^2} \\
+\xi + j\sqrt{1 - \xi^2} \\
\end{bmatrix}
\]  

(36)

Let

\[
\psi = \frac{\xi}{\sqrt{1 - \xi^2}}
\]  

(37)

\[
\begin{bmatrix}
\alpha \\
\beta \\
\lambda \\
\sigma \\
\end{bmatrix} = \frac{1}{8\xi} \begin{bmatrix}
-\psi - j \\
-\psi + j \\
+\psi - j \\
+\psi + j \\
\end{bmatrix}
\]  

(38)
\[ R_1 = +j\xi + \sqrt{1-\xi^2} \] (39)

\[
\left[ \frac{1}{(\rho-R_1)(\rho+R_1^*)(\rho-R_1^*)(\rho+R_1)} \right] = +\frac{-\psi-j}{(\rho-\sqrt{1-\xi^2}-j\xi)} \left[ \frac{1}{8\xi} \right] \\
+\frac{-\psi+j}{(\rho-\sqrt{1-\xi^2}+j\xi)} \left[ \frac{1}{8\xi} \right] \\
+\frac{+\psi-j}{(\rho+\sqrt{1-\xi^2}-j\xi)} \left[ \frac{1}{8\xi} \right] \\
+\frac{+\psi+j}{(\rho+\sqrt{1-\xi^2}+j\xi)} \left[ \frac{1}{8\xi} \right] 
\] (40)

The overall relative displacement is found by integration.

\[ [z_{RMS}(f_n, \xi)]^2 = \int_0^\infty H(\rho)H^*(\rho) \hat{Y}_{APSD}(f) df \] (41)

\[ [x_{RMS}(f_n, \xi)]^2 = f_n \int_0^\infty H(\rho)H^*(\rho) \hat{Y}_{APSD}(f) d\rho \] (42)

Assume that the acceleration PSD is constant

\[ \hat{Y}_{APSD}(f) = A \] (43)

\[ [x_{RMS}(f_n, \xi)]^2 = A f_n \int_0^\infty H(\rho)H^*(\rho) d\rho \] (44)
\[
\{ x_{\text{RMS}} (f_n, \xi) \}^2 \left[ \frac{8\xi (2\pi)^4 f_n^3}{A} \right] = + \int_0^\infty \frac{-\psi - j}{\rho - \sqrt{1 - \xi^2} - j\xi} \, dp
\]

\[
+ \int_0^\infty \frac{-\psi + j}{\rho - \sqrt{1 - \xi^2} + j\xi} \, dp
\]

\[
+ \int_0^\infty \frac{+\psi - j}{\rho + \sqrt{1 - \xi^2} + j\xi} \, dp
\]

\[
+ \int_0^\infty \frac{+\psi + j}{\rho + \sqrt{1 - \xi^2} - j\xi} \, dp
\]

(45)

Equation (45) is solved using the method in Reference 1.

\[
\{ z_{\text{RMS}} (f_n, \xi) \}^2 \left[ \frac{8\xi (2\pi)^4 f_n^3}{A} \right] = 2\pi
\]

(46)

\[
\{ z_{\text{RMS}} (f_n, \xi) \}^2 = \left[ \frac{\pi}{4\xi} \right] \frac{A}{(2\pi)^4 f_n^3}
\]

(47)

\[
z_{\text{RMS}} (f_n, \xi) = \sqrt{\frac{\pi A}{4\xi (2\pi)^4 f_n^3}}
\]

(48)
The RMS relative displacement is finally

\[ z_{\text{RMS}}(f_n, \xi) = \sqrt{\frac{A}{64\pi^3 \xi f_n^3}} \]  \hspace{2cm} (49)

\[ Q = \frac{1}{2\xi} \]  \hspace{2cm} (50)

\[ z_{\text{RMS}} = \sqrt{\frac{QA}{32\pi^3 f_n^3}} \]  \hspace{2cm} (51)

Note that equation for the absolute acceleration response in Reference 2 is

\[ \ddot{x}_{\text{GRMS}}(f_n, Q) = \sqrt{\left(\frac{\pi}{2}\right) f_n \ Q \ A} \]  \hspace{2cm} (52)

Thus

\[ \ddot{x}_{\text{GRMS}} = \left[2\pi f_n\right]^2 z_{\text{RMS}} \]  \hspace{2cm} (53)

References