PID CONTROLLER FOR A SINGLE-DEGREE-OF-FREEDOM SYSTEM
Revision A

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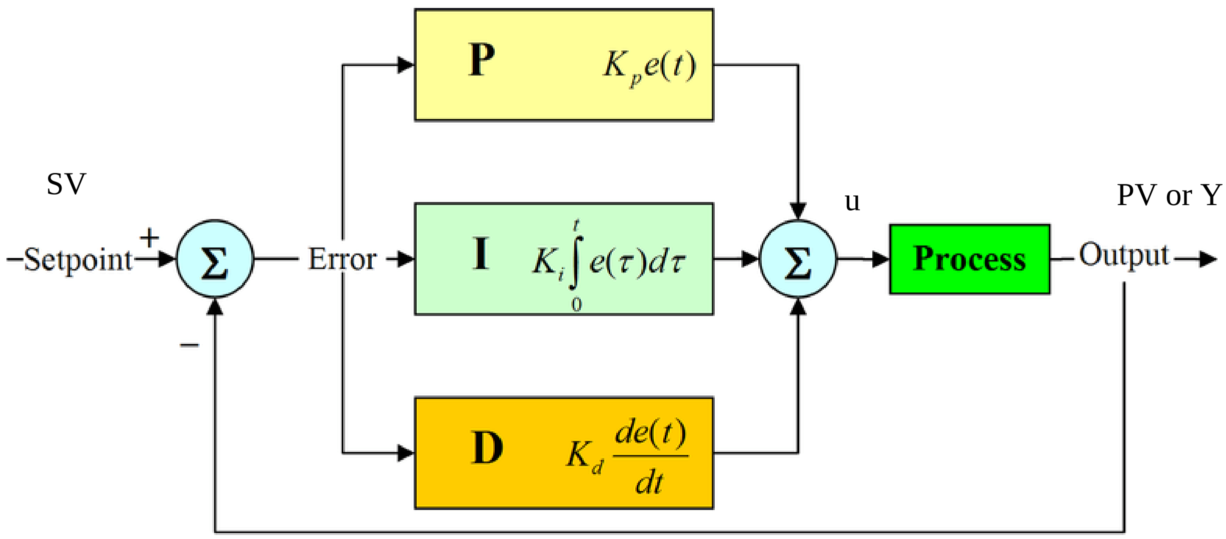

Figure 1. PID Controller Diagram

As an example, the PID controller could be used to read a sensor, and then to compute the desired actuator output by calculating proportional, integral, and derivative responses and summing those three components to compute the output.

Variables

e	tracking error
K_p	proportional gain
K_i	integral gain
K_d	derivative gain
R	desire input
u	controller output, process input
Y	process output
SP	setpoint
PV	process variable (plant output)
Y	process variable (plant output)

Introduction

A proportional–integral–derivative controller (PID controller) is a generic control loop feedback mechanism controller, widely used in industrial control systems. It is also called *a three-mode controller* or *process controller*.

A PID controller calculates an "error" value as the difference between a measured process variable and a desired setpoint. The controller attempts to minimize the error by adjusting the process control inputs.

Equations

The error is

$$e = SP - PV \tag{1}$$

The output of the PID controller is

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)dt + K_d \frac{de}{dt} \tag{2}$$

Take the Laplace transform

$$U(s) = K_p E(s) + K_i \frac{1}{s} E(s) + K_d\, s\, E(s) \tag{3}$$

The transfer function is

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d\, s = \frac{K_d\, s^2 + K_p\, s + K_i}{s} \tag{4}$$

<u>Proportional Term</u>

The proportional term makes a change to the output that is proportional to the current error value. The proportional response can be adjusted by multiplying the error by a constant K_p, called the proportional gain.

$$P_{out} = K_p e(t) \tag{5}$$

A high proportional gain results in a large change in the output for a given change in the error. If the proportional gain is too high, the system can become unstable. In contrast, a small gain results in a small output response to a large input error, and a less responsive or less sensitive controller. If the proportional gain is too low, the control action may be too small when responding to system disturbances. Tuning theory and industrial practice indicate that the proportional term should contribute the bulk of the output change.

<u>Integral Term</u>

The contribution from the integral term is proportional to both the magnitude of the error and the duration of the error. The integral in a PID controller is the sum of the instantaneous error over time and gives the accumulated offset that should have been corrected previously. The accumulated error is then multiplied by the integral gain (K_i) and added to the controller output

$$I_{out} = K_i \int_0^t e(\tau)\,dt \tag{6}$$

The integral term accelerates the movement of the process towards setpoint and eliminates the residual steady-state error that occurs with a pure proportional controller. However, since the integral term responds to accumulated errors from the past, it can cause the present value to overshoot the setpoint value

Derivative Term

The derivative of the process error is calculated by determining the slope of the error over time and multiplying this rate of change by the derivative gain K_d. The magnitude of the contribution of the derivative term to the overall control action is termed the derivative gain, K_d.

The derivative term is given by

$$D_{out} = K_d \frac{de}{dt} \tag{7}$$

The derivative term slows the rate of change of the controller output. Derivative control is used to reduce the magnitude of the overshoot produced by the integral component and improve the combined controller-process stability. However, the derivative term slows the transient response of the controller. Also, differentiation of a signal amplifies noise and thus this term in the controller is highly sensitive to noise in the error term, and can cause a process to become unstable if the noise and the derivative gain are sufficiently large. Hence an approximation to a differentiator with a limited bandwidth is more commonly used. Such a circuit is known as a phase-lead compensator.

Manual Tuning

Effects of *increasing* a parameter independently

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Decrease significantly	Degrade
K_d	Minor decrease	Minor decrease	Minor decrease	No effect in theory	Improve if K_d is small

General tips for designing a PID controller

The following steps can be used to obtain the desired response when designing a PID controller.

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until the desired overall response is obtained.

Not that a control system may not need all three modes.

For example, if a PI controller gives a good enough response, then a derivative controller is unnecessary. Keep the controller as simple as possible.

References

1. http://en.wikipedia.org/wiki/PID_controller
2. http://12000.org/my_notes/PID_ode/KERNEL/index.htm
3. T. Irvine, Table of Laplace Transforms, Revision I, Vibrationdata, 2011.

APPENDIX A

SDOF Vibration Example

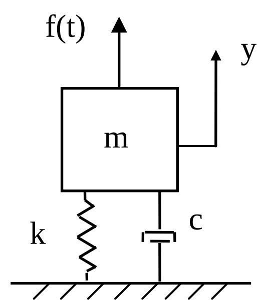

Figure A-1. SDOF System Subjected to an Applied Force

m	mass
c	viscous damping coefficient
k	stiffness
y	absolute displacement of the mass
f(t)	applied force

The SDOF system in Figure A-1 is the process block in Figure 1.

The equation of motion is

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{A-1}$$

Take the Laplace transform with zero initial conditions

$$L\{m\ddot{y} + c\dot{y} + ky\} = L\{f(t)\} \tag{A-2}$$

$$ms^2\, Y(s) + cs\, Y(s) + kY(s) = F(s) \tag{A-3}$$

$$\{ms^2 + cs + k\}Y(s) = F(s) \tag{A-4}$$

The transfer function is

$$\frac{F(s)}{Y(s)} = \frac{1}{ms^2 + cs + k} \tag{A-5}$$

The corresponding block diagram is

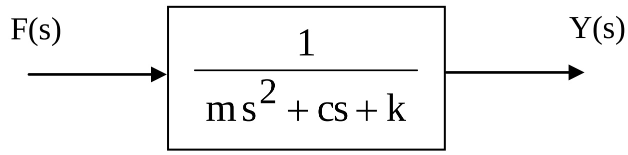

Figure A-2. SDOF System Transfer Function

Add the PID controller.

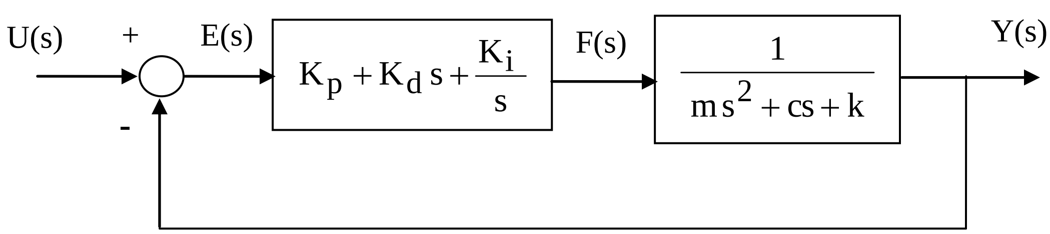

Figure A-3. SDOF System with PID Controller, System Block Diagram

Let L(s) be the open loop transfer function.

$$L(s) = \frac{K_p + K_d s + \frac{K_i}{s}}{ms^2 + cs + k} \tag{A-6}$$

$$L(s) = \frac{K_p s + K_d s^2 + K_i}{ms^3 + cs^2 + ks} \tag{A-7}$$

The closed loop transfer function G(s) is

$$G(s) = \frac{L(s)}{1 + L(s)} \tag{A-8}$$

$$G(s) = \frac{\frac{K_p s + K_d s^2 + K_i}{ms^3 + cs^2 + ks}}{1 + \frac{K_p s + K_d s^2 + K_i}{ms^3 + cs^2 + ks}} \tag{A-9}$$

$$G(s) = \frac{K_p s + K_d s^2 + K_i}{ms^3 + cs^2 + ks + K_p s + K_d s^2 + K_i} \tag{A-10}$$

$$G(s) = \frac{K_p s + K_d s^2 + K_i}{ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i} \tag{A-11}$$

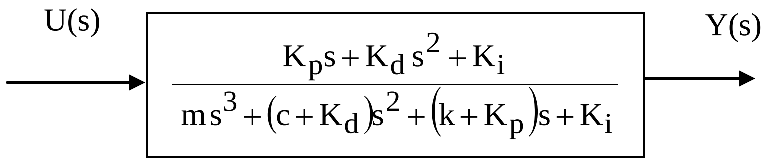

Figure A-4. Open Loop Transfer Function using PID Controller

The transfer function can also be written as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{m}\left(K_d s^2 + K_p s + K_i\right)}{s^3 + \frac{1}{m}(c + K_d)s^2 + \frac{1}{m}\left(k + K_p\right)s + \frac{K_i}{m}} \qquad \text{(A-12)}$$

There are three poles. Place one pole at $\alpha\xi\omega_n$ along the imaginary axis, as shown in Figure A-5.

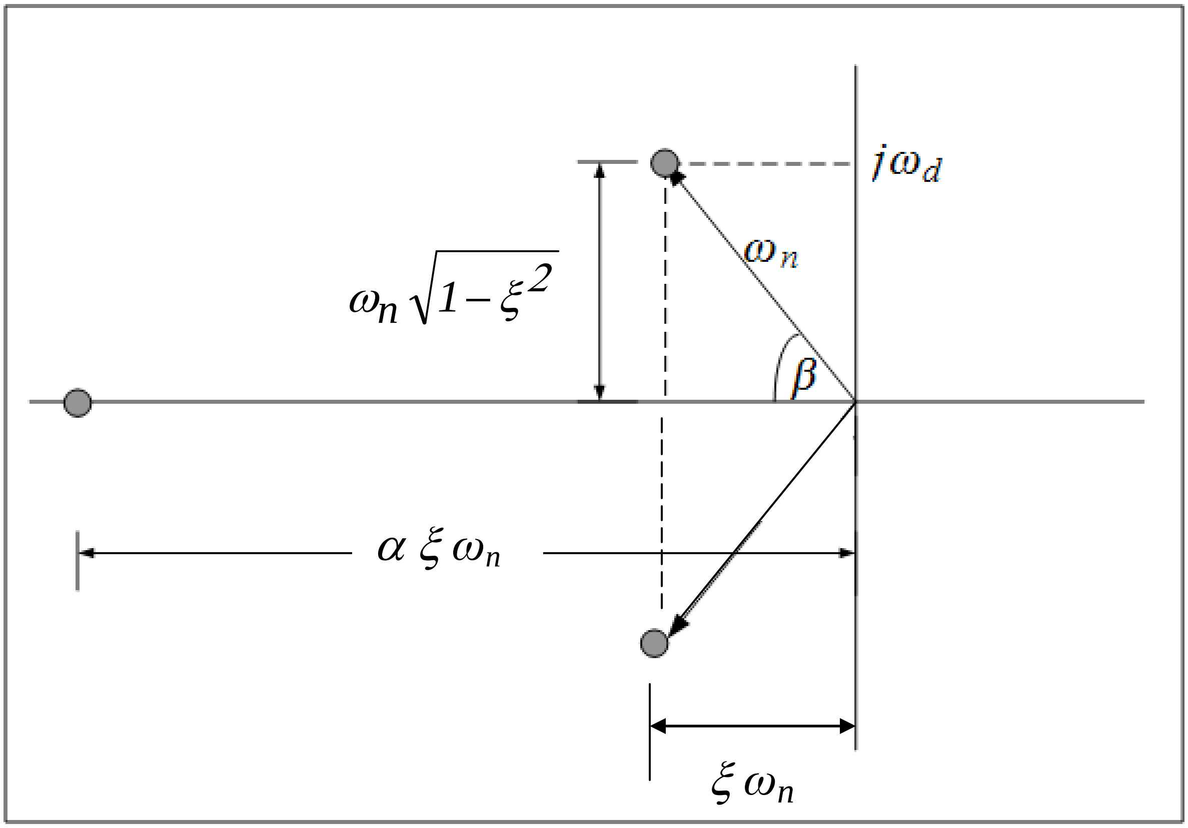

Figure A-5. Pole Placement

Model the transfer function denominator as

$$s^3+\frac{1}{m}(c+K_d)s^2+\frac{1}{m}(k+K_p)s+\frac{K_i}{m}=(s+\alpha\xi\omega_n)(s^2+2\xi\omega_n s+\omega_n{}^2) \quad \text{(A-13)}$$

$$s^3+\frac{1}{m}(c+K_d)s^2+\frac{1}{m}(k+K_p)s+\frac{K_i}{m}=s^3+2\xi\omega_n s^2+\omega_n{}^2 s$$
$$+\alpha\xi\omega_n s^2+2\alpha\xi^2\omega_n{}^2 s+\alpha\xi\omega_n{}^3$$

(A-14)

$$s^3 + \frac{1}{m}(c + K_d)s^2 + \frac{1}{m}(k + K_p)s + \frac{K_i}{m} = s^3 + 2\xi\omega_n s^2 + \omega_n{}^2 s$$
$$+ \alpha\xi\omega_n s^2 + 2\alpha\xi^2\omega_n{}^2 s + \alpha\xi\omega_n{}^3 \tag{A-15}$$

$$s^3 + \frac{1}{m}(c + K_d)s^2 + \frac{1}{m}(k + K_p)s + \frac{K_i}{m} = s^3 + (2 + \alpha)\xi\omega_n s^2 + \left(1 + 2\alpha\xi^2\right)\omega_n{}^2 s + \alpha\xi\omega_n{}^3 \tag{A-16}$$

Equate coefficients.

$$\frac{1}{m}(c + K_d) = (2 + \alpha)\xi\omega_n \tag{A-17}$$

$$\frac{1}{m}(k + K_p) = \left(1 + 2\alpha\xi^2\right)\omega_n{}^2 \tag{A-18}$$

$$\frac{K_i}{m} = \alpha\xi\omega_n{}^3 \tag{A-19}$$

The PID parameters are thus

$$K_d = (2 + \alpha)\xi\omega_n m - c \tag{A-20}$$

$$K_p = \left(1 + 2\alpha\xi^2\right)\omega_n{}^2 m - k \tag{A-21}$$

$$K_i = \alpha\xi\omega_n{}^3 m \tag{A-22}$$

The u(t) may be the unit step function or any arbitrary time-varying function.

A unit step function will be used for this example.

Set the variables for the sample system as

Variable	Value
M	1 kg
C	10 N sec/m
K	20 N/m
ω_n	17 rad/sec
ξ	0.1176
α	5
K_p	308.9680 N/m
K_i	2888.8 N /(m sec)
K_d	3.9944 N sec/m

Note that the frequency ω_n is not the natural frequency of the SDOF system. It is rather the natural frequency of the combined PID controller & SDOF system.

(The natural frequency of the SDOF system by itself is 4.5 rad/sec.)

The response of the mass is found using the formula derived in Appendix B. The resulting time history is shown in Figure A-6. The transfer function magnitude is given in Figure A-7. The calculations were made using Matlab script: pid_sdof.m.

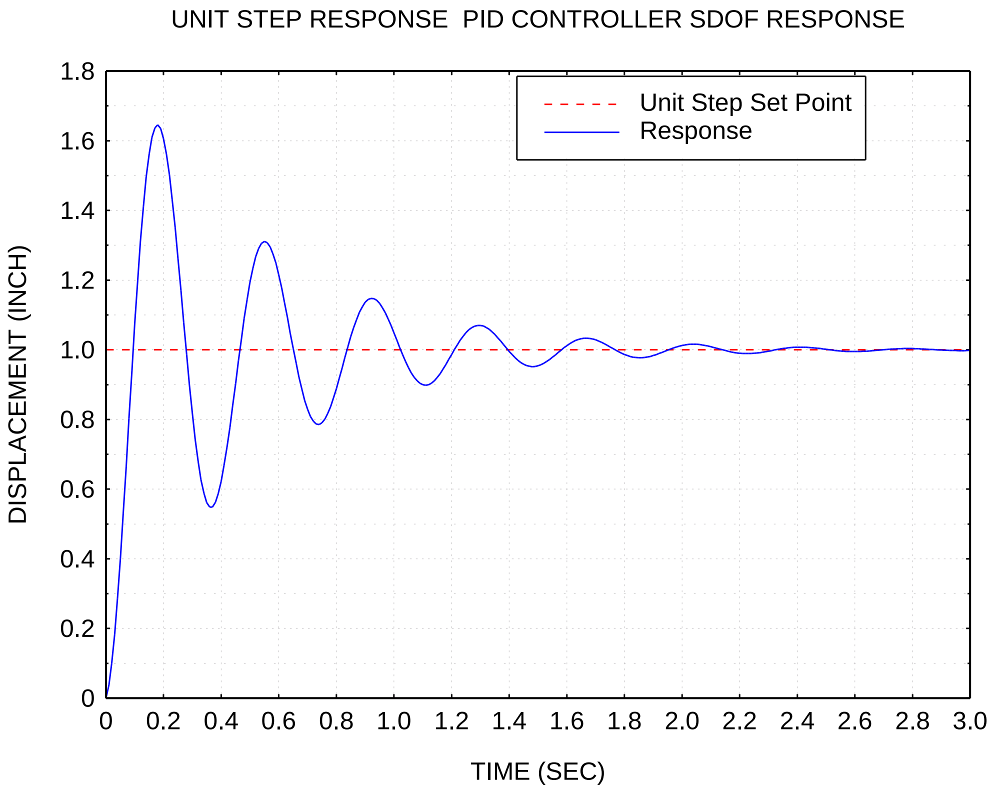
UNIT STEP RESPONSE PID CONTROLLER SDOF RESPONSE
Unit Step Set Point
Response
DISPLACEMENT (INCH)
TIME (SEC)
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0

Figure A-6.

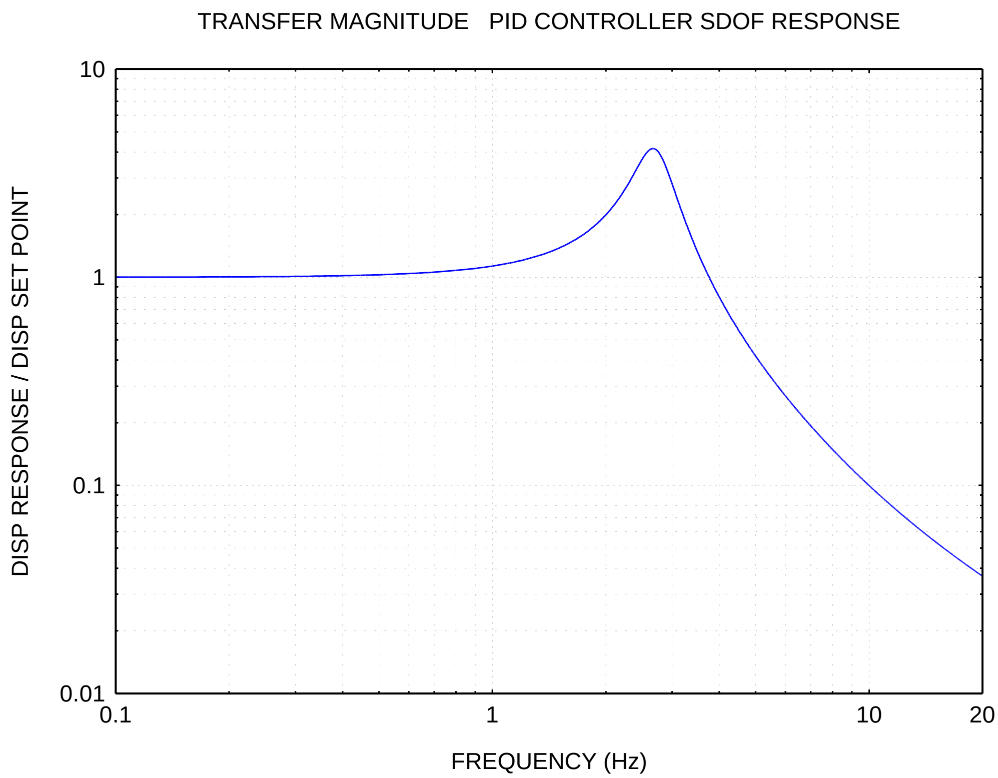
TRANSFER MAGNITUDE PID CONTROLLER SDOF RESPONSE
DISP RESPONSE / DISP SET POINT
10
1
0.1
0.01
0.1
1
10
20
FREQUENCY (Hz)

Figure A-7.

APPENDIX B

Response Time History Equation Derivation

Apply a unit set function to the transfer function. The Laplace transform of the unit step function is 1/s.

$$Y(s) = \frac{1}{s} G(s) = \left\{\frac{1}{s}\right\}\left\{\frac{\frac{1}{m}\left(K_d s^2 + K_p s + K_i\right)}{s^3 + \frac{1}{m}(c + K_d)s^2 + \frac{1}{m}\left(k + K_p\right)s + \frac{K_i}{m}}\right\} \tag{B-1}$$

Simplify the combined Laplace transform as follows.

$$\frac{1}{s} G(s) = \left\{\frac{1}{s}\right\}\left\{\frac{A s^2 + Bs + C}{(s + \alpha\xi\omega_n)\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right\} \tag{B-2}$$

$$A = \frac{K_d}{m} \tag{B-3}$$

$$B = \frac{K_p}{m} \tag{B-4}$$

$$C = \frac{K_i}{m} \tag{B-5}$$

Perform a partial fraction expansion.

$$\left\{\frac{1}{s}\right\}\left\{\frac{A s^2 + Bs + C}{(s + \alpha\xi\omega_n)\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)}\right\} = \frac{E}{s} + \frac{Q}{s + \alpha\xi\omega_n} + \frac{Rs + V}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \tag{B-6}$$

$$
\begin{aligned}
A s^2 + Bs + C = {} & E(s + \alpha\xi\omega_n)(s^2 + 2\xi\omega_n s + \omega_n{}^2) \\
& + Q\{s(s^2 + 2\xi\omega_n s + \omega_n{}^2)\} \\
& + \{Rs + V\}\{s(s + \alpha\xi\omega_n)\}
\end{aligned}
\tag{B-7}
$$

$$
\begin{aligned}
A s^2 + Bs + C = {} & E\{s^3 + 2\xi\omega_n s^2 + \omega_n{}^2 s\} \\
& + E\alpha\xi\omega_n(s^2 + 2\xi\omega_n s + \omega_n{}^2) \\
& + Q\{s^3 + 2\xi\omega_n s^2 + \omega_n{}^2 s\} \\
& + \{Rs + V\}\{s^2 + \alpha\xi\omega_n s\}
\end{aligned}
\tag{B-8}
$$

$$
\begin{aligned}
A s^2 + Bs + C = {} & Es^3 + E(2\xi\omega_n s^2) + E\omega_n{}^2 s \\
& + E\alpha\xi\omega_n s^2 + E(2\alpha(\xi\omega_n)^2)s + E\alpha\xi\omega_n{}^3 \\
& + Qs^3 + Q(2\xi\omega_n)s^2 + Q\omega_n{}^2 s \\
& + Rs^3 + R\alpha\xi\omega_n s^2 \\
& + Vs^2 + V\alpha\xi\omega_n s
\end{aligned}
\tag{B-9}
$$

$$
\begin{aligned}
A s^2 + Bs + C = {} & + (E + Q + R)s^3 \\
& + (E\,\xi\omega_n(2 + \alpha) + Q(2\xi\omega_n) + R\alpha\xi\omega_n + V)s^2 \\
& + (E\omega_n{}^2(1 + 2\alpha\xi^2) + Q\omega_n{}^2 + V\alpha\xi\omega_n)s \\
& + E\alpha\xi\omega_n{}^3
\end{aligned}
\tag{B-10}
$$

Equation (B-10) yields four equations.

$$E + Q + R = 0 \tag{B-11}$$

$$E\,\xi\omega_n(2+\alpha) + Q(2\xi\omega_n) + R\alpha\xi\omega_n + V = A \tag{B-12}$$

$$E\omega_n{}^2\left(1 + 2\alpha\xi^2\right) + Q\omega_n{}^2 + V\alpha\xi\omega_n = B \tag{B-13}$$

$$E\alpha\xi\omega_n{}^3 = C \tag{B-14}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ \xi\omega_n(2+\alpha) & 2\xi\omega_n & \alpha\xi\omega_n & 1 \\ \omega_n{}^2\left(1+2\alpha\xi^2\right) & \omega_n{}^2 & 0 & \alpha\xi\omega_n \\ \alpha\xi\omega_n{}^3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ Q \\ R \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ A \\ B \\ C \end{bmatrix} \tag{B-15}$$

Solve for E, Q, R, V using a numerical software algorithm.

The Laplace transform becomes

$$Y(s) = \frac{1}{s}G(s) \;= \frac{E}{s} + \frac{Q}{s + \alpha\xi\omega_n} + \frac{Rs + V}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \tag{B-16}$$

Let

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{B-17}$$

Thus,

$$\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \tag{B-18}$$

Substitute equation (B-18) into (B-16).

$$Y(s) = \frac{1}{s} G(s) \;\; = \frac{E}{s} + \frac{Q}{s + \alpha\xi\omega_n} + \frac{Rs + V}{(s + \xi\omega_n)^2 + \omega_d^{\,2}} \tag{B-19}$$

$$Y(s) = \frac{1}{s} G(s) \;\; = \frac{E}{s} + \frac{Q}{s + \alpha\xi\omega_n} + R\left[\frac{s + (V/R)}{(s + \xi\omega_n)^2 + \omega_d^{\,2}}\right] \tag{B-20}$$

The inverse Laplace transform via Reference 3 is

$$y(t) = E + Q\exp(-\alpha\xi\omega_n t) + R\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{(V/R) - \xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}, \qquad t \geq 0$$

(B-21)

APPENDIX C

Impulse Response Function

The transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{m}\left(K_d s^2 + K_p s + K_i\right)}{s^3 + \frac{1}{m}\left(c + K_d\right)s^2 + \frac{1}{m}\left(k + K_p\right)s + \frac{K_i}{m}} \tag{C-1}$$

Let

$$G(s) = \frac{As^2 + Bs + C}{\left(s + \alpha\xi\omega_n\right)\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)} \tag{C-2}$$

$$A = \frac{K_d}{m} \tag{C-3}$$

$$B = \frac{K_p}{m} \tag{C-4}$$

$$C = \frac{K_i}{m} \tag{C-5}$$

Expand into partial fractions.

$$\frac{As^2 + Bs + C}{\left(s + \alpha\xi\omega_n\right)\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right)} = \frac{R}{s + \alpha\xi\omega_n} + \frac{Ws + V}{s^2 + 2\xi\omega_n s + \omega_n{}^2} \tag{C-6}$$

$$A s^2 + Bs + C = R\left(s^2 + 2\xi\omega_n s + \omega_n{}^2\right) + (Ws + V)(s + \alpha\xi\omega_n) \tag{C-7}$$

$$\begin{aligned} A s^2 + Bs + C = {} & Rs^2 + R2\xi\omega_n s + R\omega_n{}^2 \\ & + Ws^2 + W\alpha\xi\omega_n s \\ & + Vs + V\alpha\xi\omega_n \end{aligned} \tag{C-8}$$

$$A = R + W \tag{C-9}$$

$$B = R(2\xi\omega_n) + W(\alpha\xi\omega_n) + V \tag{C-10}$$

$$C = R\omega_n{}^2 + V\alpha\xi\omega_n \tag{C-11}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2\xi\omega_n & \alpha\xi\omega_n & 1 \\ \omega_n{}^2 & 0 & \alpha\xi\omega_n \end{bmatrix} \begin{bmatrix} R \\ W \\ V \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{C-12}$$

The transfer function can be written as

$$G(s) \ = \frac{R}{s + \alpha\xi\omega_n} + W\left[\frac{s + (V/W)}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right] \tag{C-13}$$

The inverse Laplace transform via Reference 3 is

$$g(t) = R\exp(-\alpha\xi\omega_n t) + W\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{(V/W) - \xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\},$$

$$t \geq 0$$

(C-14)

The displacement for an arbitrary set point can then be found via a convolution integral.

$$y(t) = \int_0^t u(\tau)g(t-\tau)dt \tag{C-15}$$

The function u may be the unit step function or any arbitrary function.

Note that a small dt is needed for numerical accuracy in the case that the convolution integral is represented via a series.