RAYLEIGH-RITZ METHOD FOR BEAMS AND RODS

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Introduction

The Rayleigh-Ritz method is an extension of the Rayleigh method. The Rayleigh-Ritz method has two important benefits

1. It provides a more accurate estimate of the fundamental frequency.
2. It gives an approximation to higher frequencies and mode shapes.

The Ritz method uses a series of displacement functions multiplied by constant coefficients. The coefficients are adjusted by minimizing the frequency with respect to each of the coefficients.

Theory

Recall Rayleigh's equation for the kinetic energy T and the potential energy U.

$$T_{max} = U_{max} \tag{A-1}$$

Let

$$\omega^2 \hat{T}_{max} = T_{max} \tag{A-2}$$

where ω is the natural frequency.

By substitution,

$$\omega^2 \hat{T}_{max} = U_{max} \tag{A-3}$$

$$\omega^2 = \frac{U_{max}}{\hat{T}_{max}} \tag{A-4}$$

The Ritz method assumes that the displacement function y(x) is the sum of several displacement functions $\phi_i(x)$ multiplied by constants C_i.

$$y(x) = C_1\phi_1(x) + C_2\phi_2(x) + \ldots + C_n\phi_n(x) \quad \text{(A-5)}$$

The displacement functions $\phi_i(x)$ are admissible functions that must satisfy the geometric boundary conditions.

The energy terms can be expressed as

$$U_{max} = \frac{1}{2}\sum_i \sum_j k_{ij} C_i C_j \quad \text{(A-6a)}$$

$$\hat{T}_{max} = \frac{1}{2}\sum_i \sum_j m_{ij} C_i C_j \quad \text{(A-6b)}$$

The partial derivatives with respect to C_i are.

$$\frac{\partial}{\partial C_i} U_{max} = \sum_j k_{ij} C_j \quad \text{(A-7a)}$$

$$\frac{\partial}{\partial C_i} \hat{T}_{max} = \sum_j m_{ij} C_j \quad \text{(A-7b)}$$

The stiffness coefficients k_{ij} and the mass coefficients m_{ij} depend on the type of the problem.

The longitudinal oscillation of a rod has the following stiffness and mass coefficients

$$k_{ij} = \int EA\,\phi_i'\phi_j'\,dx \quad \text{(A-8)}$$

$$m_{ij} = \int m\,\phi_i\,\phi_j\,dx \quad \text{(A-9)}$$

Note that m is the mass per unit length.

The bending oscillation of a beam has the following stiffness and mass coefficients

$$k_{ij} = \int EI\,\phi_i''\phi_j''\,dx \quad \text{(A-10)}$$

$$m_{ij} = \int m \phi_i \phi_j \, dx \tag{A-11}$$

The next step is to minimize the natural frequency ω by differentiating it with respect to each of the constants. Apply the differentiation to equation (A-4).

$$\frac{\partial}{\partial C_i} \omega^2 = \frac{\partial}{\partial C_i} \left(\frac{U_{max}}{\hat{T}_{max}} \right) \tag{A-12}$$

Apply the chain rule.

$$\frac{\partial}{\partial C_i} \omega^2 = \left(\frac{1}{\hat{T}_{max}} \right) \frac{\partial U_{max}}{\partial C_i} + U_{max} \frac{\partial}{\partial C_i} \left(\frac{1}{\hat{T}_{max}} \right) \tag{A-13}$$

$$\frac{\partial}{\partial C_i} \omega^2 = \left(\frac{1}{\hat{T}_{max}} \right) \frac{\partial U_{max}}{\partial C_i} - \left(\frac{U_{max}}{\hat{T}_{max}{}^2} \right) \frac{\partial \hat{T}_{max}}{\partial C_i} \tag{A-14}$$

$$\frac{\partial}{\partial C_i} \omega^2 = \frac{\hat{T}_{max} \dfrac{\partial U_{max}}{\partial C_i} - U_{max} \dfrac{\partial \hat{T}_{max}}{\partial C_i}}{\hat{T}_{max}{}^2} \tag{A-15}$$

Now set the expression equal to zero for the purpose of minimizing the natural frequency.

$$\frac{\partial}{\partial C_i} \omega^2 = 0 \tag{A-16}$$

Thus

$$\frac{\hat{T}_{max} \dfrac{\partial U_{max}}{\partial C_i} - U_{max} \dfrac{\partial \hat{T}_{max}}{\partial C_i}}{\hat{T}_{max}{}^2} = 0 \tag{A-17}$$

$$\hat{T}_{max} \frac{\partial U_{max}}{\partial C_i} - U_{max} \frac{\partial \hat{T}_{max}}{\partial C_i} = 0 \tag{A-18}$$

$$\frac{\partial U_{max}}{\partial C_i} - \left(\frac{U_{max}}{\hat{T}_{max}} \right) \frac{\partial \hat{T}_{max}}{\partial C_i} = 0 \qquad \text{(A-19)}$$

Now substitute equation (A-4) into (A-19).

$$\frac{\partial U_{max}}{\partial C_i} - \omega^2 \frac{\partial \hat{T}_{max}}{\partial C_i} = 0 \qquad \text{(A-20)}$$

Substitute equations (A-7a) and (A-7b) into (A-20).

$$\sum_j k_{ij} C_j - \omega^2 \sum_j m_{ij} C_j = 0 \qquad \text{(A-21)}$$

Consider a system with n displacement functions. Equation (A-21) can be represented as

$$\left[k_{i1} C_1 + k_{i2} C_2 + \ldots + k_{in} C_n \right] - \omega^2 \left[m_{i1} C_1 + m_{i2} C_2 + \ldots + m_{in} C_n \right] = 0$$

(A-22)

$$\left(k_{i1} - \omega^2 m_{i1} \right) C_1 + \left(k_{i2} - \omega^2 m_{i2} \right) C_2 + \ldots + \left(k_{in} - \omega^2 m_{in} \right) C_n = 0 \qquad \text{(A-23)}$$

The index i also varies from 1 to n. Thus equation (A-23) can be expressed in matrix from.

$$\begin{bmatrix} \left(k_{11} - \omega^2 m_{11} \right) & \left(k_{12} - \omega^2 m_{12} \right) & \cdots & \left(k_{1n} - \omega^2 m_{1n} \right) \\ \left(k_{21} - \omega^2 m_{21} \right) & \left(k_{22} - \omega^2 m_{22} \right) & \cdots & \left(k_{2n} - \omega^2 m_{2n} \right) \\ \vdots & & & \vdots \\ \left(k_{n1} - \omega^2 m_{n1} \right) & \left(k_{n2} - \omega^2 m_{n2} \right) & \cdots & \left(k_{nn} - \omega^2 m_{nn} \right) \end{bmatrix} \begin{bmatrix} C_1 \\ C2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(A-24)

The Rayleigh-Ritz method thus reduces the system to a generalized eigenvalue problem. The natural frequencies are found by setting the determinant of the coefficient matrix equal to zero. The coefficients for each mode shape are found by substituting each natural frequency separately into the coefficient matrix and then by solving for the C_i values. Each mode shape has a set of these values.

Example 1

Consider a cantilever beam with mass per length ρ. Assume that the beam has a uniform cross section.

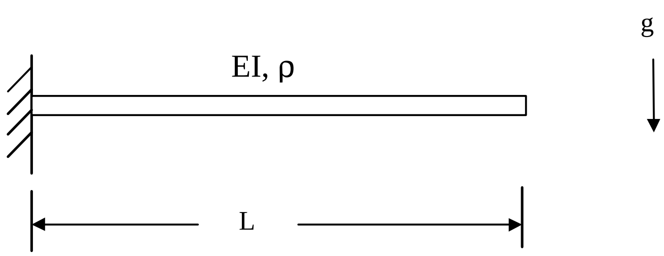

Figure B-1.

This beam was previously analyzed in Reference 1. The Rayleigh yielded a natural frequency estimate of

$$f_1 \approx \frac{1}{2\pi}\left\{\frac{3.664}{L^2}\right\}\sqrt{\frac{EI}{\rho}} \tag{B-1}$$

The "exact" solution is

$$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{B-2}$$

Use the Rayleigh-Ritz method with a three-term displacement series to refine the estimate of the fundamental frequency. Also, estimate the second and third natural frequencies.

The boundary conditions at the fixed end x = 0 are

$$y(0) = 0 \qquad \text{(zero displacement)} \tag{B-3}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{B-4}$$

The boundary conditions at the free end x = L are

$$\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{B-5}$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{B-6}$$

$$y(x) = C_1\phi_1(x) + C_2\phi_2(x) + C_3\phi_3(x) \tag{B-7}$$

Assume the following displacement functions, shown with their respective derivatives.

$$\phi_1(x) = \left[1 - \cos\left(\frac{\pi x}{2L}\right)\right] \tag{B-8}$$

$$\phi_1'(x) = \left(\frac{\pi}{2L}\right)\sin\left(\frac{\pi x}{2L}\right) \tag{B-9}$$

$$\phi_1''(x) = \left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right) \tag{B-10}$$

$$\phi_2(x) = \left[1 - \cos\left(\frac{3\pi x}{2L}\right)\right] \tag{B-11}$$

$$\phi_2'(x) = \left(\frac{3\pi}{2L}\right)\sin\left(\frac{3\pi x}{2L}\right) \tag{B-12}$$

$$\phi_2''(x) = \left(\frac{3\pi}{2L}\right)^2\cos\left(\frac{3\pi x}{2L}\right) \tag{B-13}$$

$$\phi_3(x) = \left[1 - \cos\left(\frac{5\pi x}{2L}\right)\right] \quad \text{(B-14)}$$

$$\phi_3'(x) = \left(\frac{5\pi}{2L}\right)\sin\left(\frac{5\pi x}{2L}\right) \quad \text{(B-15)}$$

$$\phi_3''(x) = \left(\frac{5\pi}{2L}\right)^2 \cos\left(\frac{5\pi x}{2L}\right) \quad \text{(B-16)}$$

The bending oscillation of a beam has the following stiffness and mass coefficients

$$k_{ij} = \int EI\, \phi_i''\phi_j''\, dx \quad \text{(B-17)}$$

$$m_{ij} = \int m\, \phi_i\, \phi_j\, dx \quad \text{(B-18)}$$

$$m_{11} = \int_0^L \rho\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right] dx \quad \text{(B-19)}$$

$$m_{11} = \int_0^L \rho\left[1 - 2\cos\left(\frac{\pi x}{2L}\right) + \cos^2\left(\frac{\pi x}{2L}\right)\right] dx \quad \text{(B-20)}$$

$$m_{11} = \int_0^L \rho\left[1 - 2\cos\left(\frac{\pi x}{2L}\right) + \frac{1}{2} + \cos\left(\frac{\pi x}{L}\right)\right] dx \quad \text{(B-21)}$$

$$m_{11} = \int_0^L \rho\left[\frac{3}{2} - 2\cos\left(\frac{\pi x}{2L}\right) + \cos\left(\frac{\pi x}{L}\right)\right] dx \quad \text{(B-22)}$$

$$m_{11} = \rho\left[\frac{3}{2}x - \left(\frac{4L}{\pi}\right)\sin\left(\frac{\pi x}{2L}\right) + \left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \quad \text{(B-23)}$$

$$m_{11} = \rho\left[\frac{3}{2}L - \left(\frac{4L}{\pi}\right)\right] \quad \text{(B-24)}$$

$$m_{11} = 0.2268\ \rho\ L \tag{B-25}$$

$$m_{22} = \int_0^L \rho\left[1-\cos\left(\frac{3\pi x}{2L}\right)\right]\left[1-\cos\left(\frac{3\pi x}{2L}\right)\right]dx \tag{B-26}$$

$$m_{22} = \int_0^L \rho\left[1-2\cos\left(\frac{3\pi x}{2L}\right)+\cos^2\left(\frac{3\pi x}{2L}\right)\right]dx \tag{B-27}$$

$$m_{22} = \int_0^L \rho\left[1-2\cos\left(\frac{3\pi x}{2L}\right)+\frac{1}{2}+\cos\left(\frac{3\pi x}{L}\right)\right]dx \tag{B-28}$$

$$m_{22} = \int_0^L \rho\left[\frac{3}{2}-2\cos\left(\frac{3\pi x}{2L}\right)+\cos\left(\frac{3\pi x}{L}\right)\right]dx \tag{B-29}$$

$$m_{22} = \rho\left[\frac{3}{2}x-\left(\frac{4L}{3\pi}\right)\sin\left(\frac{3\pi x}{2L}\right)+\left(\frac{L}{3\pi}\right)\sin\left(\frac{3\pi x}{L}\right)\right]\Bigg|_0^L \tag{B-30}$$

$$m_{22} = \rho\left[\frac{3}{2}L+\left(\frac{4L}{3\pi}\right)\right] \tag{B-31}$$

$$m_{22} = 1.9244\ \rho\ L \tag{B-32}$$

$$m_{33} = \int_0^L \rho\left[1-\cos\left(\frac{5\pi x}{2L}\right)\right]\left[1-\cos\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-33}$$

$$m_{33} = \int_0^L \rho\left[1-2\cos\left(\frac{5\pi x}{2L}\right)+\cos^2\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-34}$$

$$m_{33} = \int_0^L \rho\left[1-2\cos\left(\frac{5\pi x}{2L}\right)+\frac{1}{2}+\cos\left(\frac{5\pi x}{L}\right)\right]dx \tag{B-35}$$

$$m_{33} = \int_0^L \rho\left[\frac{3}{2}-2\cos\left(\frac{5\pi x}{2L}\right)+\cos\left(\frac{5\pi x}{L}\right)\right]dx \tag{B-36}$$

$$m_{33} = \rho \left[\frac{3}{2}x - \left(\frac{4L}{5\pi}\right)\sin\left(\frac{5\pi x}{2L}\right) + \left(\frac{L}{5\pi}\right)\sin\left(\frac{5\pi x}{L}\right)\right]\Bigg|_0^L \qquad \text{(B-37)}$$

$$m_{33} = \rho \left[\frac{3}{2}L - \left(\frac{4L}{5\pi}\right)\right] \qquad \text{(B-38)}$$

$$m_{33} = 1.2454\ \rho\ L \qquad \text{(B-39)}$$

$$m_{12} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]\left[1-\cos\left(\frac{3\pi x}{2L}\right)\right]dx \qquad \text{(B-40)}$$

$$m_{12} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)-\cos\left(\frac{3\pi x}{2L}\right)+\cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{3\pi x}{2L}\right)\right]dx \qquad \text{(B-41)}$$

$$m_{12} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)-\cos\left(\frac{3\pi x}{2L}\right)+\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{\pi x}{L}\right)\right]dx \qquad \text{(B-42)}$$

$$m_{12} = \rho\left[x-\left(\frac{2L}{\pi}\right)\sin\left(\frac{\pi x}{2L}\right)-\left(\frac{2L}{3\pi}\right)\sin\left(\frac{3\pi x}{2L}\right)+\frac{1}{2}\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L$$

(B-43)

$$m_{12} = \left[1-\left(\frac{2}{\pi}\right)+\left(\frac{2}{3\pi}\right)\right]\rho L \qquad \text{(B-44)}$$

$$m_{12} = \left[1 - \frac{4}{3\pi}\right]\rho L \qquad \text{(B-45)}$$

$$m_{12} = 0.5756\ \rho L \qquad \text{(B-46)}$$

$$m_{13} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)\right]\left[1-\cos\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-47}$$

$$m_{13} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)-\cos\left(\frac{5\pi x}{2L}\right)+\cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-48}$$

$$m_{13} = \int_0^L \rho\left[1-\cos\left(\frac{\pi x}{2L}\right)-\cos\left(\frac{5\pi x}{2L}\right)+\frac{1}{2}\cos\left(\frac{4\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{6\pi x}{L}\right)\right]dx \tag{B-49}$$

$$m_{13} = \rho\left[x-\left(\frac{2L}{\pi}\right)\sin\left(\frac{\pi x}{2L}\right)-\left(\frac{2L}{5\pi}\right)\sin\left(\frac{5\pi x}{2L}\right)+\frac{1}{2}\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)+\frac{1}{2}\left(\frac{L}{6\pi}\right)\sin\left(\frac{6\pi x}{L}\right)\right]\Bigg|_0^L \tag{B-50}$$

$$m_{13} = \left[1-\left(\frac{2}{\pi}\right)-\left(\frac{2}{5\pi}\right)\right]\rho L \tag{B-51}$$

$$m_{13} = \left[1-\left(\frac{10}{5\pi}\right)-\left(\frac{2}{5\pi}\right)\right]\rho L \tag{B-52}$$

$$m_{13} = \left[1-\frac{12}{5\pi}\right]\rho L \tag{B-53}$$

$$m_{13} = 0.2361\,\rho L \tag{B-54}$$

$$m_{23} = \int_0^L \rho\left[1-\cos\left(\frac{3\pi x}{2L}\right)\right]\left[1-\cos\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-55}$$

$$m_{23} = \int_0^L \rho\left[1-\cos\left(\frac{3\pi x}{2L}\right)-\cos\left(\frac{5\pi x}{2L}\right)+\cos\left(\frac{3\pi x}{2L}\right)\cos\left(\frac{5\pi x}{2L}\right)\right]dx \tag{B-56}$$

$$m_{23} = \int_0^L \rho\left[1-\cos\left(\frac{3\pi x}{2L}\right)-\cos\left(\frac{5\pi x}{2L}\right)+\frac{1}{2}\cos\left(\frac{8\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right]dx \tag{B-57}$$

$$m_{23} = \rho \left[x - \left(\frac{2L}{3\pi}\right)\sin\left(\frac{3\pi x}{2L}\right) - \left(\frac{2L}{5\pi}\right)\sin\left(\frac{5\pi x}{2L}\right) + \frac{1}{2}\left(\frac{L}{8\pi}\right)\sin\left(\frac{8\pi x}{L}\right) + \frac{1}{2}\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L$$

(B-58)

$$m_{23} = \left[1 + \left(\frac{2}{3\pi}\right) - \left(\frac{2}{5\pi}\right)\right]\rho L \quad \text{(B-59)}$$

$$m_{23} = \left[1 + \left(\frac{10}{15\pi}\right) - \left(\frac{6}{15\pi}\right)\right]\rho L \quad \text{(B-60)}$$

$$m_{23} = \left[1 + \left(\frac{4}{15\pi}\right)\right]\rho L \quad \text{(B-61)}$$

$$m_{23} = [1.0849]\rho L \quad \text{(B-62)}$$

Mass summary:

$$m_{11} = 0.2268\ \rho\ L \quad \text{(B-63)}$$

$$m_{12} = 0.5756\ \rho L \quad \text{(B-64)}$$

$$m_{13} = 0.2361\ \rho L \quad \text{(B-65)}$$

$$m_{22} = 1.9244\ \rho\ L \quad \text{(B-66)}$$

$$m_{23} = 1.0849\ \rho L \quad \text{(B-67)}$$

$$m_{33} = 1.2454\ \rho\ L \quad \text{(B-68)}$$

Note that equation (B-18) yields the following symmetry condition:

$$m_{ij} = m_{ji} \tag{B-69}$$

Solve for the stiffness coefficients using equation (B-17).

$$k_{11} = \int_0^L EI\left[\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right)\right]^2 dx \tag{B-70}$$

$$k_{11} = EI\left(\frac{\pi}{2L}\right)^4 \int_0^L \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi x}{L}\right)\right] dx \tag{B-71}$$

$$k_{11} = EI\left(\frac{\pi}{2L}\right)^4 \left[\frac{1}{2}x + \frac{1}{2}\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{B-72}$$

$$k_{11} = EI\left(\frac{\pi}{2L}\right)^4 \left(\frac{L}{2}\right) \tag{B-73}$$

$$k_{11} = EI\left(\frac{\pi^4}{2^5 L^3}\right) \tag{B-74}$$

$$k_{11} = 3.0440\left(\frac{EI}{L^3}\right) \tag{B-75}$$

$$k_{22} = \int_0^L EI\left[\left(\frac{3\pi}{2L}\right)^2 \cos\left(\frac{3\pi x}{2L}\right)\right]^2 dx \tag{B-76}$$

$$k_{22} = EI\left(\frac{3\pi}{2L}\right)^4 \int_0^L \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{3\pi x}{L}\right)\right] dx \tag{B-77}$$

$$k_{22} = EI\left(\frac{3\pi}{2L}\right)^4 \left[\frac{1}{2}x + \frac{1}{2}\left(\frac{L}{3\pi}\right)\sin\left(\frac{3\pi x}{L}\right)\right]\Bigg|_0^L \tag{B-78}$$

$$k_{22} = EI\left(\frac{3\pi}{2L}\right)^4\left(\frac{L}{2}\right) \tag{B-79}$$

$$k_{22} = EI\left(\frac{3^4\ \pi^4}{2^5\, L^3}\right) \tag{B-80}$$

$$k_{22} = 246.57\left(\frac{EI}{L^3}\right) \tag{B-81}$$

$$k_{33} = \int_0^L EI\left[\left(\frac{5\pi}{2L}\right)^2 \cos\left(\frac{5\pi x}{2L}\right)\right]^2 dx \tag{B-82}$$

$$k_{33} = EI\left(\frac{5\pi}{2L}\right)^4 \int_0^L \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{5\pi x}{L}\right)\right] dx \tag{B-83}$$

$$k_{33} = EI\left(\frac{5\pi}{2L}\right)^4 \left[\frac{1}{2}x + \frac{1}{2}\left(\frac{L}{5\pi}\right)\sin\left(\frac{5\pi x}{L}\right)\right]\Bigg|_0^L \tag{B-84}$$

$$k_{33} = EI\left(\frac{5\pi}{2L}\right)^4\left(\frac{L}{2}\right) \tag{B-85}$$

$$k_{33} = EI\left(\frac{5^4\ \pi^4}{2^5\, L^3}\right) \tag{B-86}$$

$$k_{33} = 1902.52\left(\frac{EI}{L^3}\right) \tag{B-87}$$

$$k_{12} = \int_0^L EI\left[\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right)\right]\left[\left(\frac{3\pi}{2L}\right)^2 \cos\left(\frac{3\pi x}{2L}\right)\right] dx \tag{B-88}$$

$$k_{12} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{3\pi}{2L}\right)^2\int_0^L \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{3\pi x}{2L}\right)\,dx \qquad \text{(B-89)}$$

$$k_{12} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{3\pi}{2L}\right)^2\int_0^L \left\{\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{\pi x}{L}\right)\right\}\,dx \qquad \text{(B-90)}$$

$$k_{12} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{3\pi}{2L}\right)^2\left\{\frac{1}{2}\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)+\frac{1}{2}\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right\}\Bigg|_0^L \qquad \text{(B-91)}$$

$$k_{12} = 0 \qquad \text{(B-92)}$$

$$k_{13} = \int_0^L EI\left[\left(\frac{\pi}{2L}\right)^2\cos\left(\frac{\pi x}{2L}\right)\right]\left[\left(\frac{5\pi}{2L}\right)^2\cos\left(\frac{5\pi x}{2L}\right)\right]dx \qquad \text{(B-93)}$$

$$k_{13} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{5\pi}{2L}\right)^2\int_0^L \cos\left(\frac{\pi x}{2L}\right)\cos\left(\frac{5\pi x}{2L}\right)\,dx \qquad \text{(B-94)}$$

$$k_{13} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{5\pi}{2L}\right)^2\int_0^L \left\{\frac{1}{2}\cos\left(\frac{\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{3\pi x}{L}\right)\right\}\,dx \qquad \text{(B-95)}$$

$$k_{13} = EI\left(\frac{\pi}{2L}\right)^2\left(\frac{3\pi}{2L}\right)^2\left\{\frac{1}{2}\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)+\frac{1}{2}\left(\frac{L}{3\pi}\right)\sin\left(\frac{3\pi x}{L}\right)\right\}\Bigg|_0^L \qquad \text{(B-96)}$$

$$k_{13} = 0 \qquad \text{(B-97)}$$

$$k_{23} = \int_0^L EI\left[\left(\frac{3\pi}{2L}\right)^2\cos\left(\frac{3\pi x}{2L}\right)\right]\left[\left(\frac{5\pi}{2L}\right)^2\cos\left(\frac{5\pi x}{2L}\right)\right]dx \qquad \text{(B-98)}$$

$$k_{23} = EI\left(\frac{3\pi}{2L}\right)^2\left(\frac{5\pi}{2L}\right)^2\int_0^L \cos\left(\frac{3\pi x}{2L}\right)\cos\left(\frac{5\pi x}{2L}\right)\,dx \qquad \text{(B-99)}$$

$$k_{23} = EI\left(\frac{3\pi}{2L}\right)^2\left(\frac{5\pi}{2L}\right)^2\int_0^L \left\{\frac{1}{2}\cos\left(\frac{\pi x}{L}\right)+\frac{1}{2}\cos\left(\frac{4\pi x}{L}\right)\right\}\, dx \tag{B-100}$$

$$k_{23} = EI\left(\frac{3\pi}{2L}\right)^2\left(\frac{3\pi}{2L}\right)^2\left\{\frac{1}{2}\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)+\frac{1}{2}\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)\right\}\Bigg|_0^L \tag{B-101}$$

$$k_{23} = 0 \tag{B-102}$$

Stiffness summary:

$$k_{11}=3.0440\left(\frac{EI}{L^3}\right) \tag{B-103}$$

$$k_{12} = 0 \tag{B-104}$$

$$k_{13} = 0 \tag{B-105}$$

$$k_{22}=246.57\left(\frac{EI}{L^3}\right) \tag{B-106}$$

$$k_{23} = 0 \tag{B-107}$$

$$k_{33}=1902.52\left(\frac{EI}{L^3}\right) \tag{B-108}$$

Note that equation (B-17) yields the following symmetry condition:

$$k_{ij} = k_{ji} \tag{B-109}$$

Recall the generalized eigenproblem.

$$\begin{bmatrix} \left(k_{11}-\omega^2 m_{11}\right) & \left(k_{12}-\omega^2 m_{12}\right) & \left(k_{13}-\omega^2 m_{13}\right) \\ \left(k_{21}-\omega^2 m_{21}\right) & \left(k_{22}-\omega^2 m_{22}\right) & \left(k_{23}-\omega^2 m_{23}\right) \\ \left(k_{31}-\omega^2 m_{31}\right) & \left(k_{32}-\omega^2 m_{32}\right) & \left(k_{33}-\omega^2 m_{33}\right) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(B-110)

Substitute the mass and stiffness coefficients into the matrix. Represent the matrix in upper triangular form due to the symmetry.

$$\left|K-\omega^2 M\right|=$$

$$\begin{bmatrix} 3.0440\left(\frac{EI}{L^3}\right)-0.2268\ \rho\ L\omega^2 & -0.5756\ \rho L\omega^2 & -0.2361\ \rho L\omega^2 \\ & 246.57\left(\frac{EI}{L^3}\right)-1.9244\ \rho\ L\ \omega^2 & -1.0849\ \rho L\omega^2 \\ & & 1902.52\left(\frac{EI}{L^3}\right)-1.2454\ \rho\ L\omega^2 \end{bmatrix}$$

(B-111)

Let

$$\lambda=\omega^2\left[\frac{\rho L^4}{EI}\right] \qquad \text{(B-112)}$$

By substitution,

$$\left|K-\omega^2 M\right|=$$

$$\begin{bmatrix} 3.0440-0.2268\ \lambda & -0.5756\ \lambda & -0.2361\ \lambda \\ & 246.57-1.9244\ \lambda & -1.0849\ \lambda \\ & & 1902.52-1.2454\ \lambda \end{bmatrix}$$

(B-113)

The eigenvalues are calculated by taking the determinant of the coefficient matrix in equation (B-113) and by setting the determinant equal to zero. Details of this solution method are given in References 2 and 3. The resulting natural frequencies are shown in Table 1. Note that the exact values are calculated from the governing partial differential equation as given in Reference 1.

Table 1.

Root Number	λ_n	Rayleigh-Ritz ω_n	Exact ω_n
1	12.39	$3.520\sqrt{\frac{EI}{\rho L^4}}$	$3.516\sqrt{\frac{EI}{\rho L^4}}$
2	493.6	$22.217\sqrt{\frac{EI}{\rho L^4}}$	$22.034\sqrt{\frac{EI}{\rho L^4}}$
3	4526.	$67.276\sqrt{\frac{EI}{\rho L^4}}$	$61.697\sqrt{\frac{EI}{\rho L^4}}$

The accuracy of the first and second natural frequencies is excellent. The third Rayleigh-Ritz frequency is 9% higher than the exact value.

References

1. T. Irvine, Natural Frequencies of Beam Bending Modes, Vibrationdata.com Publications, 1999.
2. T. Irvine, Roots of a Cubic Polynomial, Vibrationdata.com Publications, 1999.
3. T. Irvine, The Generalized Eigenvalue Problem, Vibrationdata.com Publications, 1999.