DERIVATION OF MILES EQUATION FOR
THE RELATIVE DISPLACEMENT RESPONSE TO BASE EXCITATION

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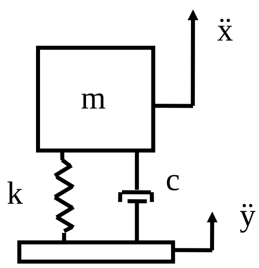
## Introduction

The objective is to derive a Miles equation which gives the overall relative displacement response of a single-degree-of-freedom system to an applied force, where the excitation is in the form of a random vibration acceleration power spectral density.

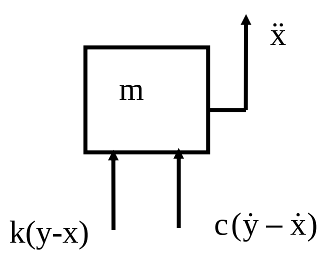
## Derivation

Consider a single degree-of-freedom system




where m equals mass, c equals the viscous damping coefficient, and k equals the stiffness. The absolute displacement of the mass equals x, and the base input displacement equals y.

The free-body diagram is




Summation of forces in the vertical direction,

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields,

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n^2$$

where $\omega_n$ is the natural frequency in (radians/sec), and $\xi$ is the damping ratio.

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \tag{6}$$

Let

$$z(t) = \hat{Z}\exp(j\omega t) \tag{7}$$

$$y(t) = \hat{Y}\exp(j\omega t) \tag{8}$$

$$\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]\hat{Z}\exp(j\omega t) = -\hat{Y}\exp(j\omega t) \tag{9}$$

Take the Fourier transform of each side.

$$\left[-\omega^2 + j2\xi\omega\omega_n + \omega_n^2\right]Z(\omega) = -Y(\omega) \tag{10}$$

$$Z(\omega) = \frac{-Y(\omega)}{\left[-\omega^2 + \omega_n^2 + j2\xi\omega\omega_n\right]} \tag{11}$$

$$Z(f) = \frac{-(1/2\pi)Y(f)}{\left[-f^2 + f_n^2 + j2\xi f f_n\right]} \tag{12}$$

Divide through by $f_n^2$ .

$$Z(f) = \frac{-\left(1/(2\pi f_n)^2\right)Y(f)}{\left[-\rho^2 + 1 + j2\xi\rho\right]} \tag{13}$$

where $\rho = f / f_n$

$$Z(f) = \frac{-\left(1/(2\pi f_n)^2\right)Y(f)}{\left[1 - \rho^2 + j2\xi\rho\right]} \tag{14}$$

Multiply each side by its complex conjugate.

$$Z(f)Z^*(f) = \frac{\left(1/(2\pi f_n)^2\right)}{\left[1 - \rho^2 + j2\xi\rho\right]}\frac{\left(1/(2\pi f_n)^2\right)}{\left[1 - \rho^2 - j2\xi\rho\right]}Y(f)Y^*(f) \tag{15}$$

$$Z(f)Z^*(f) = \frac{\left(1/(2\pi f_n)^2\right)}{\left[\rho^2 - 1 + j2\xi\rho\right]}\frac{\left(1/(2\pi f_n)^2\right)}{\left[\rho^2 - 1 - j2\xi\rho\right]}Y(f)Y^*(f) \tag{16}$$

The transfer function $H(\rho)$ times its complex conjugate is

$$H(\rho)H^*(\rho) = \left[\frac{\left(1/(2\pi f_n)^2\right)}{\rho^2 - j2\xi\rho - 1}\right]\left[\frac{\left(1/(2\pi f_n)^2\right)}{\rho^2 + j2\xi\rho - 1}\right] \tag{17}$$

Solve for the roots R1 and R2 of the first denominator.

$$R1, R2 = \frac{j2\xi \pm \sqrt{(-j2\xi)^2 - 4(-1)}}{2} \tag{18}$$

$$R1, R2 = \frac{j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{19}$$

$$R1, R2 = j\xi \pm \sqrt{1 - \xi^2} \tag{20}$$

Solve for the roots R3 and R4 of the second denominator.

$$R3, R4 = \frac{-j\xi \pm \sqrt{(j2\xi)^2 - 4(-1)}}{2} \tag{21}$$

$$R3, R4 = \frac{-j2\xi \pm \sqrt{-4\xi^2 + 4}}{2} \tag{22}$$

$$R3, R4 = -j\xi \pm \sqrt{1 - \xi^2} \tag{23}$$

(24)

Summary,

$$R1 = +j\xi + \sqrt{1 - \xi^2} \tag{25}$$

$$R2 = +j\xi - \sqrt{1 - \xi^2} \tag{26}$$

$$R3 = -j\xi + \sqrt{1 - \xi^2} \tag{27}$$

$$R4 = -j\xi - \sqrt{1-\xi^2} \tag{28}$$

Note

$$R2 = -R1^* \tag{29}$$

$$R3 = R1^* \tag{30}$$

$$R4 = -R1^* \tag{31}$$

Now substitute into the denominators.

$$H(\rho)H^*(\rho) = \left[\frac{1/(2\pi f_n)^4}{\left(\rho - j\xi - \sqrt{1-\xi^2}\right)\left(\rho - j\xi + \sqrt{1-\xi^2}\right)\left(\rho + j\xi - \sqrt{1-\xi^2}\right)\left(\rho + j\xi + \sqrt{1-\xi^2}\right)}\right] \tag{32}$$

$$H(\rho)H^*(\rho) = \left[\frac{1/(2\pi f_n)^4}{(\rho - R1)(\rho - R2)(\rho - R3)(\rho - R4)}\right] \tag{33}$$

$$H(\rho)H^*(\rho) = \left[\frac{1/(2\pi f_n)^4}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] \tag{34}$$

Expand into partial fractions.

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = \quad + \frac{\alpha}{(\rho - R1)} + \frac{\beta}{(\rho - R1^*)} + \frac{\lambda}{(\rho + R1^*)} + \frac{\sigma}{(\rho + R1)}$$

(35)

Equation (35) is solved using the method in Reference 1.

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{\left[8\xi\sqrt{1-\xi^2}\right]}\begin{bmatrix}-\xi - j\sqrt{1-\xi^2}\\ -\xi + j\sqrt{1-\xi^2}\\ +\xi - j\sqrt{1-\xi^2}\\ +\xi + j\sqrt{1-\xi^2}\end{bmatrix}$$

(36)

Let

$$\psi = \frac{\xi}{\sqrt{1-\xi^2}} \tag{37}$$

$$\begin{bmatrix}\alpha\\ \beta\\ \lambda\\ \sigma\end{bmatrix} = \frac{1}{8\xi}\begin{bmatrix}-\psi - j\\ -\psi + j\\ +\psi - j\\ +\psi + j\end{bmatrix}$$

(38)

$$R1 = +j\xi + \sqrt{1-\xi^2} \tag{39}$$

$$\left[\frac{1}{(\rho - R1)(\rho + R1^*)(\rho - R1^*)(\rho + R1)}\right] = + \frac{-\psi - j}{\left(\rho - \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{-\psi + j}{\left(\rho - \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{+\psi - j}{\left(\rho + \sqrt{1-\xi^2} - j\xi\right)}\left[\frac{1}{8\xi}\right] + \frac{+\psi + j}{\left(\rho + \sqrt{1-\xi^2} + j\xi\right)}\left[\frac{1}{8\xi}\right] \tag{40}$$

The overall relative displacement is found by integration.

$$\left[z_{RMS}(f_n, \xi)\right]^2 = \int_0^\infty H(\rho) H^*(\rho)\, \hat{Y}_{APSD}(f)\, df \tag{41}$$

$$\left[z_{RMS}(f_n, \xi)\right]^2 = f_n \int_0^\infty H(\rho) H^*(\rho)\, \hat{Y}_{APSD}(f)\, d\rho \tag{42}$$

Assume that the acceleration PSD is constant

$$\hat{Y}_{APSD}(f) = A \tag{43}$$

$$\left[x_{RMS}(f_n, \xi)\right]^2 = A\, f_n \int_0^\infty H(\rho) H^*(\rho)\, d\rho \tag{44}$$

$$\{ x_{RMS}(f_n, \xi) \}^2 \left[ \frac{8\xi (2\pi)^4 f_n{}^3}{A} \right] = \quad + \int_0^{\infty} \frac{-\psi - j}{\left( \rho - \sqrt{1-\xi^2} - j\xi \right)} d\rho$$

$$+ \int_0^{\infty} \frac{-\psi + j}{\left( \rho - \sqrt{1-\xi^2} + j\xi \right)} d\rho$$

$$+ \int_0^{\infty} \frac{+\psi - j}{\left( \rho + \sqrt{1-\xi^2} - j\xi \right)} d\rho$$

$$+ \int_0^{\infty} \frac{+\psi + j}{\left( \rho + \sqrt{1-\xi^2} + j\xi \right)} d\rho \tag{45}$$

Equation (45) is solved using the method in Reference 1.

$$\{ z_{RMS}(f_n, \xi) \}^2 \left[ \frac{8\xi (2\pi)^4 f_n{}^3}{A} \right] = 2\pi \tag{46}$$

$$\{ z_{RMS}(f_n, \xi) \}^2 = \left[ \frac{\pi}{4\xi} \right] \frac{A}{(2\pi)^4 f_n{}^3} \tag{47}$$

$$z_{RMS}(f_n, \xi) = \sqrt{\frac{\pi A}{4\xi (2\pi)^4 f_n{}^3}} \tag{48}$$

The RMS relative displacement is finally

$$z_{RMS}(f_n, \xi) = \sqrt{\frac{A}{64\pi^3 \xi f_n^3}} \tag{49}$$

$$Q = \frac{1}{2\xi} \tag{50}$$

$$z_{RMS} = \sqrt{\frac{QA}{32\pi^3 f_n^3}} \tag{51}$$

Note that equation for the absolute acceleration response in Reference 2 is

$$\ddot{x}_{GRMS}(f_n, Q) = \sqrt{\left(\frac{\pi}{2}\right) f_n \; Q \; A} \tag{52}$$

Thus

$$\ddot{x}_{GRMS} = [2\pi f_n]^2 \; z_{RMS} \tag{53}$$

<u>References</u>


1. T. Irvine, Derivation of Miles Equation for an Applied Force, Vibrationdata, 2008.

2. T. Irvine, Derivation of Miles Equation Revision D, Vibrationdata, 2008.